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POSITIVE-ADDITIVE FUNCTIONAL EQUATIONS IN C^*-ALGEBRAS

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Abstract. In this paper, we introduce a positive-additive functional equation in C^*-algebras. Using fixed point methods, we prove the stability of the positive-additive functional equation in C^*-algebras. Moreover, we prove the Hyers-Ulam stability of the positive-additive functional equation in C^*-algebras by the direct method of Hyers and Ulam.
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1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [35] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [31] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [31] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M. Rassias [28]–[30] followed the innovative approach of the Th.M. Rassias' theorem [31] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 9, 10, 17, 19]).

Definition 1.1. [8] Let A be a C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [8]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [14]).

Kenary [20] introduced the following functional equation

$$f\left(\left(\sqrt{x} + \sqrt{y}\right)^2\right) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$$

in the set of non-negative real numbers.

In this paper, we introduce the following functional equation

$$T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) = \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m \tag{1.1}$$

for all $x, y \in A^+$ and a fixed integer m greater than 1, which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*.

Note that the function $f(x) = cx,\ \ c \geq 0$, in the set of non-negative real numbers is a solution of the functional equation (1.1).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.2. [7, 32, 34] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set*

$$Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\};$$

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In 1991, Baker [2] used the Banach fixed point theorem to give generalized Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu [27] applied the fixed point alternative theorem to prove the generalized Hyers-Ulam stability. Miheţ [21] applied the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the generalized Hyers-Ulam stability for two functional equations in a single variable and L. Găvruta [11] used the Matkowski's fixed point theorem to obtain a new general result concerning the generalized Hyers-Ulam stability of a functional equation in a single variable. For a new method, see [13]. In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 15], [23]–[25], [33]).

This paper is organized as follows: In Section 2, using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.1) in C^*-algebras. In Section 3, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1.1) in C^*-algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^*-algebras A and B, respectively. Assume that m is a fixed integer greater than 1.

2. Stability of the positive-additive functional equation (1.1): fixed point approach

In this section, we investigate the positive-additive functional equation (1.1) in C^*-algebras.

Lemma 2.1. *Let $T : A^+ \to B^+$ be a positive-additive mapping satisfying* (1.1). *Then T satisfies*

$$T(2^{mn}x) = 2^{mn}T(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

Proof. Putting $x = y$ in (1.1), we obtain

$$T(2^m x) = 2^m T(x)$$

for all $x \in A^+$. So one can show that

$$T(2^{mn}x) = 2^{mn}T(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1.1) in C^*-algebras.

Note that the fundamental ideas in the proofs of the main results in this section are contained in [3, 4, 5].

Theorem 2.2. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{2^m}\varphi\left(2^m x, 2^m y\right) \tag{2.1}$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}}\right)^m \right\| \leq \varphi(x,y) \tag{2.2}$$

for all $x, y \in A^+$.

Then there exists a unique positive-additive mapping $T : A^+ \to A^+$ satisfying (1.1) *and*

$$\|f(x) - T(x)\| \leq \frac{L}{2^m - 2^m L}\varphi(x,x) \tag{2.3}$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (2.2), we get

$$\|f(2^m x) - 2^m f(x)\| \leq \varphi(x,x) \tag{2.4}$$

for all $x \in A^+$.

Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on X:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x,x),\ \forall x \in A^+\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [22]).

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2^m g\left(\frac{x}{2^m}\right)$$

for all $x \in A^+$.

Let $g, h \in X$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varphi(x,x)$$

for all $x \in A^+$. Hence

$$\|Jg(x) - Jh(x)\| = \|2^m g\left(\frac{x}{2^m}\right) - 2^m h\left(\frac{x}{2^m}\right)\| \leq L\varphi(x,x)$$

for all $x \in A^+$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in X$.

It follows from (2.4) that

$$\|f(x) - 2^m f\left(\frac{x}{2^m}\right)\| \leq \frac{L}{2^m}\varphi(x,x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{L}{2^m}$.

By Theorem 1.2, there exists a mapping $T : A^+ \to B^+$ satisfying the following:

(1) T is a fixed point of J, i.e.,

$$T\left(\frac{x}{2^m}\right) = \frac{1}{2^m}T(x) \tag{2.5}$$

for all $x \in A^+$. The mapping T is a unique fixed point of J in the set

$$M = \{g \in X : d(f,g) < \infty\}.$$

This implies that T is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - T(x)\| \leq \mu\varphi(x, x)$$

for all $x \in A^+$;

(2) $d(J^n f, T) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^{mn}}\right) = T(x)$$

for all $x \in A^+$;

(3) $d(f, T) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, T) \leq \frac{L}{2^m - 2^m L}.$$

This implies that the inequality (2.3) holds.

By (2.1) and (2.2),

$$2^{mn} \left\| f\left(\frac{\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m}{2^{mn}}\right) - \left(\left(2^{mn} f\left(\frac{x}{2^{mn}}\right)\right)^{\frac{1}{m}} + \left(2^{mn} f\left(\frac{y}{2^{mn}}\right)\right)^{\frac{1}{m}}\right)^m \right\|$$
$$\leq 2^{mn}\varphi\left(\frac{x}{2^{mn}}, \frac{y}{2^{mn}}\right) \leq L^{mn}\varphi(x, y)$$

for all $x, y \in A^+$ and all $n \in \mathbb{N}$. So

$$\left\|T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m\right\| = 0$$

for all $x, y \in A^+$. Thus the mapping $T : A^+ \to B^+$ is positive-additive, as desired. □

Corollary 2.3. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\left\|f\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(f(x)^{\frac{1}{m}} + f(y)^{\frac{1}{m}}\right)^m\right\| \tag{2.6}$$
$$\leq \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$$

for all $x, y \in A^+$.

Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{2^{mp} - 2^m}||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \text{ for all } x, y \in A^+.$$

Then we can choose $L = 2^{m-mp}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2^m L \varphi\left(\frac{x}{2^m}, \frac{y}{2^m}\right)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying (2.2). Then there exists a unique positive-additive mapping $T : A^+ \to A^+$ satisfying (1.1) and

$$\|f(x) - T(x)\| \leq \frac{1}{2^m - 2^m L}\varphi(x, x)$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2^m} g(2^m x)$$

for all $x \in A^+$.

It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2^m} f(2^m x) \right\| \leq \frac{1}{2^m}\varphi(x, x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{1}{2^m}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying (2.6). Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{2^m - 2^{mp}} ||x||^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \text{ for all } x, y \in A^+.$$

Then we can choose $L = 2^{mp-m}$ and we get the desired result. □

3. Stability of the positive-additive functional equation (1.1): direct method

In this section, using the direct method of Hyers and Ulam, we prove the Hyers-Ulam stability of the positive-additive functional equation (1.1) in C^*-algebras.

Theorem 3.1. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying* (2.2) *and*

$$\widetilde{\varphi}(x,y) := \sum_{j=1}^{\infty} 2^{mj} \varphi\left(\frac{x}{2^{mj}}, \frac{y}{2^{mj}}\right) \quad < \quad \infty \tag{3.1}$$

for all $x, y \in A^+$.

Then there exists a unique positive-additive mapping $T : A^+ \to A^+$ satisfying (1.1) *and*

$$\|f(x) - T(x)\| \le \frac{1}{2^m} \widetilde{\varphi}(x,y) \tag{3.2}$$

for all $x \in A^+$.

Proof. It follows from (2.4) that

$$\left\| f(x) - 2^m f\left(\frac{x}{2^m}\right) \right\| \le \varphi\left(\frac{x}{2^m}, \frac{x}{2^m}\right)$$

for all $x \in A^+$. Hence

$$\left\| 2^{ml} f\left(\frac{x}{2^{ml}}\right) - 2^{mk} f\left(\frac{x}{2^{mk}}\right) \right\| \quad \le \quad \frac{1}{2^m} \sum_{j=l+1}^{k} 2^{mj} \varphi\left(\frac{x}{2^{mj}}, \frac{x}{2^{mj}}\right) \tag{3.3}$$

for all nonnegative integers k and l with $k > l$ and all $x \in A^+$. It follows from (3.1) and (3.3) that the sequence $\left\{2^{mj} f\left(\frac{x}{2^{mj}}\right)\right\}$ is Cauchy for all $x \in A^+$. Since B^+ is complete, the sequence $\left\{2^{mj} f\left(\frac{x}{2^{mj}}\right)\right\}$ converges. So one can define the mapping $T : A^+ \to B^+$ by

$$T(x) := \lim_{j \to \infty} 2^{mj} f\left(\frac{x}{2^{mj}}\right)$$

for all $x \in A^+$.

By (2.2) and (3.1),

$$\begin{aligned}
&\left\| T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m \right\| \\
&= \lim_{j\to\infty} 2^{mj} \left\| f\left(\frac{\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m}{2^{mj}}\right) - \left(\left(2^{mj} f\left(\frac{x}{2^{mj}}\right)\right)^{\frac{1}{m}} + \left(2^{mj} f\left(\frac{y}{2^{mj}}\right)\right)^{\frac{1}{m}}\right)^m \right\| \\
&\le \lim_{j\to\infty} 2^{mj} \varphi\left(\frac{x}{2^{mj}}, \frac{y}{2^{mj}}\right) = 0
\end{aligned}$$

for all $x, y \in A^+$. So

$$T\left(\left(x^{\frac{1}{m}} + y^{\frac{1}{m}}\right)^m\right) - \left(T(x)^{\frac{1}{m}} + T(y)^{\frac{1}{m}}\right)^m = 0$$

for all $x, y \in A^+$. Hence the mapping $T : A^+ \to B^+$ is positive-additive. Moreover, letting $l = 0$ and passing the limit $k \to \infty$ in (3.3), we get (3.2). So there exists a positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) and (3.2).

Now, let $T' : A^+ \to B^+$ be another positive-additive mapping satisfying (1.1) and (3.2). Then we have

$$\begin{aligned}
\|T(x) - T'(x)\| &= 2^{mq} \left\| T\left(\frac{x}{2^{mq}}\right) - T'\left(\frac{x}{2^{mq}}\right) \right\| \\
&\le 2^{mq} \left\| T\left(\frac{x}{2^{mq}}\right) - f\left(\frac{x}{2^{mq}}\right) \right\| + 2^{mq} \left\| T'\left(\frac{x}{2^{mq}}\right) - f\left(\frac{x}{2^{mq}}\right) \right\| \\
&\le \frac{2 \cdot 2^{mq}}{2^m} \widetilde{\varphi}\left(\frac{x}{2^{mq}}, \frac{x}{2^{mq}}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in A^+$. So we can conclude that $T(x) = T'(x)$ for all $x \in A^+$. This proves the uniqueness of T. □

Corollary 3.2. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (2.6).

Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - T(x)\| \le \frac{2\theta_1 + \theta_2}{2^{mp} - 2^m} ||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.1. Then we get the desired result. □

Theorem 3.3. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying* (2.2) *such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-mj} \varphi(2^{mj}x, 2^{mj}y) < \infty$$

for all $x, y \in A^+$.

Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - T(x)\| \le \frac{1}{2^m} \widetilde{\varphi}(x, x)$$

for all $x \in A^+$.

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{1}{2^m} f(2^m x) \right\| \le \frac{1}{2^m} \varphi(x, x)$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (2.6). *Then there exists a unique positive-additive mapping $T : A^+ \to B^+$ satisfying* (1.1) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{2^m - 2^{mp}} ||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.3. Then we get the desired result. □

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