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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR SECOND-ORDER SELF-ADJOINT BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE

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Abstract. In this paper, we are concerned with the second order self-adjoint boundary value problem at resonance

$$-(p(t)x'(t))' = f(t, x(t)), \quad t \in (0,1),$$
$$x'(0) = 0, \quad x(1) = \int_0^1 x(s)g(s)ds.$$

A few new results are given for the existence of at least one, two, three and n positive solutions of the above boundary value problem by using the theory of a fixed point index for A-proper semilinear operators defined on cones, where n is an arbitrary natural number.

Key Words and Phrases: Boundary value problem, positive solution, resonance, multiplicity, A-proper, fixed point index.

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1. Introduction

In this paper, we study the existence and multiplicity of positive solutions to the following self-adjoint boundary value problem (BVP) at resonance

$$-(p(t)x'(t))' = f(t, x(t)), \quad t \in (0,1), \tag{1.1}$$

$$x'(0) = 0, \quad x(1) = \int_0^1 x(s)g(s)ds. \tag{1.2}$$

Throughout this paper, we assume that the following conditions hold without further mention.

(A_1) $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous, $p \in C[0,1] \cap C^1(0,1)$, $p(t) > t(2-t)$ on $[0,1]$,

$$\int_0^1 \frac{1}{p(t)}dt < e \quad \text{and} \quad \int_0^1 \int_s^1 \frac{\tau}{p(\tau)} d\tau g(s)ds > 0.$$

(A_2) $g \in L^1[0,1]$ with $g(t) \geq 0$ on $[0,1]$,

$$\int_0^1 g(s)ds = 1,\ g(t) \geq \frac{\displaystyle\int_0^1 \Big(\int_s^1 \frac{\tau}{p(\tau)} d\tau\Big) g(s)ds}{\displaystyle p(t)\int_0^1 \frac{\tau}{p(\tau)} d\tau \int_t^1 \frac{d\tau}{p(\tau)}}.$$

We note that condition (A_2) means that the self-adjoint boundary value problem $(1.1), (1.2)$ happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} -x''(t) = 0, \quad t \in (0,1), \\ x'(0) = 0, \quad x(1) = \displaystyle\int_0^1 x(s)g(s)ds, \end{cases}$$

has $x(t) \equiv c, t \in [0,1], c \in \mathbb{R}$, as a nontrivial solution.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multi-point and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers by Gallardo [1], Karakostas and Tsamatos [2], Lomtatidze and Malaguti [3] and the references therein. Moreover, boundary value problems with integral boundary conditions have been studied by many authors. For details, see, for example, [4–15] and references therein. The classical tools for such problems include the coincidence degree theory of Mawhin [16], the Leray-Schauder continuation theorem [16], the fixed point index theory [12, 14], fixed point theorem of cone expansion and compression [9]. However, as far as the positive solutions are concerned, most results are for the non-resonant case [4, 5, 9]. In most real problems, only the positive solution is significant. It is well known that the problem of existence of positive solutions to boundary value problem is very difficult when the resonant case is considered. For the existence of positive solutions of multi-point boundary value problems at resonance, there are only few related works; one can see [15, 17–24]. Recently, Yang and Ge [15] obtained the existence of a positive solution for problem (1.1), (1.2). The main method is the Leggett-Williams norm-type theorem established by O'Regan and Zima [25]. But there are no results concerning with the multiplicity of positive solutions for (1.1), (1.2). Being directly motivated by [15], in this paper, we study the existence of n positive solutions for boundary value problem (1.1) with integral boundary conditions (1.2), where n is an arbitrary natural number. This will be done by applying the theory of a fixed point index for A-proper semilinear operators defined on cones due to Cremins [26]. The method we used here is different from the paper [15] and the main results of this paper are also new.

The paper is divided into four sections. In Section 2, we provide some notation and lemmas, which play key roles in this paper. In Section 3, we use the theory of the fixed point index for A-proper semilinear operators defined on cones to establish several existence results of at least one, two, three, and n positive solutions to the BVP (1.1) and (1.2). Finally, in Section 4, we give an example to illustrate our results.

2. Notation and preliminaries

We start by introducing some basic notation relative to theory of the fixed point index for A-proper semilinear operators defined on cones established by Cremins (see [26]).

Let X and Y be Banach spaces, D a linear subspace of X, $\{X_n\} \subset D$, and $\{Y_n\} \subset Y$ sequences of oriented finite dimensional subspaces such that $Q_n y \to y$ in Y for every y and $\operatorname{dist}(x, X_n) \to 0$ for every $x \in D$ where $Q_n : Y \to Y_n$ and $P_n : X \to X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y.

Definition 2.1 [26]. *A map $T : D \subset X \to Y$ is called approximation-proper (abbreviated A-proper) at a point $y \in Y$ with respect to Γ, if $T_n \equiv Q_n T|_{D \cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever $\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$ is bounded with $T_{n_j} x_{n_j} \to y$, then there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \to x \in D$, and $Tx = y$. T is said to be A-proper on a set Ω if it is A-proper at all points of Ω.*

Let K be a cone in a finite dimensional Banach space X and $\Omega \subset X$ be open and bounded with $\Omega \cap K = \Omega_K \neq \emptyset$. Let $T : \overline{\Omega}_K \to K$ be continuous such that $Tx \neq x$ on $\partial\Omega_K = \partial\Omega \cap K$ where $\overline{\Omega}_K$ and $\partial\Omega_K$ denote the closure and boundary, respectively, of Ω_K relative K. Let $\rho : X \to K$ be an arbitrary retraction.

The following definition of finite dimensional index forms the basis of generalized index for A-proper maps $I - T$.

Definition 2.2 [26]. *We define*

$$i_K(T, \Omega) = \deg_B(I - T\rho, \rho^{-1}(\Omega) \cap B_R, 0),$$

where the degree is the Brouwer degree and B_R is a ball containing Ω_K.

Now let K be a cone in an infinite dimensional Banach space X with projection scheme Γ such that $Q_n(K) \subseteq K$ for every $n \in \mathbb{N}$. Let $\rho : X \to K$ be an arbitrary retraction and $\Omega \subset X$ an open bounded set such that $\Omega_K = \Omega \cap K \neq \emptyset$. Let $T : \overline{\Omega}_K \to K$ be such that $I - T$ is A-proper at 0. Write $K_n = K \cap X_n = Q_n K$ and $\Omega_n = \Omega_K \cap X_n$. Then $Q_n \rho : X_n \to K_n$ is a finite dimensional retraction.

Definition 2.3 [26]. *If $Tx \neq x$ on $\partial\Omega_K$, then we define*

$$\operatorname{ind}_K(T, \Omega) = \{k \in \mathbb{Z} \cup \{\pm\infty\} : i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j}) \to k \text{ for some } n_j \to \infty\},$$

that is, the index is the set of limit points of $i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j})$, where the finite dimensional index is that defined above.

Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm map of index zero and $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $X = \operatorname{Ker} L \bigoplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \bigoplus \operatorname{Im} Q$. The restriction of L to $\operatorname{dom} L \cap \operatorname{Ker} P$, denote L_1, is a bijection onto $\operatorname{Im} L$ with continuous inverse $L_1^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$. Since $\dim \operatorname{Im} Q = \dim \operatorname{Ker} L$, there exists a continuous bijection $J : \operatorname{Im} Q \to \operatorname{Ker} L$. let K be a cone in an infinite dimensional Banach space X with projection scheme Γ. If we let $H = L + J^{-1} P$, then $H : \operatorname{dom} L \subset X \to Y$ is a linear bijection with bounded inverse. Thus $K_1 = H(K \cap \operatorname{dom} L)$ is a cone in the Banach space Y.

Let $\Omega \subset X$ be open and bounded with $\Omega_K \cap \operatorname{dom} L \neq \emptyset$, $L : \operatorname{dom} L \subset X \to Y$ a bounded Fredholm operator of index zero, $N : \overline{\Omega}_K \cap \operatorname{dom} L \to Y$ a bounded continuous nonlinear operator such that $L - N$ is A-proper at 0.

We can now extend the definition of the index to A-proper maps of the form $L-N$ acting on cones.

Definition 2.4. [26] *Let ρ_1 be a retraction from Y to K_1 and assume $Q_nK_1 \subset K_1$, $P+JQN+L_1^{-1}(I-Q)N$ maps $K\cap \mathrm{dom}L$ to $K\cap \mathrm{dom}L$ and $Lx \neq Nx$ on $\partial\Omega_K$. We define the fixed point index of $L-N$ over Ω_K as*

$$\mathrm{ind}_K([L,N],\Omega) = \mathrm{ind}_{K_1}(T,U),$$

where $U = H(\Omega_K)$, $T : Y \to Y$ be defined as $Ty = (N+J^{-1}P)H^{-1}y$ for each $y \in Y$, and the index on the right is that of Definition 2.3.

For convenience, we recall some properties of ind_K.

Proposition 2.1. [26] *Let $L : \mathrm{dom}L \to Y$ be Fredholm of index zero, $\Omega \subset X$ be open and bounded. Assume that $P+JQN+L_1^{-1}(I-Q)N$ maps K to K, and $Lx \neq Nx$ on $\partial\Omega_K$. Then we have*

(P_1) *(Existence property) if $\mathrm{ind}_K([L,N],\Omega) \neq \{0\}$, then there exists $x \in \Omega_K$ such that $Lx = Nx$.*

(P_2) *(Normality property) if $x_0 \in \Omega_K$, then $\mathrm{ind}_K([L,-J^{-1}P+\hat{y}_0],\Omega) = \{1\}$, where $\hat{y}_0 = Hx_0$ and $\hat{y}_0(y) = y_0$ for every $y \in H(\Omega_K)$.*

(P_3) *(Additivity property) if $Lx \neq Nx$ for $x \in \overline{\Omega}_K\backslash(\Omega_1 \cup \Omega_2)$, where Ω_1 and Ω_2 are disjoint relatively open subsets of Ω_K, then*

$$\mathrm{ind}_K([L,N],\Omega) \subseteq \mathrm{ind}_K([L,N],\Omega_1) + \mathrm{ind}_K([L,N],\Omega_2)$$

with equality if either of indices on the right is a singleton.

(P_4) *(Homotopy invariance property) if $L-N(\lambda,x)$ is an A-proper homotopy on Ω_K for $\lambda \in [0,1]$ and $(N(\lambda,x)+J^{-1}P)H^{-1} : K_1 \to K_1$ and $\theta \notin (L-N(\lambda,x))(\partial\Omega_K)$ for $\lambda \in [0,1]$, then $\mathrm{ind}_K([L,N(\lambda,x)],\Omega) = \mathrm{ind}_{K_1}(T_\lambda,U)$ is independent of $\lambda \in [0,1]$, where $T_\lambda = (N(\lambda,x)+J^{-1}P)H^{-1}$.*

The following two lemmas will be used in this paper.

Lemma 2.1. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, Ω is an open bounded set, and $\Omega_K \cap \mathrm{dom}L \neq \emptyset$, and let $L-\lambda N$ be A-proper for $\lambda \in [0,1]$. Assume that N is bounded and $P+JQN+L_1^{-1}(I-Q)N$ maps K to K. If there exists $\varpi \in K_1\backslash\{\theta\}$, such that*

$$Lx - Nx \neq \mu\varpi, \tag{2.1}$$

for every $x \in \partial\Omega_K$ and all $\mu \geq 0$, then $ind_K([L,N],\Omega) = \{0\}$.

Proof. Choose a real number l such that

$$l > \sup_{x\in\Omega} \frac{\|Lx-Nx\|}{\|\varpi\|}, \tag{2.2}$$

and define $N(\mu,x) : [0,1]\times\overline{\Omega}_K \to Y$ by

$$N(\mu,x) = Nx + l\mu\varpi.$$

Trivially, $(N(\mu,x)+J^{-1}P)H^{-1} : K_1 \to K_1$ and from (2.1) we obtain

$$Nx + l\mu\varpi \neq Lx, \text{ for any } (\mu,x) \in [0,1]\times\partial\Omega_K.$$

Again, by homotopy invariance property in Proposition 2.1, we have

$$\mathrm{ind}_K([L,N(0,x)],\Omega) = \mathrm{ind}_K([L,N],\Omega) = \mathrm{ind}_K([L,N(1,x)],\Omega).$$

However

$$\mathrm{ind}_K([L, N(1,x)], \Omega) = \{0\}.$$

In fact, if $\mathrm{ind}_K([L, N(1,x)], \Omega) \neq \{0\}$, the existence property in Proposition 2.1 implies that there exists $x_0 \in \Omega_K$ such that

$$Lx_0 = Nx_0 + l\varpi.$$

Then

$$l = \frac{\|Lx_0 - Nx_0\|}{\|\varpi\|},$$

which contradicts (2.2). So

$$\mathrm{ind}_K([L, N], \Omega) = \{0\}.$$

Remark 2.1. The original condition of Theorem 5 in [26] was given with $\theta \neq \varpi \in L(K \cap \mathrm{dom}L)$ instead of $\varpi \in K_1 \backslash \{\theta\}$. The modification is necessary since otherwise it can not guarantee that $(N + \mu\varpi + J^{-1}P)H^{-1} : K_1 \to K_1$.

We assume that there is a continuous bilinear form $[y, x]$ on $Y \times X$ such that $y \in \mathrm{Im}L$ iff $[y, x] = 0$ for each $x \in \mathrm{Ker}L$. This condition implies that if $\{x_1, x_2, \cdots, x_n\}$ is a basis in $\mathrm{Ker}L$, then the linear map $J : \mathrm{Im}Q \to \mathrm{Ker}L$ define by $Jy = \beta \sum\limits_{i=1}^{n} [y, x_i]x_i,\ \beta \in \mathbb{R}^+$ is an isomorphism and that if $y = \sum\limits_{i=1}^{n} y_i x_i$ then $[J^{-1}y, x_i] = \dfrac{y_i}{\beta}$ for $1 \leq i \leq n$ and $[J^{-1}x_0, x_0] > 0$ for $x_0 \in \mathrm{Ker}L$.

In [26], Cremins extended a continuation theorem related to that of Mawhin [27] and Petryshyn [28] for semilinear equations to cones refer to [26, Corollary 1] for the details. By Lemma 2.1 and [26, Corollary 1], we obtain that following existence theorem of *positive* solutions to a semilinear equation $Lx = Nx$ in cones.

Lemma 2.2. *If $L : \mathrm{dom}L \to Y$ is Fredholm of index zero, $K \subset X$ is a cone, Ω_1 and Ω_2 are open bounded sets such that $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and $\Omega_2 \cap K \cap \mathrm{dom}L \neq \emptyset$. Suppose that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ with $N : \overline{\Omega_2 \cap K} \to Y$ bounded. Assume that*

(C_1) *$(P + JQN)(K) \subset K$ and $(P + JQN + L_1^{-1}(I - Q)N)(K) \subset K$,*
(C_2) *$Lx \neq \lambda Nx$ for $x \in \partial\Omega_2 \cap K,\ \lambda \in (0,1]$,*
(C_3) *$QNx \neq 0$ for $x \in \partial\Omega_2 \cap K \cap \mathrm{Ker}L$,*
(C_4) *$[QNx, x] \leq 0$, for all $x \in \partial\Omega_2 \cap K \cap \mathrm{Ker}L$,*
(C_5) *there exists $\varpi \in K_1 \backslash \{\theta\}$, such that*

$$Lx - Nx \neq \mu\varpi, \quad \textit{for every } \mu \geq 0,\ x \in \partial\Omega_1 \cap K.$$

Then there exists $x \in \mathrm{dom}L \cap K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $Lx = Nx$.

Corollary 2.1. *Assume all conditions of Lemma 2.2 hold except (C_2) and assume*

$(C_2)'$ *$\|Lx - Nx\|^2 \geq \|Nx\|^2 - \|Lx\|^2$ for each $x \in \partial\Omega_2 \cap K$. Then the same conclusion holds.*

Proof. We show that $(C_2)'$ implies (C_2), i.e., $Lx \neq \lambda Nx$, for each $x \in \partial\Omega_2 \cap K,\ \lambda \in (0,1]$. Here $\lambda \in (0,1)$. Otherwise, the proof is finished. If $x \in \mathrm{Ker}L \cap \partial\Omega_2 \cap K$, then it follows from $Lx = \lambda Nx = \theta$ that $Lx = Nx$ has a solution in $\mathrm{dom}L \cap K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and Corollary 2.1 is proved. If $x \in \mathrm{dom}L\backslash\mathrm{Ker}L \cap \partial\Omega_2 \cap K$ and $Lx = \lambda Nx$ for some $\lambda \in (0,1)$, then $Nx = \lambda^{-1}Lx$ and

$$(\lambda-1)^2\|Nx\|^2 = \|Lx - Nx\|^2 \geq \|Nx\|^2 - \|Lx\|^2 = (1-\lambda^2)\|Nx\|^2,$$

by condition $(C_2)'$; that is $(\lambda-1)^2 \geq 1-\lambda^2$, contradiction the fact that $\lambda \in (0,1)$. This completes the proof of Corollary 2.1.

3. Main results

The goal of this section is to apply Lemma 2.2 to discuss the existence and multiplicity of positive solutions for the BVP (1.1), (1.2). For simplicity of notation, we set

$$\omega := \int_0^1 \int_s^1 \frac{\tau}{p(\tau)} d\tau g(s) ds,$$

$$l(s) := \int_s^1 \int_\tau^1 \frac{1}{p(r)} dr g(\tau) d\tau + \int_s^1 \frac{1}{p(\tau)} d\tau \int_0^s g(\tau) d\tau,$$

$$G(t,s) := \begin{cases} \dfrac{1}{\omega} l(s)\Big(1 - \displaystyle\int_t^1 \frac{\tau}{p(\tau)} d\tau + \int_0^1 \frac{\tau^2}{p(\tau)} d\tau\Big) - \int_s^1 \frac{\tau}{p(\tau)} d\tau + \int_t^1 \frac{1}{p(\tau)} d\tau, \\ \qquad 0 \leq s \leq t \leq 1, \\ \dfrac{1}{\omega} l(s)\Big(1 - \displaystyle\int_t^1 \frac{\tau}{p(\tau)} d\tau + \int_0^1 \frac{\tau^2}{p(\tau)} d\tau\Big) + \int_s^1 \frac{1-\tau}{p(\tau)} d\tau, \\ \qquad 0 \leq t < s \leq 1. \end{cases}$$

It follows from (A_1) and (A_2) that $G(t,s) \geq 0,\ t,s \in [0,1]$, and

$$1 - \frac{\kappa}{\omega} l(s) \geq 0,\ s \in [0,1],$$

for every $\kappa \in \left(0, \dfrac{\omega}{\displaystyle\int_0^1 \int_s^1 \frac{1}{p(\tau)} d\tau g(s) ds}\right]$. We also set

$$\kappa := \min\left\{\frac{\omega}{\displaystyle\int_0^1 \int_s^1 \frac{1}{p(\tau)} d\tau g(s) ds}, \frac{1}{\max\limits_{t,s\in[0,1]} G(t,s)}\right\}.$$

Note that $\kappa < 1$.

Let $X = \{x \in C[0,1] : (px')' \in C[0,1],\ x'(0) = 0,\ x(1) = \int_0^1 x(s)g(s)ds\}$ endowed with the norm $\|x\|_X = \max\limits_{t\in[0,1]} |x(t)|$ and let $Y = C[0,1]$ with the norm $\|y\|_Y = \max\limits_{t\in[0,1]} |y(t)|$ and $K = \{x \in X : x(t) \geq 0,\ t \in [0,1]\}$, then K is a cone of X.

We define

$$\begin{aligned} &\mathrm{dom}L = X, \\ &L : \mathrm{dom}L \to Y, \quad (Lx)(t) = -(p(t)x'(t))', \\ &N : X \to Y, \quad (Nx)(t) = f(t, x(t)), \end{aligned}$$

then BVP (1.1), (1.2) can be written

$$Lx = Nx, \qquad x \in K.$$

It is easy to check that

$$\begin{aligned}
&\mathrm{Ker}L = \{x \in \mathrm{dom}L : x(t) \equiv c \text{ on } [0,1], c \in \mathbb{R}\},\\
&\mathrm{Im}L = \Big\{y \in Y : \int_0^1 l(s)y(s)ds = 0\Big\},\\
&\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L = 1,
\end{aligned}$$

so that L is a Fredholm operator of index zero.

Next, define the projections $P : X \to X$ by

$$Px = \int_0^1 x(s)ds,$$

and $Q : Y \to Y$ by

$$Qy = \frac{1}{\omega}\int_0^1 l(s)y(s)ds.$$

Furthermore, we define the isomorphism $J : \mathrm{Im}Q \to \mathrm{Im}P$ as $Jy = \beta y$, where $\beta = 1$. We are easy to verify that the inverse operator $L_1^{-1} : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ of $L|_{\mathrm{dom}L\cap\mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ as $(L_1^{-1}y)(t) = \displaystyle\int_0^1 k(t,s)y(s)ds$, where

$$k(t,s) = \begin{cases} \displaystyle\int_t^1 \frac{1}{p(\tau)}d\tau - \int_s^1 \frac{\tau}{p(\tau)}d\tau, & 0 \le s \le t \le 1,\\ \displaystyle\int_s^1 \frac{1-\tau}{p(\tau)}d\tau, & 0 \le t < s \le 1. \end{cases}$$

The following Theorem 3.1 is a basic existence criterion of BVP (1.1), (1.2).

Theorem 3.1. *Assume that there exist two positive numbers a, b such that*

(H_1) $f(t,x) \ge -\kappa x,$ *for all* $t \in [0,1],\ x \ge 0.$

(H_2) *If one of the two conditions*

(i) $f(t,x) > 0,\ \forall\ t \in [0,1],\ x \ge 0, \|x\|_X = a;\ f(t,b) < 0,\ \forall\ t \in [0,1],$

and

(ii) $f(t,a) < 0,\ \forall\ t \in [0,1],\ f(t,x) > 0,\ \forall\ t \in [0,1],\ x \ge 0, \|x\|_X = b$

is satisfied, then the BVP (1.1), (1.2) has at least one positive solution $x^* \in K$ satisfying $\min\{a,b\} \le \|x^*\|_X \le \max\{a,b\}$.

Proof. It is easy to see $a \ne b$. Without loss of generality, let $a < b$.

First, we note that L, as so defined, is Fredholm of index zero, L_1^{-1} is compact by Arzela-Ascoli theorem and thus $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ by (a) of Lemma 2 in [28].

For each $x \in K$, then by condition (H_1) that

$$\begin{aligned}
Px + JQNx &= \int_0^1 x(s)ds + \frac{1}{\omega}\int_0^1 l(s)f(s,x(s))ds\\
&\ge \int_0^1 (1 - \frac{\kappa}{\omega}l(s))x(s)ds\\
&\ge 0,
\end{aligned}$$

$$\begin{aligned} Px+JQNx+L_1^{-1}(I-Q)Nx &= \int_0^1 x(s)ds+\frac{1}{\omega}\int_0^1 l(s)f(s,x(s))ds \\ &\quad+\int_0^1 k(t,s)\Big[f(s,x(s))-\frac{1}{\omega}\int_0^1 l(\tau)f(\tau,x(\tau))d\tau\Big]ds \\ &=\int_0^1 x(s)ds+\int_0^1 G(t,s)f(s,x(s))ds \\ &\geq \int_0^1 (1-\kappa G(t,s))x(s)ds\geq 0. \end{aligned}$$

This implies that condition (C_1) of Lemma 2.2 is satisfied. To apply Lemma 2.2, we should define two open bounded subsets Ω_1,Ω_2 of X so that $(C_2)-(C_5)$ of Lemma 2.2 hold.

We prove only Case (H_2)(i). In the same way, we can prove the case (H_2)(ii). Let

$$\Omega_1=\{x\in X:\|x\|_X<a\},\ \Omega_2=\{x\in X:\|x\|_X<b\}.$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\theta\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2.$$

Next we show that (H_2)(i) implies (C_2). For this purpose, suppose that there exist $x_1\in K\cap\partial\Omega_2$ and $\lambda_1\in(0,1]$ such that $Lx_1=\lambda_1 Nx_1$ then $(p(t)x_1'(t))'=-\lambda_1 f(t,x_1(t))$ for all $t\in[0,1]$. Let $t_1\in[0,1]$, such that $x_1(t_1)=\max_{t\in[0,1]}x_1(t)=b$. From boundary conditions, we have $t_1\in[0,1)$. To continue with the proof, we distinguish between two cases.

Case 1. $t_1=0$. Here, $x_1'(0)=0$, we see from (H_2)(i) that

$$0<-\lambda_1 f(0,x_1(0))=(p(0)x_1'(0))'=p'(0)x_1'(0)+p(0)x_1''(0)=p(0)x_1''(0),$$

and consequently, $x_1''(0)>0$. It follows from $x_1''(t)$ is continuous in $[0,1]$ that there exists $\delta\in(0,1)$, such that $x_1''(t)>0$ when $t\in(0,\delta]$. This, together with boundary condition $x_1'(0)=0$, imply $x_1'(t)=x_1'(0)+\int_0^t x_1''(s)ds>0$. Hence

$$x_1(t)=x_1(0)+\int_0^t x_1'(s)ds>x_1(0),\quad t\in(0,\delta],$$

and $x_1(0)$ is not the maximum on $[0,1]$, a contradiction.

Case 2. $t_1\in(0,1)$. In this case, $x_1'(t_1)=0,\ x_1''(t_1)\leq 0$. This gives

$$-\lambda_1 f(t_1,x_1(t_1))=(p(t_1)x_1'(t_1))'=p'(t_1)x_1'(t_1)+p(t_1)x_1''(t_1)=p(t_1)x_1''(t_1)\leq 0,$$

which contradicts (H_2)(i). So for each $x\in\partial\Omega_2\cap K$ and $\lambda\in(0,1]$, we have $Lx\neq\lambda Nx$. Thus (C_2) of Lemma 2.2 is satisfied.

To prove (C_4) of Lemma 2.2, we define the bilinear form $[\cdot,\cdot]:Y\times X\to\mathbb{R}$ as

$$[y,x]=\int_0^1 l(s)y(s)x(s)ds.$$

It is clear that $[\cdot,\cdot]$ is continuous and satisfies $[y,x]=0$ for every $x\in\mathrm{Ker}L,\ y\in\mathrm{Im}L$. In fact, for any $x\in\mathrm{Ker}L$ and $y\in\mathrm{Im}L$, we have $x\equiv c$, a constant, and there exists $x\in X$ such that $y(s)=-(p(s)x'(s))'$ for each $s\in[0,1]$.

By $x'(0)=0,\ x(1)=\int_0^1 x(s)g(s)ds$, we get

$$[y,x]=\int_0^1 l(s)y(s)x(s)ds=-c\int_0^1 l(s)(p(s)x'(s))'ds=0.$$

Let $x\in \mathrm{Ker}L\cap\partial\Omega_2\cap K$, then $x(t)\equiv b$, so we have by condition (H_2)(i)

$$QNx=\frac{1}{\omega}\int_0^1 l(s)f(s,b)ds\neq 0,$$

$$\begin{aligned}[QNx,x] &= \int_0^1 l(s)\Big(\frac{1}{\omega}\int_0^1 l(\tau)f(\tau,b)d\tau\Big)\cdot b\ ds\\ &= b\int_0^1 l(s)ds\frac{1}{\omega}\int_0^1 l(\tau)f(\tau,b)d\tau\\ &< 0.\end{aligned}$$

Thus (C_3) and (C_4) of Lemma 2.2 are verified.

Finally, we prove (C_5) of Lemma 2.2 is satisfied. We may suppose that

$$Lx\neq Nx,\ \forall\ x\in\partial\Omega_1\cap K.$$

Otherwise, the proof is completed. Let $\varpi\equiv 1\in K_1\backslash\{\theta\}$. We claim that

$$Lx-Nx\neq\mu\varpi,\ \forall\ x\in\partial\Omega_1\cap K,\ \mu\geq 0. \tag{3.2}$$

In fact, if not, there exist $x_2\in\partial\Omega_1\cap K$ and $\mu_1>0$, such that

$$Lx_2-Nx_2=\mu_1.$$

Operating on both sides of the latter equation by Q and using $QL=\theta$, we have

$$QNx_2+Q\mu_1=0,$$

that is

$$\frac{1}{\omega}\int_0^1 l(s)(f(s,x_2(s))+\mu_1)ds=0. \tag{3.3}$$

For any $x_2\in\partial\Omega_1\cap K$, we have $\|x_2\|_X=a$. By condition (H_2)(i) and $\mu_1>0$,

$$\frac{1}{\omega}\int_0^1 l(s)(f(s,x_2(s))+\mu_1)ds>0,$$

in contradiction to (3.3). So (3.2) holds, that is (C_5) of Lemma 2.2 is verified.

Thus all conditions of Lemma 2.2 are satisfied and there exists $x^*\in K\cap(\overline{\Omega}_2\backslash\Omega_1)$ such that $Lx^*=Nx^*$ and the assertion follows. Thus $x^*\in K$ and $a\leq\|x^*\|_X\leq b$.

Let $[c]$ be the integer part of c. The following result concerns the existence of n positive solutions.

Theorem 3.2. *Assume that there exist $n+1$ positive numbers $a_1<a_2<\cdots<a_{n+1}$ such that*

$(H_1)'$ $f(t,x)\geq-\kappa x,\ for\ all\ t\in[0,1],\ x\geq 0.$

$(H_2)'$ *If one of the two conditions*

(i) $f(t,x)>0,\ \forall\ t\in[0,1],\ x\geq 0,\|x\|_X=a_{2i-1},\ i=1,2,\cdots,[\frac{n+2}{2}];$

$f(t,a_{2i})<0,\ \forall\ t\in[0,1],\ i=1,2,\cdots,[\frac{n+1}{2}]$

and

(ii) $f(t, a_{2i-1}) < 0,\ \forall\ t \in [0,1],\ i = 1, 2, \cdots, [\frac{n+2}{2}]$;
$f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_{2i},\ i = 1, 2, \cdots, [\frac{n+1}{2}]$
is satisfied, then the BVP(1.1), (1.2) has at least n positive solutions $x_i^* \in K,\ i = 1, 2, \cdots, n$ satisfying $a_i < \|x_i^*\|_X < a_{i+1}$.

Proof. Modeling the proof of Theorem 3.1, we can prove that if there exist two positive numbers a, b such that $f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = b$ and $f(t, a) < 0,\ \forall\ t \in [0,1]$, then BVP (1.1), (1.2) has at least one positive solution $x^* \in K$ satisfying $\min\{a, b\} < \|x^*\|_X < \max\{a, b\}$.

By the claim, for every pair of positive numbers $\{a_i, a_{i+1}\}, i = 1, 2, \cdots, n$, (1.1), (1.2) has at least n positive solutions $x_i^* \in K$ satisfying $a_i < \|x_i^*\|_X < a_{i+1}$.

We have the following existence result for two positive solutions.

Corollary 3.1. *Assume that there exist three positive numbers $a_1 < a_2 < a_3$ such that*

$(H_1)''$ $f(t, x) \geq -\kappa x,$ *for all* $t \in [0,1],\ x \geq 0.$

$(H_2)''$ *If one of the two conditions*

(i) $f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_1$;
$f(t, a_2) < 0,\ \forall\ t \in [0,1],\ f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_3$

and

(ii) $f(t, a_1) < 0,\ \forall\ t \in [0,1],\ f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_2$;
$f(t, a_3) < 0,\ \forall\ t \in [0,1]$

is satisfied, then the BVP (1.1), (1.2) has at least two positive solutions $x_1^*, x_2^* \in K$ satisfying $a_1 \leq \|x_1^*\|_X < a_2 < \|x_2^*\|_X \leq a_3$.

We also have the following existence result for three positive solutions.

Corollary 3.2. *Assume that there exist four positive numbers $a_1 < a_2 < a_3 < a_4$ such that*

$(H_1)'''$ $f(t, x) \geq -\kappa x,$ *for all* $t \in [0,1],\ x \geq 0.$

$(H_2)'''$ *If one of the two conditions*

(i) $f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_1;\ f(t, a_2) < 0,\ \forall\ t \in [0,1]$,
$f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_3;\ f(t, a_4) < 0,\ \forall\ t \in [0,1]$

and

(ii) $f(t, a_1) < 0,\ \forall\ t \in [0,1],\ f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_2$;
$f(t, a_3) < 0,\ \forall\ t \in [0,1],\ f(t, x) > 0,\ \forall\ t \in [0,1],\ x \geq 0, \|x\|_X = a_4$

is satisfied, then the BVP (1.1), (1.2) has at least three positive solutions $x_1^*, x_2^*, x_3^* \in K$ satisfying $a_1 \leq \|x_1^*\|_X < a_2 < \|x_2^*\|_X < a_3 < \|x_3^*\|_X \leq a_4$.

Remark 3.1. Using the method above, we can deal with the following self-adjoint boundary value problem (BVP)

$$\begin{cases} (p(t)x'(t))' = f(t, x(t)), & t \in (0,1), \\ x'(0) = 0, \quad x(1) = \int_0^1 x(s)g(s)ds. \end{cases} \tag{3.4}$$

We can also verify the similar results presented in this paper are valid for BVP (3.4), we omit the details here.

4. Example

In this section, we give an example to illustrate the main results of the paper. Consider the following second-order three-point boundary value problem (BVP)

$$\begin{cases} -(e^t x'(t))' = -\frac{(5t+2)(3e-8)}{10e-25}\sin x, \quad 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \displaystyle\int_0^1 2tx(t)dt. \end{cases} \tag{4.1}$$

Corresponding to the BVP (1.1), (1.2), $p(t) = e^t, g(t) = 2t$, and

$$f(t,x) = -\frac{(5t+2)(3e-8)}{10e-25}\sin x.$$

After direct computations, we get $\kappa = \frac{2(3e-8)}{2e-5}$ and $f(t,x) \geq -\kappa x,\ x \geq 0,\ t \in [0,1]$.

(1) We take $x \equiv \frac{\pi}{2} = a,\ x \equiv \frac{3\pi}{2} = b$, then all the conditions of Theorem 3.1 are satisfied. Thus BVP (4.1) has at least one positive solution x^* satisfying

$$\frac{\pi}{2} \leq \|x^*\|_X \leq \frac{3\pi}{2}.$$

(2) We take $x \equiv \frac{\pi}{2} = a_1,\ x \equiv \frac{3\pi}{2} = a_2,\ x \equiv \frac{5\pi}{2} = a_3$, then all the conditions of Corollary 3.1 are satisfied. Thus BVP (4.1) has at least two positive solutions x_1^*, x_2^* satisfying $\frac{\pi}{2} \leq \|x_1^*\|_X < \frac{3\pi}{2} < \|x_2^*\|_X \leq \frac{5\pi}{2}$.

(3) We take $x \equiv \frac{\pi}{2} = a_1,\ x \equiv \frac{3\pi}{2} = a_2,\ x \equiv \frac{5\pi}{2} = a_3,\ \ x \equiv \frac{7\pi}{2} = a_4$, then all the conditions of Corollary 3.2 are satisfied. Thus BVP (4.1) has at least three positive solutions x_1^*, x_2^*, x_3^* satisfying $\frac{\pi}{2} \leq \|x_1^*\|_X < \frac{3\pi}{2} < \|x_2^*\|_X < \frac{5\pi}{2} < \|x_3^*\|_X \leq \frac{7\pi}{2}$.

(4) We take $x \equiv \frac{(4i-3)\pi}{2} = a_{2i-1},\ i = 1, 2, \cdots, [\frac{n+2}{2}];\ x \equiv \frac{(4i-1)\pi}{2} = a_{2i},\ i = 1, 2, \cdots, [\frac{n+1}{2}]$, then all the conditions of Theorem 3.2 are satisfied. Thus BVP (4.1) has at least n positive solutions $x_i^*,\ i = 1, 2, \cdots, n$ satisfying $a_i < \|x_i^*\|_X < a_{i+1},\ i = 1, 2, \cdots, n$.

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