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P-CONVEXITY AND PROPERTIES IMPLYING THE FIXED POINT PROPERTY

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Abstract. We show that several properties implying the FPP do not follow from P-convexity.
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1. Introduction

Kottman introduced in 1970 the concept of P-convexity in [8]. He proved that every P-convex space is reflexive and also that P-convexity follows from uniform convexity, as well as from uniform smoothness. In 2008 Saejung proved in [13] that if a Banach space X is P-convex then X^* has uniform normal structure and in particular X^* has the fixed point property for nonexpansive mappings (FPP). So far, it seems to be unknown if P-convex Banach spaces have the FPP.

There are many geometrical conditions related to the FPP. Among these are the following: Brodskii and Milman introduced in 1948 the concept of normal structure of a Banach space [2] and Kirk proved in 1965 [7] that every Banach space with normal structure has the weak fixed point property for nonexpansive mappings (WFPP). Huff defined in 1980 the uniform Kadec-Klee property [6]. van Dulst and Sims proved in 1981 that every Banach space with the uniform Kadec-Klee property has the WFPP [14]. Property (S_m) was introduced by Wiśnicki in 2001 [15]. He proved that if X is a superreflexive space and there exists a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ such that the ultrapower $\{X\}_{\mathfrak{U}}$ has property (S_m) then X has the FPP. He also introduced in [15] another property stronger than property (S_m) called property (S). In 2006 in [4] García Falset, Llorens-Fuster and Mazcuñán Navarro defined the coefficient $MW(X)$ and proved that the condition $MW(X) > 1$ ensures the WFPP of X.

In this paper we show that normal structure, condition $MW(X) > 1$, property (S) and the Kadec-Klee property do not follow from P-convexity. We also prove that if X is a Banach space whose dual space X^* is P-convex then X has property (S).

2. P-convex Banach spaces

Throughout this paper $(X, \|\cdot\|)$ will be a Banach space and when there is no possible confusion, we simply write X. The unit ball $\{x \in X : \|x\| \leq 1\}$ and the unit sphere $\{x \in X : \|x\| = 1\}$ of X are denoted, respectively, by B_X and S_X. The topological dual space of X is denoted by X^*.

In [8], Kottman defined the concept of a P-convex Banach space as follows.

Definition 2.1. *Let X be a Banach space. For each $n \in \mathbb{N}$ let*

$$P(n, X) = \sup\{r > 0 : \textit{there exist } n \textit{ disjoint balls of radius } r \textit{ in } B_X\}.$$

In the same paper it is shown that $\frac{1}{3} \leq P(n, X) \leq \frac{1}{2}$ for each $n \geq 2$.

Definition 2.2. *X is said to be P-convex if $P(n, X) < \frac{1}{2}$ for some $n \in \mathbb{N}$.*

Recall that the *characteristic of convexity of X* is defined by

$$\varepsilon_0(X) = \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\},$$

where $\delta_X(\varepsilon) = \inf\{1 - \frac{1}{2}\|x - y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$ is the Clarkson *modulus of convexity of X*.

Among other interesting results, Kottman [8] proved that P-convexity is a sufficient condition for reflexivity. Furthermore, for a uniformly convex space ($\varepsilon_0(X) = 0$) one has $P(3, X) < \frac{1}{2}$ and hence, it is P-convex.

The following useful characterization of a P-convex space is also found in [8].

Lemma 2.3. *Let X be a Banach space and $n \in \mathbb{N}$. Then $P(n, X) < \frac{1}{2}$ if and only if there exists $\varepsilon > 0$ such that for any $x_1, x_2, ..., x_n \in S_X$*

$$\min\{\|x_i - x_j\| : 1 \leq i, j \leq n,\ i \neq j\} \leq 2 - \varepsilon. \tag{1}$$

That is, X is P-convex if and only if X satisfies condition (1) for some $n \in \mathbb{N}$ and some $\varepsilon > 0$.

Definition 2.4. *Given $n \in \mathbb{N}$ and $\varepsilon > 0$ we say that X is $P(\varepsilon, n)$-convex if X satisfies (1). For each $n \in \mathbb{N}$, X is said to be $P(n)$-convex if it is $P(\varepsilon, n)$-convex for some $\varepsilon > 0$.*

Recently, in [11] the author obtained the following result.

Theorem 2.5. *Let X be a Banach space which satisfies $\varepsilon_0(X) < 1$. Then X is P(3)-convex.*

3. P-convexity and properties implying the WFPP

In this section we will separate P-convexity from some geometric conditions on a Banach space sufficient for the FPP.

3.1. **P-convexity and normal structure.** Brodskii and Milman introduced in 1948 [2] the concept of normal structure.

Let C be a bounded subset of a Banach space X. For each $x \in C$ we define the radius of C relative to x by

$$r_x(C) = \sup\{\| x - y \| : \ y \in C\}.$$

Obviously $r_x(C) \leq diam(C)$. We say that $x \in C$ is a diametral point of C if the equality $r_x(C) = diam(C)$ holds, otherwise we say that x is a nondiametral point of C and C is called diametral if every point in C is diametral.

Definition 3.1. *A bounded and convex subset C of a Banach space has normal structure if each bounded, convex subset S of C with $diam(S) > 0$ contains a nondiametral point.*

There is another property stronger than normal structure called uniform normal structure defined by Bynum in 1980 [3]. For each $S \subset X$ we define

$$r(S) = \inf\{r_x(S) : x \in S\}.$$

Definition 3.2. *A nonempty, bounded and convex set C in X is said to have uniform normal structure if there exists a constant $k \in (0,1)$ such that*

$$r(S) \leq k \ diam(S)$$

for any closed convex subset $S \subset C$. The space X is said to have uniform normal structure if each of its nonempty convex subsets has this property.

Maluta proved in 1984 (see [9]) that every Banach space with uniform normal structure is reflexive. The normal structure has been widely studied in the fixed point theory for nonexpansive mappings since Kirk proved in 1965 that every Banach space with normal structure has the WFPP [7]. In 2008 Saejung proved in [13] that if a Banach space X is P-convex then X^* has uniform normal structure.

In the next example we present a P-convex space which fails to have normal structure. To do that, we will use the following lemma proved by Brodskii and Milman in [2].

Lemma 3.3. *A Banach space X does not have normal structure if and only if there exists a bounded sequence $\{x_n\}$ of elements of X such that*

$$\lim_{k\to\infty} d\big(x_{k+1}, conv\{x_i\}_{i=1}^k\big) = diam\{x_n\}_n.$$

Such a sequence is called a *diametral sequence*. The following example shows that normal structure does not follow from the P-convexity.

Example 3.4. *There is a P-convex space lacking normal structure.*

Consider the space X where X is l_2 with the norm

$$\|x\| = \max\left\{ \sup_{i\neq j}\{|x_i + x_j|\}, \|x\|_2 \right\}$$

where $\|\cdot\|_2$ is the l_2-norm. Naidu and Sastry proved in [12], example 3.6, that X is P-convex. We will see that X does not have normal structure. Define the sequence

$\{x_n\}_n$ as $x_k = \frac{1}{2}(e_{2k-1} + e_{2k})$ for each k, where $\{e_n\}_n$ is the canonical basis in l_2. It is easy to see that $\{x_n\}_n \subset S_X$ and $\|x_i - x_j\| = 1$ for all $i \neq j$. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be such that $\sum_{i=1}^n \alpha_i = 1$ and $\alpha_i \geq 0$ for each i. We have that

$$\left\| x_{n+1} - \sum_{i=1}^{n} \alpha_i x_i \right\| = \max\left\{1,\ \frac{\sqrt{2}}{2}\left(1 + \sum_{i=1}^{n} \alpha_i^2\right)^{1/2}\right\} = 1.$$

Then $\lim_{k\to\infty} d\big(x_{k+1}, conv\{x_i\}_{i=1}^k\big) = diam\{x_n\}_n$, that is, $\{x_n\}_n$ is a diametral sequence. Consecuently by Lemma 3.3 X does not have normal structure.

However we have that X has the FPP. To see that we recall that the Banach-Mazur distance between two isomorphic Banach spaces X and Y is

$$d(X,Y) = \inf\big\{\ \|U\|\|U^{-1}\| \mid U : X \to Y \text{ is a linear isomorphism}\big\}.$$

Mazcuñán-Navarro proved in 2005 [10] the following result on stability of the FPP.

Theorem 3.5. *Let H be a Hilbert space. If X is a Banach space such that*

$$d(X,H) < \sqrt{\frac{5+\sqrt{17}}{2}},$$

then X has the FPP.

Since $\|x\|_2 \leq \|x\| \leq \sqrt{2}\|x\|_2$ we get by Theorem 3.5 that X has the FPP.

3.2. **P-convexity and coefficient $MW(X)$.** In 2006 in [4] García Falset, Llorens-Fuster and Mazcuñán Navarro defined for each $a > 0$ the parameter

$$RW(a,X) = \sup\big\{(\liminf \|x_n + x\|) \wedge (\liminf \|x_n - x\|) : \{x_n\} \subset B_X, x_n \rightharpoonup 0, \|x\| \leq a\big\},$$

and the coefficient

$$MW(X) = \sup\left\{\frac{1+a}{RW(a,X)} : a > 0\right\}.$$

We have that $\max\{a,1\} \leq RW(a,X) \leq 1 + a$ for each $a > 0$ and $1 \leq MW(X) \leq 2$. They proved that if a Banach space X satisfies $MW(X) > 1$ then X has the WFPP. In particular, a uniformly nonsquare Banach space ($\varepsilon_0(X) < 2$) has this property.

The next example shows that P-convexity does not imply $MW(X) > 1$.

Example 3.6. *There is a P-convex Banach space X satisfying $MW(X) = 1$.*

Consider the space $X = (l_2, \|\cdot\|)$ obtained by renorming the space l_2 as follows. For each $x = (x_n)_n \in l_2$ we define

$$\|x\| = |x_1| + \|(x_2, x_3, ...)\|_2$$

where $\|\cdot\|_2$ is the l_2-norm. Since l_2 is uniformly convex, it is also P(ε,3)-convex for some $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $N\frac{\varepsilon}{2} \geq 1$. We will verify that X is P$\big(\frac{\varepsilon}{2}, 2N+1\big)$-convex. Let $x^{(1)}, x^{(2)}, ..., x^{(2N+1)} \in S_X$, $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, ...)$ for each $1 \leq m \leq 2N+1$. From

$$[0,1] \subset \big[0, N\frac{\varepsilon}{2}\big] = \bigcup_{k=1}^{N}\left[(k-1)\frac{\varepsilon}{2}, k\frac{\varepsilon}{2}\right]$$

we have that there are different $1 \leq i, j, k \leq 2N+1$ so that

$$\max\big\{|x_1^i - x_1^j|, |x_1^j - x_1^k|, |x_1^i - x_1^k|\big\} \leq \frac{\varepsilon}{2}.$$

On the other hand, define $y^{(m)} = (x_2^{(m)}, x_3^{(m)}, \ldots)$, $1 \leq m \leq 2N+1$. It is clear that $y^{(m)} \in B_{l_2}$ for every $1 \leq m \leq 2N+1$. Since l_2 is P(ε,3)-convex the next inequality holds

$$\min\big\{\|y^{(i)} - y^{(j)}\|_2, \|y^{(j)} - y^{(k)}\|_2, \|y^{(i)} - y^{(k)}\|_2\big\} \leq 2 - \varepsilon$$

and thus

$$\min\big\{\|x^{(i)} - x^{(j)}\|, \|x^{(j)} - x^{(k)}\|, \|x^{(i)} - x^{(k)}\|\big\} \leq 2 - \frac{\varepsilon}{2}.$$

Then X is P$\big(\frac{\varepsilon}{2}, 2N+1\big)$-convex. Now consider the canonical basis $\{e_n\}_n$ in l_2. It is clear that $e_n \in S_X$ for each n and $e_n \rightharpoonup 0$. Furthermore for every $a > 0$ we have that $\|ae_1 + e_i\| = 1 + a$ and $\|ae_1 - e_i\| = 1 + a$ for all $i > 1$. Then $RW(a, X) = 1 + a$ for every $a > 0$ and consecuently $MW(X) = 1$.

It is not difficult to see that $\|x\|_2 \leq \|x\| \leq \sqrt{2}\|x\|_2$ and by Theorem 3.5 we have that X has the FPP.

3.3. **P-convexity and property (S).** Before talking about property (S) we recall the definition and some results regarding ultrapowers which can be found in [1].

Let $\mathfrak{U}$ be a nontrivial ultrafilter on $\mathbb{N}$ and let X be a Banach space. A sequence $\{x_n\}$ in X converges to x with respect to $\mathfrak{U}$, denoted by $\lim_{\mathfrak{U}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathfrak{U}$. Let $l_\infty(X)$ be the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm $\|\{x_n\}\| = \sup_{n\in\mathbb{N}} \|x_n\| < \infty$, and let

$$\mathcal{N}_{\mathfrak{U}} = \left\{\{x_i\} \in l_\infty(X) : \lim_{\mathfrak{U}} ||x_i|| = 0\right\}.$$

The ultrapower of X, denoted by $\tilde{X}$, is the quotient space $l_\infty(X)/\mathcal{N}_{\mathfrak{U}}$ equipped with the quotient norm. Write $\{x_n\}_{\mathfrak{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that $\|\{x_n\}_{\mathfrak{U}}\| = \lim_{\mathfrak{U}} \|x_n\|$. We will write $\tilde{x}$ instead of $\{x_n\}_{\mathfrak{U}}$ unless we need to specify the ultrafilter we are talking about. Note that X can be embedded into $\tilde{X}$ isometrically.

In [15] Wiśnicki defined the following property of Banach spaces which he called property (S).

Definition 3.7. *We say that a Banach space X has property (S) if for every $A \subset S_X$ with diam$(A) \leq 1$ there exists a functional $F \in X^*$ such that $F(x) > 0$ for all $x \in A$.*

He showed that there exist superreflexive spaces which do not possess this property. He defined another property slightly weaker than property (S) and he called it property (S_m).

Definition 3.8. *A closed set A is said to be metrically convex if for every $x, y \in A$ there exists $z \in A$ such that*

$$\|x - z\| = \|y - z\| = \frac{\|x - y\|}{2}.$$

We say that a Banach space X has property (S_m) if for every metrically convex $A \subset X$ with $diam(A) \leq 1$ there exists a functional $F \in X^*$ such that $F(x) > 0$ for all $x \in A$.

Moreover Wiśnicki in [15] proved the next theorem.

Theorem 3.9. *If X is a superreflexive space and there exists a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ such that the ultrapower $\{X\}_{\mathfrak{U}}$ has property (S_m) then X has the FPP.*

He also proved that every separable space, every strictly convex space and every Banach space X satisfying $\varepsilon_0(X) < 1$ has property (S). We will show that P-convexity does not necessarily imply property (S).

Example 3.10. *There exists a P-convex Banach space lacking property (S).*
Consider the space X where X is $l_2(\mathbb{R})$ with the equivalent norm

$$\|x\| = \max\left\{ \sup_{\alpha \neq \beta}\{|x_\alpha + x_\beta|\}, \|x\|_2 \right\}$$

and $\|\cdot\|_2$ is the l_2-norm. Because for each $x = \{x_\alpha\}_{\alpha\in\mathbb{R}} \in l_2(\mathbb{R})$ we have that all except possibly a countable number of x_α's are equal to zero, we can prove as in [12], example 3.6, that X is P-convex. We will see that X lacks property (S). For each $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define the element $x_{\alpha,\beta} = \frac{1}{2}(e_\alpha + e_\beta)$, where $\{e_\alpha\}_\alpha$ is the canonical basis in $l_2(\mathbb{R})$. Let $A = \{x_{\alpha,\beta} : \alpha, \beta \in \mathbb{R}, \alpha < \beta\}$. Clearly $A \subset S_X$. We will verify that $diam(A) = 1$. Indeed, let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha < \beta$, $\gamma < \delta$. If $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are all different then $\|x_{\alpha,\beta} - x_{\gamma,\delta}\| = 1$. Otherwise, only two of them can be equal. In this case we have that $\|x_{\alpha,\beta} - x_{\gamma,\delta}\| = \frac{\sqrt{2}}{2}$ and so $diam(A) = 1$. Finally let $y \in \left(l_2(\mathbb{R})\right)^* = l_2(\mathbb{R})$, $y = \{y_\alpha\}_{\alpha\in\mathbb{R}}$. Choosing $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, such that $y_\alpha = y_\beta = 0$ and considering $x_{\alpha,\beta} \in A$ we get that $y(x_{\alpha,\beta}) = 0$ and thus X does not have property (S).

By Theorem 3.5 we have that X has the FPP because $\|x\|_2 \leq \|x\| \leq \sqrt{2}\|x\|_2$.
Now we will prove that if X is a Banach space whose dual space X^* is P-convex then X has property (S). In order to prove it we need the following lemma which is a slight modification of Lemma 4.6 from [15].

Lemma 3.11. *Let X be a reflexive space and suppose there exists $A \subset S_X$ with $diam(A) \leq 1$ which can not be separated from zero, that is, for each $F \in X^*$ there exists $x \in A$ such that $F(x) = 0$. Then for every $\varepsilon > 0$ there exist sequences $\{x_n\}_n \subset A$ and $\{f_n\}_n \subset S_{X^*}$ so that $f_i(x_i) = 1$ and $0 \leq f_i(x_j) < \varepsilon$ for $i \neq j$.*

Proof. In Lemma 4.6 from [15], Wiśnicki proved that if for each $x \in A$ we take a supporting functional $f_x \in S_{X^*}$ with $f_x(x) = 1$ then, there exists $y \in \bar{A}^w$ such that $f_x(y) = 0$ for all $x \in A$, where $\bar{A}^w$ denotes the weak closure of A. Let $\varepsilon > 0$. We choose a sequence $\{x_n\}_n \subset A$ such that $x_n \rightharpoonup y$. For each n denote $f_{x_n} = f_n$. Because B_{X^*} is weakly compact there exists a subsequence of $\{f_n\}_n$, which we denote again as $\{f_n\}_n$, such that $f_n \rightharpoonup f \in B_{X^*}$. In particular $f(y) = 0$. Since $x_n \rightharpoonup y$, $f(y) = f_n(y) = 0$ for each n and $f_n \rightharpoonup f$, there exists an integer $k_1 > 1$ such that $|f_1(x_k - y)| = |f_1(x_k)| < \varepsilon$, $|f(x_k - y)| = |f(x_k)| < \frac{\varepsilon}{2}$ and $|(f_k - f)(x_1)| < \frac{\varepsilon}{2}$ for each $k > k_1$. Similarly there exists an integer $k_2 > k_1$ such that $|f_{k_1}(x_k)| < \varepsilon$,

$|f(x_k)| < \frac{\varepsilon}{2}$ and $|(f_k - f)(x_{k_1})| < \frac{\varepsilon}{2}$ for every $k > k_2$. Proceeding inductively we find an increasing subsequence of natural numbers $\{k_i\}_i$ so that $|f_{k_i}(x_{k_j})| < \varepsilon$, $|f(x_{k_i})| < \frac{\varepsilon}{2}$ and $|(f_{k_j} - f)(x_{k_i})| < \frac{\varepsilon}{2}$ for each $1 \le i < j$. By the last two inequalities we get $|f_{k_j}(x_{k_i})| < |f(x_{k_i})| + \frac{\varepsilon}{2} < \varepsilon$ for every $1 \le i < j$. Finally we note that for $i \ne j$ we have that $1 - f_j(x_i) = f_j(x_j - x_i) \le \|x_j - x_i\| \le 1$ and hence $f_j(x_i) \ge 0$. From above we obtain that $f_{k_i}(x_{k_i}) = 1$ and $0 \le f_{k_i}(x_{k_j}) < \varepsilon$ for every $i \ne j$. $\square$

We also need the next result which is proved in [11].

Proposition 3.12. *Let X be a Banach space. We have that X^* is P(n)-convex if and only if there exists $\varepsilon > 0$ such that for each $f_1, f_2, ..., f_n \in S_{X^*}$ there exist $1 \le i, j \le n,\ i \ne j$, so that*

$$S(f_i,\ -f_j,\ \varepsilon) = \emptyset,$$

where for each $f, g \in X^*$ we define

$$S(f,\ g,\ \delta) = \big\{x \in B_X : f(x) \ge 1 - \delta,\ g(x) \ge 1 - \delta\big\}.$$

Proposition 3.13. *If X is a Banach space whose dual space X^* is P-convex then X has property (S).*

Proof. Suppose that X^* is P-convex but X does not satisfy property (S). Then there exists $A \subset S_X$ with $diam(A) \le 1$ which can not be separated from zero. Let $\varepsilon > 0$ be as in Proposition 3.12. By Lemma 3.11 there exist sequences $\{x_n\}_n \subset A$ and $\{f_n\}_n \subset S_{X^*}$ such that $f_i(x_i) = 1$ and $0 \le f_i(x_j) < \varepsilon$ for $i \ne j$. Since $f_i(x_i - x_j) > 1 - \varepsilon$ and $x_i - x_j \in B_X$ for $i \ne j$ we have that $x_i - x_j \in S(f_i, -f_j, \varepsilon)$, $i \ne j$, and by Proposition 3.12 the above contradicts that X^* is P-convex. $\square$

Example 3.10 and Proposition 3.13 show that property (S) is not autodual. In [11] it is proved that for every Banach space X we have that X is P(n)-convex if and only if $\tilde{X}$ is P(n)-convex. Hence we obtain the next corollary:

Corollary 3.14. *Let X be a Banach space and $n \in \mathbb{N}$. Then X^* is P(n)-convex if and only if $\widetilde{(X^*)}$ is P(n)-convex.*

In 2008 Saejung proved in [13] that every Banach space X whose dual space X^* is P-convex has uniform normal structure and in particular X has the FPP. This fact can also be proved from our results.

Corollary 3.15. *Let X be a Banach space whose dual space X^* is P-convex. Then X has the FPP.*

Proof. By Corollary 3.14 we have that $\widetilde{(X^*)}$ is P-convex. Since X is reflexive, $\widetilde{(X^*)} = \big(\tilde{X}\big)^*$ and by Proposition 3.13 we get that $\tilde{X}$ has property (S) and in particular it has property (S_m). Thus, by Theorem 3.9 we obtain that X has the FPP. $\square$

3.4. P-convexity and Kadec-Klee property.

Definition 3.16. *We say that X has the Kadec-Klee property if every weakly convergent sequence contained in the unit sphere of X is norm convergent.*

X has the uniform Kadec-Klee property if for every $\varepsilon > 0$ there exists $0 < \delta < 1$ such that if $\{x_n\}$ is a sequence contained in the unit ball of X so that $\inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$ and $\{x_n\}$ converges weakly to x then $\|x\| \leq \delta$.

Clearly every Banach space with the uniform Kadec-Klee property has the Kadec-Klee property. The uniform Kadec-Klee property was introduced by Huff [6]. van Dulst and Sims proved that every Banach space with the uniform Kadec-Klee property has the WFPP [14]. Now we will give a example of a P-convex space lacking the Kadec-Klee property.

Example 3.17. *There is a P-convex Banach space lacking the Kadec-Klee property.*

Indeed, let $\lambda \in \left(1, \frac{\sqrt{5}}{2}\right)$ and consider the space $X_\lambda = (l_2, \|\cdot\|_\lambda)$, where

$$\|x\|_\lambda = \max\left\{\|x\|_\infty, \frac{1}{\lambda}\|x\|_2\right\}.$$

It is known that $\varepsilon_0(X_\lambda) = 2\sqrt{\lambda^2 - 1} < 1$ (see [5]) and by Theorem 2.5 X_λ is P(3)-convex. On the other hand, we define $x = e_1$, $x_n = e_1 + c\,e_n$ for each $n \in \mathbb{N}$, where $\{e_n\}_n$ is the canonical basis in l_2 and $c = \sqrt{\lambda^2 - 1}$. It is easy to see that $\{x_n\}_{n \geq 2} \subset S_X$, $x \in S_X$, $\{x_n\}$ converges weakly to x and $\|x_n - x\| = c > 0$ for all n. Thus X_λ does not have the Kadec-Klee property and consequently it does not have uniform Kadec-Klee property.

It is clear that $\|x\|_\lambda \leq \|x\|_2 \leq \frac{\sqrt{5}}{2}\|x\|_\lambda$, and thus, by Theorem 3.5 X has the FPP.

These examples show that we need other properties than those mentioned here, in order to determine if P-convex Banach spaces have the FPP.

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