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AN EXISTENCE-UNIQUENESS THEOREM FOR A CLASS OF BOUNDARY VALUE PROBLEMS

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Abstract. In this paper the solutions of a two–endpoint boundary value problem is studied and under suitable assumptions the existence and uniqueness of a solution is proved. As a consequence, a condition to guarantee the existence of at least one periodic solution for a class of Liénard equations is presented.
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1. Introduction and the statement of the main result

It is well known that Liénard equations are considered in several problems in mechanics, engineering, and electrical circuits theory. There are some existence and multiplicity results for such equations with nonconstant forced terms; see for example [6, 7, 8, 10, 13, 14, 15, 16, 17, 21]. In the following we state and prove an existence–uniqueness type theorem for a class of two–endpoint boundary value problems associated with the second order forced Liénard equations.

Theorem 1.1 *Let $a_1 < a_2$ and $B > 0$ be real numbers and put $A = \max\{2|a_1|, 2|a_2|\}$. Suppose f and g are real functions on $\mathbb{R}$ which are locally Lipschitz and at least one of the f or g is nonconstant on $|x| \leq A$; and p is a continuous real function on $[0, T]$, $T > 0$. Also suppose M_0 is the maximum value of $|p|$ on $[0, T]$; M_1, M_2 are the maximum values of $|f|$, $|g|$ on $|x| \leq A$; and M_1', M_2' are the Lipschitz constants of f, g on*

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$|x| \le A$, respectively. Consider $M = 2/(M_1'B + M_2' + M_1)$, $N = 1/(M_1B + M_2 + M_0)$, and $0 < T_0 < \min\{T, 2\sqrt{AN}, 2BN, 2\sqrt{M+1} - 2\}$. Then for each $a_1 \le b \le a_2$ the boundary value problem

$$\begin{cases} x'' + f(x)x' + g(x) = p(t) & : 0 \le t \le T_0 \\ x(0) = x(T_0) = b & \\ |x(t)| \le A,\ |x'(t)| \le B & : 0 \le t \le T_0 \end{cases} \tag{1.1}$$

has a unique solution.

Proof. Consider the equation $x'' = 0$ with boundary condition $x(0) = x(T_0) = b$. The existence of Green's function for a typical two–endpoint problem was suggested by a simple physical example in [1] and is as follows:

$$G(t,s) = \begin{cases} s(t - T_0)/T_0 & : 0 \le s \le t \le T_0 \\ t(s - T_0)/T_0 & : 0 \le t \le s \le T_0 \end{cases}$$

If we now consider the integral equation

$$x(t) = b + \int_0^{T_0} G(t,s)\Big(f(x(s))x'(s) + g(x(s)) - p(s)\Big)ds,$$

then it is easy to see that the solutions $x(t)$ of this integral equation which are satisfied in $|x(t)| \le A$ and $|x'(t)| \le B$ for each $0 \le t \le T_0$ are exactly the solutions of given boundary value problem. Hence, to prove the theorem, it is enough to show that the above integral equation has a unique solution $x(t)$ satisfying $|x(t)| \le A$ and $|x'(t)| \le B$ for each $0 \le t \le T_0$. In order to do so, suppose $X = C^1([0, T_0], \mathbb{R})$, and for $\phi \in X$ define $||\phi|| = \max_{0 \le t \le T_0} |\phi(t)| + \max_{0 \le t \le T_0} |\phi'(t)|$. It is clear that X is a Banach space. Now, consider

$$\Omega = \Big\{\phi \in X :\ |\phi(t)| \le A \text{ and } |\phi'(t)| \le B \text{ hold for each } 0 \le t \le T_0\Big\},$$

which is obviously a closed, bounded, and convex subspace of X. Define the operator $S : \Omega \to X$ by mapping ϕ to $S(\phi)$, where $S(\phi)$ is defined by

$$S(\phi)(t) = b + \int_0^{T_0} G(t,s)\Big(f(\phi(s))\phi'(s) + g(\phi(s)) - p(s)\Big)ds.$$

First, we show that S maps Ω into itself. In order to do this, note that for each x, x', and t such that $|x| \le A$, $|x'| \le B$, and $0 \le t \le T_0$ we have

$$\Big|f(x)x' + g(x) - p(t)\Big| \le M_1B + M_2 + M_0 = \frac{1}{N}.$$

Also for each $0 \le t \le T_0$ we have

$$\textstyle\int_0^{T_0} |G(t,s)|ds = \frac{1}{2}t(T_0 - t) \le \frac{{T_0}^2}{8},$$

$$\textstyle\int_0^{T_0} |\frac{\partial}{\partial t}G(t,s)|ds = \frac{1}{T_0}t^2 - t + \frac{1}{2}T_0 \le \frac{T_0}{2}.$$

Hence we conclude that for each $\phi \in \Omega$ and $0 \le t \le T_0$,

$$|S(\phi)(t)| \le |b| + \tfrac{1}{N}\int_0^{T_0} |G(t,s)|ds \le |b| + \tfrac{{T_0}^2}{8N} \le \tfrac{A}{2} + \tfrac{A}{2} = A,$$

$$|S(\phi)'(t)| \le \tfrac{1}{N}\int_0^{T_0} |\tfrac{\partial}{\partial t}G(t,s)|ds \le \tfrac{T_0}{2N} \le B.$$

These mean that for each $\phi \in \Omega$, $S(\phi) \in \Omega$ and therefore S is an operator from Ω to Ω.

Next, we show that S is a compact operator on Ω. For this, it is enough to show that each bounded sequence $\{\phi_n\}$ on Ω has a subsequence $\{\phi_{n_i}\}$ for which $\{S(\phi_{n_i})\}$ is convergent on Ω. Therefore, let $\{\phi_n\}$ be a given sequence on Ω which is automatically bounded by definition of Ω. Suppose $\epsilon > 0$ is given. Since G is a uniformly continuous function on $[0,T_0] \times [0,T_0]$, there exists δ, $0 < \delta < \epsilon N$, such that $(t_1,s_1),(t_2,s_2) \in [0,T_0] \times [0,T_0]$ and $\sqrt{(t_1-t_2)^2 + (s_1-s_2)^2} < \delta$ imply that $|G(t_1,s_1) - G(t_2,s_2)| < \epsilon N/2T_0$. We now conclude that for each n and for each $t_1,t_2 \in [0,T_0]$, if $|t_1 - t_2| < \delta$, then

$$|S(\phi_n)(t_1) - S(\phi_n)(t_2)| \le \tfrac{1}{N}\int_0^{T_0} |G(t_1,s) - G(t_2,s)|ds < \epsilon,$$

$$|S(\phi_n)'(t_1) - S(\phi_n)'(t_2)| \le \tfrac{1}{N}\int_0^{T_0} |\tfrac{\partial}{\partial t}G(t_1,s) - \tfrac{\partial}{\partial t}G(t_2,s)|ds = \tfrac{1}{N}|t_1 - t_2| < \epsilon.$$

Hence $\{S(\phi_n)(t)\}$ and $\{S(\phi_n)'(t)\}$ are equicontinuous family of functions on $[0,T_0]$ and by classical Ascoli–Arzela theorem, there exists a subsequence $\{\phi_{n_i}(t)\}$ of $\{\phi_n(t)\}$ for which $\{S(\phi_{n_i})(t)\}$ and $\{S(\phi_{n_i})'(t)\}$ are uniformly convergent on $[0,T_0]$. This shows that $\{S(\phi_{n_i})\}$ is convergent on Ω and so S is a compact operator.

Therefore, by Schauder's fixed point theorem, there exists $\phi \in \Omega$ such that $S(\phi) = \phi$. So for each $0 \le t \le T_0$, we have $S(\phi)(t) = \phi(t)$ which is to say

$$\phi(t) = b + \int_0^{T_0} G(t,s)\Big(f(\phi(s))\phi'(s) + g(\phi(s)) - p(s)\Big)ds.$$

This means that $\phi \in \Omega$ is a solution of the mentioned integral equation with restrictions $|\phi(t)| \le A$ and $|\phi'(t)| \le B$ for each $0 \le t \le T_0$ and therefore is a solution of the given boundary value problem.

We now suppose that ψ is another solution of the given boundary value problem. This means that $\psi \in \Omega$, $\psi \ne \phi$, and $S(\psi) = \psi$. By the locally Lipschitz condition for f and g, note that for each $x,\ y,\ x'$, and y' such that $|x| \le A,\ |y| \le A,\ |x'| \le B$, and $|y'| \le B$ we have

$$\begin{aligned}\big|\big(f(x)x' + g(x)\big) - \big(f(y)y' + g(y)\big)\big| &= \big|\big(f(x) - f(y)\big)x' + f(y)(x' - y') + g(x) - g(y)\big| \\ &\le (M_1'B + M_2')|x-y| + M_1|x' - y'|.\end{aligned}$$

Therefore by the above inequality, for each $0 \le t \le T_0$,

$$\begin{aligned}|S(\phi)(t) - S(\psi)(t)| &\le \tfrac{{T_0}^2}{8}\big(M_1'B + M_2' + M_1\big)||\phi - \psi|| \\ &= \tfrac{{T_0}^2}{8}\tfrac{2}{M}||\phi - \psi|| \\ &= \tfrac{{T_0}^2}{4M}||\phi - \psi||,\end{aligned}$$

$$\begin{aligned} |S(\phi)'(t) - S(\psi)'(t)| &\leq \tfrac{T_0}{2}\big(M_1'B + M_2' + M_1\big)||\phi - \psi|| \\ &= \tfrac{T_0}{2}\tfrac{2}{M}||\phi - \psi|| \\ &= \tfrac{T_0}{M}||\phi - \psi||. \end{aligned}$$

Hence,

$$\begin{aligned} ||\phi - \psi|| &= ||S(\phi) - S(\psi)|| \\ &= \max_{0\leq t\leq T_0}|S(\phi)(t) - S(\psi)(t)| + \max_{0\leq t\leq T_0}|S(\phi)'(t) - S(\psi)'(t)| \\ &\leq (\tfrac{{T_0}^2}{4M} + \tfrac{T_0}{M})||\phi - \psi||. \end{aligned}$$

Therefore, we obtain ${T_0}^2 + 4T_0 \geq 4M$, or $T_0 \geq 2\sqrt{M+1} - 2$ which is contradictory with the definition of T_0. So ϕ is the unique solution of the given boundary value problem $\Box$.

Remark 1.2 In the above proof, we obtained a more deeper property for the operator S, which is contractivity condition. Therefore, we can apply the Banach's fixed point theorem directly to Eq. (1.1). We will verify this property in Section 3.

2. An application

The analysis of periodic Liénard equations have long been a topic of interest. In this direction, an important question, which has been studied extensively by a number of authors (see, for example [2, 3, 5, 9, 11, 12, 18, 19, 20]), is whether Liénard equations can support periodic solutions or not. In this section, as a consequence of Theorem 1.1, we investigate the existence of periodic solutions for a class of the second order forced Liénard equations

$$x'' + f(x)x' + g(x) = p(t),$$

where f and g are real functions on $\mathbb{R}$ and p is a real function on $[0, T]$, $T > 0$. These equations appear in a number of physical models and one important question is whether these equations can support periodic solutions. In the following we state and prove a result which yields a condition to guarantee the existence of at least one periodic solution for the above equation.

Theorem 2.1 *Suppose f and g are real functions on $\mathbb{R}$ which are locally Lipschitz and p is a nonconstant, continuous, real function on $[0, T]$, $T > 0$. Also suppose all solutions of the initial value problems associated with $x'' + f(x)x' + g(x) = p(t)$ can be extended to $[0, T]$. If there exist real numbers a_1 and a_2 for which $g(a_1) \leq p(t) \leq g(a_2)$ holds for each $0 \leq t \leq T$, then there exists T_0, $0 < T_0 < T$, such that if p is T_0–periodic, $x'' + f(x)x' + g(x) = p(t)$ has at least one T_0–periodic solution.*

Proof. By the assumption we conclude that $a_1 \neq a_2$ and so without loss of generality we can suppose that $a_1 < a_2$. Define the functions $\tilde{g}$ and $\hat{g}$ as follows which are obviously locally Lipschitz:

$$\tilde{g}(x) = \begin{cases} g(x) & : x \leq a_1 \\ g(a_1) + a_1 - x & : x > a_1 \end{cases}$$

and

$$\hat{g}(x) = \begin{cases} g(x) & : x \geq a_2 \\ g(a_2) + a_2 - x & : x < a_2 \end{cases}$$

Consider $A = \max\{2|a_1|, 2|a_2|\}$ and suppose $B = 1$. Let M_0 be the maximum value of $|p|$ on $[0, T]$; M_1, M_2, $\tilde{M}_2$, $\hat{M}_2$ be the maximum values of $|f|$, $|g|$, $|\tilde{g}|$, $|\hat{g}|$ on $|x| \leq A$; and M_1', M_2', $\tilde{M}_2'$, $\hat{M}_2'$ be the Lipschitz constants of f, g, $\tilde{g}$, $\hat{g}$ on $|x| \leq A$, respectively. Consider $M = 2/(M_1' + M_2' + M_1)$, $N = 1/(M_1 + M_2 + M_0)$, $\tilde{M} = 2/(M_1' + \tilde{M}_2' + M_1)$, $\tilde{N} = 1/(M_1 + \tilde{M}_2 + M_0)$, $\hat{M} = 2/(M_1' + \hat{M}_2' + M_1)$, $\hat{N} = 1/(M_1 + \hat{M}_2 + M_0)$, and $0 < T_0 < \min\{L, \tilde{L}, \hat{L}\}$, where

$$L = \min\left\{T, 2\sqrt{AN}, 2N, 2\sqrt{M+1} - 2\right\},$$

$$\tilde{L} = \min\left\{T, 2\sqrt{A\tilde{N}}, 2\tilde{N}, 2\sqrt{\tilde{M}+1} - 2\right\},$$

$$\hat{L} = \min\left\{T, 2\sqrt{A\hat{N}}, 2\hat{N}, 2\sqrt{\hat{M}+1} - 2\right\}.$$

Theorem 1.1 now implies that for each $a_1 \leq b \leq a_2$, the boundary value problem

$$\begin{cases} x'' + f(x)x' + g(x) = p(t) & : 0 \leq t \leq T_0 \\ x(0) = x(T_0) = b & \\ |x(t)| \leq A, \ |x'(t)| \leq 1 & : 0 \leq t \leq T_0 \end{cases}$$

has a unique solution, say $x(t, b)$.

Lemma 2.2 *For each $0 \leq t \leq T_0$, we have $x(t, a_1) \leq a_1 < a_2 \leq x(t, a_2)$.*

Proof. We prove that $x(t, a_1) \leq a_1$ holds for each $0 \leq t \leq T_0$. By Theorem 1.1, the boundary value problem

$$\begin{cases} x'' + f(x)x' + \tilde{g}(x) = p(t) & : 0 \leq t \leq T_0 \\ x(0) = x(T_0) = a_1 & \\ |x(t)| \leq A, \ |x'(t)| \leq 1 & : 0 \leq t \leq T_0 \end{cases}$$

has a unique solution $x(t)$. We claim that $x(t) \leq a_1$ holds for each $0 \leq t \leq T_0$. Suppose for the purpose of a contradiction, there exists a point $0 \leq \tilde{t} \leq T_0$ such that $x(\tilde{t}) > a_1$. Therefore the function $x(t) - a_1$ has a positive maximum on the interval $(0, T_0)$, say at t_1. Hence $(x(t) - a_1)'\big|_{t=t_1} = 0$, or $x'(t_1) = 0$. Therefore we established

$$\begin{aligned} (x(t) - a_1)'' = & \quad x''(t_1) \\ = & \quad -f(x(t_1))x'(t_1) - \tilde{g}(x(t_1)) + p(t_1) \\ = & \quad -\tilde{g}(x(t_1)) + p(t_1) \\ = & \quad -g(a_1) - a_1 + x(t_1) + p(t_1) \\ = & \quad \big(p(t_1) - g(a_1)\big) + \big(x(t_1) - a_1\big) \\ > & \quad 0, \end{aligned}$$

which is a contradiction since $x(t) - a_1$ has a maximum at t_1. Therefore for each $0 \le t \le T_0$, $x(t) \le a_1$ and so by the definition of $\tilde{g}$, $\tilde{g}(x(t)) = g(x(t))$ holds for each $0 \le t \le T_0$. This means that $x(t)$ is a solution of

$$\begin{cases} x'' + f(x)x' + g(x) = p(t) & : 0 \le t \le T_0 \\ x(0) = x(T_0) = a_1 \\ |x(t)| \le A, \ |x'(t)| \le 1 & : 0 \le t \le T_0 \end{cases}$$

The uniqueness property now implies that for each $0 \le t \le T_0$, $x(t) = x(t, a_1)$ and so $x(t, a_1) \le a_1$ holds for each $0 \le t \le T_0$.

A similar argument applying to the function $\hat{g}$ gives us the other inequality. □

Lemma 2.3 *There exists $\hat{b}$, $a_1 \le \hat{b} \le a_2$, such that $x'(0, \hat{b}) = x'(T_0, \hat{b})$.*

Proof. Define the function θ on $[a_1, a_2]$ by

$$\theta(b) = x'(0, b) - x'(T_0, b).$$

Using the Ascoli–Arzela theorem, one may easily verify that both $x(t, b)$ and $x'(t, b)$ are continuous on $[0, T_0] \times [a_1, a_2]$. This implies that θ is continuous also. On the other hand, note that for $i \in \{1, 2\}$,

$$x'(0, a_i) = \lim_{t \to 0^+} \frac{x(t, a_i) - a_i}{t}, \qquad x'(T_0, a_i) = \lim_{t \to 0^+} \frac{a_i - x(T_0 - t, a_i)}{t},$$

and therefore,

$$\begin{aligned} \theta(a_i) = &\quad x'(0, a_i) - x'(T_0, a_i) \\ = &\quad \lim_{t \to 0^+} \frac{x(t, a_i) + x(T_0 - t, a_i) - 2a_i}{t}. \end{aligned}$$

So by Lemma 2.2, we obtain $\theta(a_1) \le 0$ and $\theta(a_2) \ge 0$. Hence there exists $\hat{b}$, $a_1 \le \hat{b} \le a_2$, such that $\theta(\hat{b}) = 0$, or $x'(0, \hat{b}) = x'(T_0, \hat{b})$. □

Lemma 2.3 now implies that $x(t, \hat{b})$ is a solution of the periodic boundary value problem

$$\begin{cases} x'' + f(x)x' + g(x) = p(t) & : 0 \le t \le T_0 \\ x(0, \hat{b}) = x(T_0, \hat{b}) \\ x'(0, \hat{b}) = x'(T_0, \hat{b}) \end{cases}$$

and therefore, by a similar method as the one used in [4], we can extend $x(t, \hat{b})$ periodically with period T_0 to obtain a periodic solution of the equation $x'' + f(x)x' + g(x) = p(t)$. Note that this periodic solution is nontrivial, since p is a nonconstant forced function. □

3. An illustrative example

In this section, we give a concrete example satisfying the assumptions of the main result. In order to do this, consider the initial value problem

$$\begin{cases} x'' = p(t,x,x') = \frac{x}{10}x' - (16x + \frac{x^3}{10}) - \frac{(1119999+480\cos(924t)+\cos(48t))\sin(24t)}{20000} \\ x(0) = 0 \\ x'(0) = \frac{24}{10} \end{cases} \quad (3.1)$$

There are several numerical methods to solve Eq. (3.1) in the standard texts of numerical analysis and numerical solutions of ordinary differential equations. For example using Runge-Kutta method, the error of approximation is about 10^{-8}, knowing the exact solution of Eq. (3.1) which is $x(t) = 0.1\sin(24t)$.

We now present a rather nonstandard symbolic-numeric scheme for generating approximate solution for this example. This method is based on transformation of the second order initial value problem to a system of the first order equations and then use Picard's iteration method, with controlling the number of terms at each step. More precisely, at each step we ignore all the terms with an upper bound less than 10^{-12}. Using this method, we show that the corresponding Picard's iteration converges, and also we give a crude approximation to the contraction factor of the Picard's method.

Let $C^1([0,T_0],\mathbb{R})$ be the Banach space equipped with the norm

$$\|x\|_\infty = \max\left\{ \max_{0\le t\le T_0} |x(t)|, \max_{0\le t\le T_0} |x'(t)| \right\}.$$

Assuming $x = u_1$, $x' = u_2$ and

$$u = [u_1, u_2]^T, u' = [u_1', u_2']^T, F = [u_2, p(t, u_1, u_2)]^T,$$

Eq. (3.1) is equivalent to the system

$$\begin{cases} u' = F(t,u) & : 0 < t \le T_0 \\ u(0) = [0, 2.4]^T \end{cases}$$

with corresponding Picard's iteration formula given by

$$\begin{cases} u_n(t) = u(0) + \int_0^t F(s, u_{n-1}(s))ds \\ u_0(t) = u(0) \end{cases} \quad (3.2)$$

For $n = 1, \dots, 7$, we generate the sequence of the functions $u_1, \dots, u_7$ and $x_1, \dots, x_7$ and the approximation to Eq. (1.1) with exact solution $x(t) = 0.1\sin(24t)$. Numerical values for expressions $\|x_n(t) - x_{n-1}(t)\|_\infty$ and $\|x_n(t) - x(t)\|_\infty$ are given in the following table.

n	3	4	5	6	7
$\|x_n(t) - x_{n-1}(t)\|_\infty$	0.0692	0.0025	0.0000486	5.88×10^{-7}	2.6306×10^{-9}
$\|x_n(t) - x(t)\|_\infty$	2.11	0.0717	0.00255	0.0000492	5.91×10^{-7}

Note that error decrease at most with the factor 0.1. This verifies numerically that the operator corresponding to the Picard's iteration given in Eq. (3.2) is contraction mapping with contraction factor 0.1.

Numerical example given by Eq. (3.1) is generated by trail and error method designed in Mathematica version 5, and implemented in a cluster environment at Laboratory of Scientific Computation in Institute for Studies in Theoretical Physics and Mathematics (see http://www.scc.ipm.ac.ir/ganglia/).

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