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NON-SMOOTH GUIDING FUNCTIONS AND PERIODIC SOLUTIONS OF FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY IN HILBERT SPACES

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Abstract. In this paper we develop the method of non-smooth integral guiding functions to deal with the problem of existence of periodic solutions for functional differential inclusions with infinite delay in Hilbert spaces. As an example we study the periodic problem for a gradient functional differential inclusion with infinite delay.
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1. Introduction

It is well known that the method of guiding functions developed by M.A. Krasnosel'skii, A.I. Perov and other researchers (see, e.g., [16] - [18]) is an effective tool to investigate the problems of periodic oscillations in nonlinear systems. Using the method of integral guiding functions A. Fonda ([7]) studied the periodic problem for functional differential equations. The method of guiding functions was extended to differential inclusions (see, e.g., [2, 9]).

In many problems of nonlinear oscillations there arises the necessity to use guiding functions which are non-smooth. To study such problems for systems admitting forced oscillations S. Kornev and V. Obukhovskii in [13] - [15] developed the notion of non-smooth guiding functions by using the methods of non-smooth analysis.

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It should be noted also that until now the method of guiding functions was applied only to systems governed by differential equations and inclusions in finite dimensional spaces. Recently, N.V. Loi in [19] presented an approach to extend this method to differential inclusions in Hilbert spaces.

In the present paper, developing this approach, we define the notion of a non-smooth integral guiding function for a system governed by a functional differential inclusion with infinite delay in a Hilbert space and study the existence of periodic oscillations in such systems. The paper is organized in the following way. In the next section we recall some basic facts from multivalued analysis, theory of Fredholm operators and the phase space theory. In Section 3, after the statement of the problem, we introduce the notion of non-smooth integral guiding function and present the main result (Theorem 8) on the existence of a periodic solution for a functional differential inclusion in a Hilbert space satisfying condition of A-approximation solvability and admitting a guiding function with a non-trivial index. Some sufficient conditions for A-approximation solvability of inclusion are given in the same section (see Theorems 11 and 12). In the last section, by applying the abstract results, we study the periodic problem for a gradient functional differential inclusion with infinite delay.

2. Preliminaries

2.1. **Multimaps.** Let X and Y be Banach spaces. Denote by $P(Y)$ $[Cv(Y), Kv(Y)]$ the collections of all nonempty [respectively, nonempty closed convex, nonempty compact convex] subsets of Y. By $B_X(0,r)$ [respectively, $\partial B_X(0,r)$] we will denote a ball [a sphere] in X of radius r centered at the origin.

Definition 1. *(see, e.g.,* [2], [9], [12]*). A multivalued map (multimap) $F\colon X \to P(Y)$ is said to be:*

(i) *upper semicontinuous (u.s.c.), if for every open subset $V \subset Y$ the set*

$$F_+^{-1}(V) = \{x \in X \colon F(x) \subset V\}$$

is open in X;

(ii) *closed if its graph*

$$\{(x,y) \in X \times Y : y \in F(x)\}$$

is a closed subset of $X \times Y$;

(iii) *compact, if the set*

$$F(X') := \bigcup_{x \in X'} F(x)$$

is relatively compact in Y for every bounded subset $X' \subset X$.

Recall (see, e.g., [2], [9], [12]) that if u.s.c. and compact multimap $F\colon \overline{U} \to Kv(X)$ has no fixed points on the boundary ∂U of an open bounded subset $U \subset X$, then the topological degree $deg(i - F, \overline{U})$ of the corresponding multivalued vector field $i - F$ (i here denotes the inclusion map) is well defined and has all standard properties of the Leray–Schauder topological degree.

2.2. **Fredholm Operators.**

Definition 2. *(see, e.g., [8]). A linear bounded map $\ell\colon X \to Y$ is said to be a Fredholm operator of index zero, if*

(*i*) *$Im\ell$ is closed in Y;*

(*ii*) *$Ker\ell$ and $Coker\ell$ have the finite dimensions and dim $Ker\ell =$ dim $Coker\ell$.*

Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. For every $n \in \mathbb{N}$, let H_n be an $n-$dimensional subspace of H with the basis $\{e_k\}_{k=1}^{n}$ and P_n be a projection of H onto H_n. By $\langle \cdot,\cdot \rangle_H$ we denote the inner product in H. The symbol I denotes the interval $[0,T]$. By $C(I,H)$ $[L_2(I,H)]$ we denote the spaces of all continuous [respectively, square summable] functions $u\colon I \to H$ with usual norms

$$\|u\|_C = \max_{t\in I} \|u(t)\|_H \quad \text{and} \quad \|u\|_2 = \left(\int_0^T \|u(t)\|_H^2 dt \right)^{\frac{1}{2}}.$$

The symbol $\langle \cdot,\cdot \rangle_L$ will denote the inner product in $L_2(I,H)$.

Consider the space of all absolutely continuous functions $u : I \to H$ whose derivatives belong to $L_2(I,H)$. It is known (see, e.g., [1]) that this space can be identified with the Sobolev space $W^{1,2}(I,H)$ endowed with the norm

$$\|u\|_W = \|u\|_2 + \|u'\|_2.$$

The embedding $W^{1,2}(I,H) \hookrightarrow C(I,H)$ is continuous, and for every $n \geq 1$ the space $W^{1,2}(I,H_n)$ is compactly embedded in $C(I,H_n)$. The weak convergence in $W^{1,2}(I,H)$ $[L_2(I,H)]$ is denoted by $x_n \overset{W}{\rightharpoonup} x_0$ [respectively , $f_n \overset{L}{\rightharpoonup} f_0$].

By $W_T^{1,2}(I,H)$ $[C_T(I,H)]$ we denote the subspaces of all functions $x \in W^{1,2}(I,H)$ [respectively, $C(I,H)$] satisfying the boundary condition $x(0) = x(T)$.

Let $n \in \mathbb{N}$, and $\ell\colon W_T^{1,2}(I,H_n) \to L_2(I,H_n)$ be a linear Fredholm operator of index zero. Then there exist the projections (see, e.g., [8]):

$$C_n\colon W_T^{1,2}(I,H_n) \to W_T^{1,2}(I,H_n)$$

and

$$Q_n\colon L_2(I,H_n) \to L_2(I,H_n)$$

such that $Im\ C_n = Ker\,\ell$ and $Ker\ Q_n = Im\,\ell$. If the operator

$$\ell_{C_n}\colon dom\,\ell \cap Ker\,C_n \to Im\,\ell$$

is defined as the restriction of ℓ on $dom\,\ell \cap Ker\,C_n$, then ℓ_{C_n} is a linear isomorphism and we can define the operator $K_{C_n}\colon Im\,\ell \to dom\,\ell$, $K_{C_n} = \ell_{C_n}^{-1}$. Now, set $Coker\,\ell = L_2(I,H_n)/Im\,\ell$; and let $\Pi_n\colon L_2(I,H_n) \to Coker\,\ell$ be the canonical projection

$$\Pi_n(z) = z + Im\,\ell$$

and $\Lambda_n\colon Coker\,\ell \to Ker\,\ell$ be the linear continuous isomorphism. Then the equation

$$\ell x = y,\ y \in L_2(I,H_n)$$

is equivalent to

$$(i - C_n)x = (\Lambda_n \Pi_n + K_{C_n,Q_n})y,$$

where $K_{C_n,Q_n}: L_2(I,H_n) \to W_T^{1,2}(I,H_n)$ is given as

$$K_{C_n,Q_n} = K_{C_n}(i - Q_n).$$

The following notion will play an important role in the sequel.

Let $\mathcal{A}\colon W_T^{1,2}(I,H) \to L_2(I,H)$ be a linear operator; $\mathcal{F}\colon C_T(I,H) \to P(L_2(I,H))$ a multimap. For $n \in \mathbb{N}$, define the projection $\mathbb{P}_n : L_2(I,H) \to L_2(I,H_n)$ generated by P_n as

$$(\mathbb{P}_n f)(t) = P_n f(t), \quad \text{for a.e. } t \in I.$$

Definition 3. *(cf. Definition 21.2* [4]*). An inclusion*

$$\mathcal{A}x \in \mathcal{F}(x)$$

is said to be $\mathcal{A}-$approximation solvable, if from the existence of sequences $\{n_k\}$ and $\{x^{(k)}\}$, $x^{(k)} \in W_T^{1,2}(I,H_{n_k})$ such that $\sup_k \|x^{(k)}\|_C < +\infty$ and $\mathcal{A}x^{(k)} \in \mathbb{P}_{n_k}\mathcal{F}(x^{(k)})$ it follows that there is a subsequence $\{x^{(k_m)}\}$ such that

$$x^{(k_m)} \xrightarrow{W} x^* \in W_T^{1,2}(I,H), \text{ and } \mathcal{A}x^* \in \mathcal{F}(x^*).$$

2.3. **Phase Space.** We will use an axiomatical definition of the *phase space* $\mathcal{B}$, introduced by J.K. Hale and J. Kato (see [10], [11]) for studying functional differential equations and inclusions with infinite delay. The space $\mathcal{B}$ will be considered as a linear topological space of functions mapping $(-\infty,0]$ into a Hilbert space H endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$.

For any function $y : (-\infty;T] \to H$ and for every $t \in I$, y_t represents the function from $(-\infty,0]$ into H defined by

$$y_t(\theta) = y(t+\theta),\ \theta \in (-\infty;0].$$

We will assume that $\mathcal{B}$ satisfies the following axioms.

($\mathcal{B}1$) If $y : (-\infty;T] \to H$ is such that $y_{|I} \in C(I;H)$ and $y_0 \in \mathcal{B}$, then we have

- (i) $y_t \in \mathcal{B}$ for $t \in I$;
- (ii) function $t \in I \mapsto y_t \in \mathcal{B}$ is continuous;
- (iii) $\|y_t\|_{\mathcal{B}} \le K(t) \sup\limits_{0\le\tau\le t} \|y(\tau)\| + N(t)\|y_0\|_{\mathcal{B}}$ for $t \in [0,T]$, where $K(\cdot), N(\cdot) : [0;\infty) \to [0;\infty)$ are independent of y, $K(\cdot)$ is strictly positive and continuous, and $N(\cdot)$ is bounded.

($\mathcal{B}2$) There exists $l > 0$ such that

$$\|\psi(0)\|_H \le l\|\psi\|_{\mathcal{B}}$$

for all $\psi \in \mathcal{B}$.

Let us mention that under above hypotheses the space C_{00} of all continuous functions from $(-\infty,0]$ into H with compact support is a subset of each phase space $\mathcal{B}$ ([11], Proposition 1.2.1). We will assume, additionally, that the following hypothesis holds.

($\mathcal{B}3$) If a uniformly bounded sequence $\{\psi_n\}_{n=1}^{+\infty} \subset C_{00}$ converges to a function ψ compactly (i.e. uniformly on each compact subset of $(-\infty,0]$), then $\psi \in \mathcal{B}$ and

$$\lim_{n\to+\infty} \|\psi_n - \psi\|_{\mathcal{B}} = 0.$$

The hypothesis ($\mathcal{B}3$) implies that the Banach space $BC((-\infty,0];H)$ of bounded continuous functions is continuously embedded into $\mathcal{B}$.

We may consider the following examples of phase spaces satisfying all above properties.

(1) For $\nu>0$, let $\mathcal{B}=C_\nu$ be the space of functions $\psi:(-\infty;0]\to H$ such that: (i) $\psi_{|[-r,0]}\in C([-r,0];E)$ for each $r>0$; (ii) the limit $\lim\limits_{\theta\to-\infty} e^{\nu\theta}\|\psi(\theta)\|$ is finite. Then we set

$$\|\psi\|_{\mathcal{B}}=\sup_{-\infty<\theta\le 0} e^{\nu\theta}\|\psi(\theta)\|.$$

(2) *Spaces of "fading memory".* Let $\mathcal{B}=C_\rho$ be the space of functions $\psi:(-\infty;0]\to E$ such that

(a) $\psi\in C([-r;0];E)$ for some $r>0$;
(b) ψ is Lebesgue measurable on $(-\infty;-r)$ and there exists a positive Lebesgue integrable function $\rho:(-\infty;-r)\to\mathbb{R}^+$ such that $\rho\psi$ is Lebesgue integrable on $(-\infty;-r)$; moreover, there exists a locally bounded function $P:(-\infty;0]\to\mathbb{R}^+$ such that, for all $\xi\le 0$, $\rho(\xi+\theta)\le P(\xi)\rho(\theta)$ a.e. $\theta\in(-\infty;-r)$. Then,

$$\|\psi\|_{\mathcal{B}}=\sup_{-r\le\theta\le 0}\|\psi(\theta)\|+\int_{-\infty}^{-r}\rho(\theta)\|\psi(\theta)\|d\theta.$$

A simple example of such a space can be obtained by taking the function $\rho(\theta)=e^{\mu\theta}$, $\mu\in\mathbb{R}$.

3. Existence of Solutions of Functional Differential Inclusions with Infinite Delay in Hilbert Spaces

3.1. **The Statement of the Problem.** Let H be a Hilbert space. The Banach space $BC((-\infty,0];H)$ of bounded continuous functions will be denoted by $\mathcal{BC}(H)$.

We will study the functional differential inclusion in H with the infinite delay of the following form

$$x'(t)\in F(t,x_t)\quad \text{for } a.e.\quad t\in I. \tag{3.1}$$

We assume that a multimap $F\colon\mathbb{R}\times\mathcal{BC}(H)\to Kv(H)$ satisfies the following conditions:

(F_T) multimap $F\colon\mathbb{R}\times\mathcal{BC}(H)\to Kv(H)$ is $T-$periodic with respect to the first argument, i.e.,

$$F(t,\psi)=F(t+T,\psi) \text{ for a.e. } t\in\mathbb{R} \text{ and for all } \psi\in\mathcal{BC}(H);$$

($F1$) for every $\psi\in\mathcal{BC}(H)$ multifunction $F(\cdot,\psi)\colon[0,T]\to Kv(H)$ has a measurable selection;
($F2$) for a.e. $t\in[0,T]$ multimap $F(t,\cdot)\colon\mathcal{BC}(H)\to Kv(H)$ is u.s.c.;
($F3$) for every $r>0$ there exists a function $\nu_r\in L_2^+[0,T]$ such that for each $x\in C_T(I,H)$ with $\|x\|_2\le r$ we have

$$\|F(s,\widetilde{x}_s)\|_H:=\sup\{\|y\|_H : y\in F(s,\widetilde{x}_s)\}\le\nu_r(s) \text{ for a.e. } s\in[0,T],$$

where $\widetilde{x}$ denotes the T-periodic extension of x on $(-\infty,T]$.

From the above conditions it follows that the superposition multioperator

$$\mathcal{P}_F\colon C_T(I,H) \to Cv(L_2(I,H)),$$
$$\mathcal{P}_F(x) = \{f \in L_2(I,H)\colon f(s) \in F(s,\widetilde{x}_s) \text{ for a.e. } s \in I\},$$

is well-defined and closed (see, e.g., [2],[12]).

Consider the operator of differentiation

$$A\colon W_T^{1,2}(I,H) \to L_2(I,H), \quad Ax = x'.$$

Then we will treat the problem of existence of $T-$periodic solutions of inclusion (3.1) as the problem of existence of solutions of the following operator inclusion

$$Ax \in \mathcal{P}_F(x). \tag{3.2}$$

Recall now some notions of non-smooth analysis (see, e.g., [3]).

Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. For every $y_0 \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$ the generalized directional derivative $V^0(y_0;\nu)$ of function V at the point y_0 in the direction ν is defined as

$$V^0(y_0;\nu) = \overline{\lim_{\substack{y \to y_0 \\ t \downarrow 0}}} \frac{V(y+t\nu)-V(y)}{t}. \tag{3.3}$$

The subdifferential $\partial V(y_0)$ of function V at y_0 is defined by:

$$\partial V(y_0) = \left\{y \in \mathbb{R}^n\colon \langle y,\nu\rangle \le V^0(y_0;\nu) \textit{ for every } \nu \in \mathbb{R}^n\right\},$$

It is well known (see, e.g., [3]) that the multimap $\partial V\colon \mathbb{R}^n \to P(\mathbb{R}^n)$ is u.s.c. and has compact convex values. In particular, it means that for every continuous function $x\colon [0,T] \to \mathbb{R}^n$ the set $\mathcal{P}_{\partial V}(x)$ of all summable selections of the multifunction $\partial V(x(t))$ is non-empty.

A locally Lipschitz functional $V\colon H \to \mathbb{R}$ is called *regular,* if for every $y \in H$ and $\nu \in H$ there exists the directional derivative $V'(y,\nu)$ and $V'(y,\nu) = V^0(y,\nu)$. It is known (see, e.g., [3]) that locally bounded convex functionals are regular.

Given a regular functional $V\colon H \to \mathbb{R}$, for each $i = 1,2,...$, define the function

$$V_i\colon \mathbb{R} \to \mathbb{R},\ V_i(y) = V(0,\cdots,0,y,0,\cdots),$$

where y is placed in the i-th position. It is clear that V_i is also regular.

We define *the generalized gradient* $\partial^* V(x)$ of a regular functional V at $x = (x_1,x_2,\cdots) \in H$ in the following way:

$$\partial^* V(x) = \partial V_1(x_1) \times \partial V_2(x_2) \times ... \times \partial V_i(x_i) \times ... \subset \mathbb{R}^\infty,$$

where ∂V_i, $i = 1,2,...$ is the subdifferential of the function V_i.
Notice that our definition of generalized gradient is different from the classical Clarke definition (see [3]) and its calculation is easier.

For example, let $V\colon \ell_2 \to \mathbb{R}$ be defined as

$$V(x) = |x_1| + x_1x_2 + \sum_2^\infty x_k^2,\ x = (x_1,x_2,\cdots). \tag{3.4}$$

We have

$$\partial V_1(x_1) = \begin{cases} 1 & \text{if } x_1 > 0, \\ [-1,1] & \text{if } x_1 = 0, \\ -1 & \text{if } x_1 < 0, \end{cases}$$

and for every $i \geq 2$, $\partial V_i(x_i) = 2x_i$.

Definition 4. *A regular functional $V\colon H \to \mathbb{R}$ is said to be a projectively homogeneous potential, if there exists $n_0 \in \mathbb{N}$ such that*

$$Pr_n \partial^* V(x) = \partial^* V(P_n x) \tag{3.5}$$

for all $n \geq n_0$ and $x \in H$, where $Pr_n : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is the natural projection on first n coordinates.

It is easy to see that functional (3.4) is projectively homogeneous.

Definition 5. *A regular functional $V\colon H \to \mathbb{R}$ is said to be a non-degenerate potential, if there exists $R_0 > 0$ such that*

$$(0, 0, \cdots, 0, \cdots) \notin \partial^* V(x)$$

for all $x \in H$ such that $\|x\|_H \geq R_0$.

For each $n \in \mathbb{N}$, let us make the natural identification $H_n \cong Pr_n \mathbb{R}^\infty \cong \mathbb{R}^n$. Then, restricting the multifield $Pr_n \partial^* V$ on H_n, we can consider it as the u.s.c. multifield $Pr_n \partial^* V : \mathbb{R}^n \to Kv(\mathbb{R}^n)$.

From Definitions 4 and 5 it follows that if V is a non-degenerate projectively homogeneous potential then the multifields $Pr_n \partial^* V$ have no zeros on spheres $\partial B_{\mathbb{R}^n}(0, R)$ for all $n \geq n_0$ and $R \geq R_0$. So the topological degrees

$$\gamma_n = deg(Pr_n \partial^* V, \partial B_{\mathbb{R}^n}(0, R)),\ n \geq n_0,$$

are well-defined and do not depend on $R \geq R_0$.

The index of the non-degenerate projectively homogeneous potential V is defined by:

$$ind\, V = (\gamma_{n_0}, \gamma_{n_0+1}, \cdots).$$

By $ind\, V \neq 0$ we mean that there exists a subsequence $\{n_k\}$ such that $\gamma_{n_k} \neq 0$ for all n_k.

For every continuous function $x \in C(I, H)$, $x(t) = \big(x_1(t), x_2(t), \cdots\big)$, $t \in I$, by a selection $\upsilon(t) \in \partial^* V(x(t))$ we mean

$$\upsilon(t) = \big(\upsilon_1(t), \upsilon_2(t), \cdots\big),\ t \in I,$$

where $\upsilon_i(t) \in \partial V_i(x_i(t))$, for a.e. $t \in I$, $i \geq 1$, are summable selections.

Definition 6. *A projectively homogeneous potential $V\colon H \to \mathbb{R}$ is said to be a non-smooth integral guiding function for inclusion* (3.1)*, if there exists $N > 0$ such that for every $x \in W_T^{1,2}(I, H)$ with*

$$\|x\|_2 \geq N,\ \|x'(s)\|_H \leq \|F(s, \widetilde{x}_s)\|_H \textit{ for a.e. } s \in [0, T]$$

the following relation holds:

$$\overline{\lim}_{m\to\infty}\ sign\left(\sum_{k=1}^{m}\int_0^T \upsilon_k(s) f_k(s)\,ds\right) = 1,$$

for all $f \in \mathcal{P}_F(x)$, $f(s) = (f_1(s), f_2(s), ...)$ *and all selections* $\upsilon(s) \in \partial^* V(x(s))$.

Lemma 7. *If* V *is a non-smooth integral guiding function for inclusion* (3.1) *then* V *is the non-degenerate potential.*

Proof. In fact, for every $y = (y_1, y_2, \cdots) \in H$, $\|y\|_H \geq \frac{N}{\sqrt{T}}$, considering y as the constant function we have that

$$\|y\|_2 \geq N,\ \|y'\|_H \leq \|F(t,y)\|_H \text{ for all } t \in I.$$

Hence,

$$\overline{\lim}_{m\to\infty}\ sign\left(\sum_{k=1}^{m}\int_0^T \upsilon_k f_k(s)\,ds\right) = 1,$$

for all $f \in \mathcal{P}_F(y)$ and all $\upsilon = (\upsilon_1, \upsilon_2, \cdots) \in \partial^* V(y)$. So $\upsilon \neq (0, 0, \cdots, 0, \cdots)$. □

3.2. **Main results.**

Theorem 8. *Let conditions* (F_T) *and* $(F1)-(F3)$ *hold. Assume that there exists a non-smooth integral guiding function* V *for inclusion* (3.1) *such that* $ind\, V \neq 0$. *If inclusion* (3.2) *is* $A-$*approximation solvable then inclusion* (3.1) *has a* $T-$*periodic solution.*

Remark 9. *Some sufficient conditions of A-approximation solvability of inclusion* (3.2) *will be given in Theorems 11 and 12.*

For the proof of the theorem we will need the next assertion which may be proved by following the same reasonings as in [5], Section 1.5.

Lemma 10. *Let a function* $\mathcal{V}\colon H_n \to \mathbb{R}$ *be regular,* $x\colon [0,T] \to H_n$ *an absolutely continuous function. Then the function* $\mathcal{V}(x(t))$ *is absolutely continuous and*

$$\mathcal{V}(x(t)) - \mathcal{V}(x(0)) = \int_0^t \mathcal{V}^0(x(s), x'(s))\,ds,\quad t \in [0,T].$$

Proof of Theorem 8. It is easy to see that for each $n \in \mathbb{N}$ the restriction

$$A_n = A_{|_{W_T^{1,2}(I,H_n)}} : W_T^{1,2}(I,H_n) \to L_2(I,H_n)$$

is the linear Fredholm operator of index zero and

$$ker\, A_n \cong H_n \cong coker\, A_n.$$

The spaces $W_T^{1,2}(I,H_n)$ and $L_2(I,H_n)$ can be decomposed as:

$$W_T^{1,2}(I,H_n) = W_0^{(n)} \oplus W_1^{(n)},$$

and

$$L_2(I,H_n) = \mathcal{L}_0^{(n)} \oplus \mathcal{L}_1^{(n)},$$

where $W_0^{(n)} = ker\, A_n$, $\mathcal{L}_0^{(n)} = coker\, A_n$, $W_1^{(n)} = (W_0^{(n)})^{\perp}$ and $\mathcal{L}_1^{(n)} = Im A_n$.
For every $u \in W_T^{1,2}(I, H_n)$ and $f \in L_2(I, H_n)$ we denote their corresponding decompositions by

$$u = u_{(0)}^{(n)} + u_{(1)}^{(n)},$$

and

$$f = f_{(0)}^{(n)} + f_{(1)}^{(n)}.$$

Notice that a function $x \in W_T^{1,2}(I, H_n)$ is a solution of the inclusion

$$A_n x \in \mathbb{P}_n \mathcal{P}_F(x)$$

if and only if it is a fixed point

$$x \in G_n(x), \tag{3.6}$$

of the multimap

$$G_n \colon C_T(I, H_n) \to Cv\big(C_T(I, H_n)\big),$$
$$G_n(x) = C_n x + (\Lambda_n \Pi_n + K_{C_n,Q_n}) \circ \mathbb{P}_n \mathcal{P}_F(x),$$

where projection $\Pi_n \colon L_2(I, H_n) \to H_n$ is defined as

$$\Pi_n f = \frac{1}{T} \int_0^T f(s)\, ds$$

and the homomorphism $\Lambda_n \colon H_n \to H_n$ is the identity operator.

We will show that the multioperator G_n is u.s.c. and compact. Indeed, from the fact that the multioperator $\mathcal{P}_F$ is closed and the operator $(\Lambda_n \Pi_n + K_{C_n,Q_n}) \circ \mathbb{P}_n$ is linear and continuous it follows that the multimap $(\Lambda_n \Pi_n + K_{C_n,Q_n}) \circ \mathbb{P}_n \mathcal{P}_F$ is closed (see, e.g., Theorem 1.5.30 [2]). Further, for every bounded subset $U \subset C_T(I, H_n)$ the set $\mathbb{P}_n \mathcal{P}_F(U)$ is bounded in $L_2(I, H_n)$. Then the set $(\Lambda_n \Pi_n + K_{C_n,Q_n}) \circ \mathbb{P}_n \mathcal{P}_F(U)$ is bounded in $W_T^{1,2}(I, H_n)$ and by the compact embedding property, the set $(\Lambda_n \Pi_n + K_{C_n,Q_n}) \circ \mathbb{P}_n \mathcal{P}_F(U)$ is relatively compact in $C_T(I, H_n)$. Finally, our assertion follows from the fact that the operator C_n is continuous and takes values in a finite dimensional space.

Now let us demonstrate that solutions of inclusion (3.2) are a priori bounded in the space $C_T(I, H)$. In fact, assume that $x \in W_T^{1,2}(I, H)$ is a solution of inclusion (3.2). Then there is a function $f \in \mathcal{P}_F(x)$ such that $x'(t) = f(t)$ for a.e. $t \in I$. For every selection $\upsilon(s) \in \partial^* V(x(s))$ we have

$$\overline{\lim}_{m\to\infty}\, sign \left(\sum_{k=1}^{m} \int_0^T \upsilon_k(s) f_k(s)\, ds \right) = \overline{\lim}_{m\to\infty}\, sign \left(\sum_{k=1}^{m} \int_0^T \upsilon_k(s) x_k^{'}(s)\, ds \right) \leq$$
$$\leq \overline{\lim}_{m\to\infty}\, sign \left(\sum_{k=1}^{m} \int_0^T V_k^0\big(x_k(s), x_k^{'}(s)\big)\, ds \right) =$$
$$= \overline{\lim}_{m\to\infty}\, sign \left(\sum_{k=1}^{m} \left(V_k\left(x_k(T)\right) - V_k\left(x_k(0)\right)\right) \right) = 0,$$

where $x(t) = (x_1(t), x_2(t), \cdots)$ and $f(t) = (f_1(t), f_2(t), \cdots)$, $t \in I$.
Hence, $\|x\|_2 < N$. From $(F3)$ it follows that there exists $K > 0$ such that $\|x'\|_2 < K$. Then there is a number $M > 0$, independent of x, such that $\|x\|_C < M$.

Choose an arbitrary $R \geq \max\{R_0, M\}$, where R_0 is the constant in Definition 5. Then inclusion (3.2) has no solutions on $\partial B_C(0, R)$. Let us show that for each $n \geq n_0$

$$x \notin G_n(x)$$

provided $x \in \partial B_C^{(n)}(0, R) = \partial B_C(0, R) \cap C_T(I, H_n)$.

To the contrary, assume that $x^* \in \partial B_C^{(n_*)}(0, R)$, $n_* \geq n_0$, is a solution of inclusion (3.6). Then there is a function $f^* \in \mathcal{P}_F(x^*)$ such that $Ax^* = \mathbb{P}_{n_*} f^*$. Therefore, for a.e. $t \in I$

$$\|x^{*\prime}(t)\|_H = \|\mathbb{P}_{n_*} f^*(t)\|_H \leq \|f^*(t)\|_H \leq \|F(t, \widetilde{x}_t^*)\|_H.$$

On the other hand, from the choice of R it follows that $\|x^*\|_2 \geq N$. Then we obtain

$$\overline{\lim}_{m\to\infty}\ sign \left(\sum_{k=1}^{m} \int_0^T \upsilon_k(s) f_k^*(s)\, ds \right) = 1,$$

for all selections $\upsilon(s) \in \partial^* V(x^*(s))$, $s \in I$.
Since the function x^* takes values in H_{n_*} and V is projectively homogeneous, we have

$$\overline{\lim}_{m\to\infty}\ sign \left(\sum_{k=1}^{m} \int_0^T \upsilon_k(s) f_k^*(s)\, ds \right) = sign \left(\sum_{k=1}^{n_*} \int_0^T \upsilon_k(s) f_k^*(s)\, ds \right) =$$

$$= sign \left(\sum_{k=1}^{n_*} \int_0^T \upsilon_k(s) x_k^{*\prime}(s)\, ds \right) \leq sign \left(\sum_{k=1}^{n_*} \int_0^T V_k^0\ (x_k^*(s), x_k^{*\prime}(s))\ ds \right) =$$

$$= sign \Big(\sum_{k=1}^{n_*} \big(V_k(x_k^*(T) - V_k(x_k^*(0)) \big) \Big) = 0,$$

that is the contradiction.
Thus, for each $n \geq n_0$ the topological degree

$$\omega_n = deg(i - G_n, B_C^{(n)}(0, R))$$

is well-defined.
Now we will evaluate ω_n. To this aim we consider the multimap

$$\Sigma_n \colon C_T(I, H_n) \times [0, 1] \to Kv(C_T(I, H_n)),$$
$$\Sigma_n(x, \lambda) = C_n x + (\Lambda_n \Pi_n + K_{C_n, Q_n}) \circ \alpha_n(\mathbb{P}_n \mathcal{P}_F(x), \lambda),$$

where $\alpha_n \colon L_2(I, H_n) \times [0, 1] \to L_2(I, H_n)$ is defined as

$$\alpha_n(f_{(0)}^{(n)} + f_{(1)}^{(n)}, \lambda) = f_{(0)}^{(n)} + \lambda f_{(1)}^{(n)}.$$

It is easy to see that the multimap Σ_n is u.s.c. and compact. Let us show that the set

$$Fix\big(\Sigma_n, \partial B_C^{(n)}(0, R) \times [0, 1]\big)$$

of fixed points of the family $\Sigma_n(\cdot,\lambda)$ on $\partial B_C^{(n)}(0,R)$ is empty. To the contrary, assume that there exists $(x^*,\lambda^*)\in\partial B_C^{(n)}(0,R)\times[0,1]$ such that

$$x^*\in\Sigma_n(x^*,\lambda^*).$$

Then there is a function $f^*\in\mathcal{P}_F(x^*)$ such that

$$\begin{cases} A_n x^* = \lambda^* f_{(1)}^{*(n)} \\ 0 = f_{(0)}^{*(n)}, \end{cases}$$

where $f_{(0)}^{*(n)}+f_{(1)}^{*(n)}=\mathbb{P}_n f^*$, $f_{(0)}^{*(n)}\in\mathcal{L}_0^{(n)}$ and $f_{(1)}^{*(n)}\in\mathcal{L}_1^{(n)}$.
It is clear that $\|x^*\|_2\geq N$ and $\|x^{*\prime}(t)\|_H\leq\|f^*(t)\|_H\leq\|F(t,\widetilde{x}_t^*)\|_H$ for a.e. $t\in I$.
Then we have

$$\overline{\lim}_{m\to\infty}\ sign\left(\sum_{k=1}^{m}\int_0^T \upsilon_k(s)f_k^*(s)\,ds\right)=1,$$

for all selections $\upsilon(s)\in\partial^*V(x^*(s))$, $s\in I$.
Since $x^*\in C_T(I,H_n)$ we obtain

$$\overline{\lim}_{m\to\infty}\ sign\left(\sum_{k=1}^{m}\int_0^T \upsilon_k(s)f_k^*(s)\,ds\right)=sign\left(\sum_{k=1}^{n}\int_0^T \upsilon_k(s)f_k^*(s)\,ds\right),$$

where $f^*(t)=(f_1^*(t),f_2^*(t),\cdots)$ and $x^*(t)=(x_1^*(t),\cdots,x_n^*(t),0,0,\cdots)$.

If $\lambda^*\neq 0$, then

$$sign\left(\sum_{k=1}^{n}\int_0^T \upsilon_k(s)f_k^*(s)\,ds\right)=sign\left(\frac{1}{\lambda^*}\sum_{k=1}^{n}\int_0^T \upsilon_k(s)x_k^{*\prime}(s)\,ds\right)\leq$$

$$\leq sign\left(\sum_{k=1}^{n}\int_0^T V_k^0(x_k^*(s),x_k^{*\prime}(s))\,ds\right)=sign\left(\sum_{k=1}^{n}\big(V_k(x_k^*(T))-V_k(x_k^*(0))\big)\right)=0,$$

that is the contradiction.

In case $\lambda^*=0$, we have $A_n x^*=0$. Therefore, $x^*\in ker A_n$, i.e.,

$$x^*(t)\equiv y=(y_1,\cdots,y_n,0,0,\cdots),\quad t\in I,$$

where $\|y\|_H=R$.
From the fact that $\|y'\|_2=0\leq\|f\|_2$ for all $f\in\mathcal{P}_F(y)$ it follows that

$$\overline{\lim}_{m\to\infty}\ sign\left(\sum_{k=1}^{m}\int_0^T \upsilon_k f_k(s)\,ds\right)=1,$$

for all $f\in\mathcal{P}_F(y)$ and all elements $\upsilon=(\upsilon_1,\cdots,\upsilon_n,0,0,\cdots)\in\partial^*V(y)$.
On the other hand

$$\overline{\lim}_{m\to\infty}\ sign\left(\sum_{k=1}^{m}\int_0^T \upsilon_k f_k(s)\,ds\right)=sign\left(\sum_{k=1}^{n}\int_0^T \upsilon_k f_k(s)\,ds\right)=$$

$$=sign\left\langle\upsilon,\int_0^T(\mathbb{P}_n f)(s)ds\right\rangle_{\mathbb{R}^n}=sign\langle\upsilon,\Pi_n f^{(n)}\rangle_{\mathbb{R}^n},$$

where $f^{(n)} = \mathbb{P}_n f \in \mathbb{P}_n \mathcal{P}_F(y)$. So

$$\langle v, \Pi_n f^{(n)} \rangle_{\mathbb{R}^n} > 0, \tag{3.7}$$

and hence, $\Pi_n f^{(n)} \neq 0$ for all $f \in \mathcal{P}_F(y)$. In particular, $\Pi_n f^{*(n)} \neq 0$. But $\Pi_n f^{*(n)} = \Pi_n f^{*(n)}_{(0)} = 0$, giving the contradiction.

Thus, Σ_n is a homotopy connecting the multioperators $\Sigma_n(x,1) = G_n$ and $\Sigma_n(x,0) = C_n + \Pi_n \mathbb{P}_n \mathcal{P}_F$. Then we obtain

$$deg\big(i - G_n, B^{(n)}_C(0,R)\big) = deg\big(i - C_n - \Pi_n \mathbb{P}_n \mathcal{P}_F, B^{(n)}_C(0,R)\big).$$

The operator $C_n + \Pi_n \mathbb{P}_n \mathcal{P}_F$ takes values in $H_n \cong \mathbb{R}^n$, so, by the map restriction property of the topological degree we obtain

$$deg\big(i - C_n - \Pi_n \mathbb{P}_n \mathcal{P}_F, B^{(n)}_C(0,R)\big) = deg\big(i - C_n - \Pi_n \mathbb{P}_n \mathcal{P}_F, B_{\mathbb{R}^n}(0,R)\big).$$

In the space $H_n \cong \mathbb{R}^n$ the multifield $i - C_n - \Pi_n \mathbb{P}_n \mathcal{P}_F$ has the form

$$i - C_n - \Pi_n \mathbb{P}_n \mathcal{P}_F = -\Pi_n \mathbb{P}_n \mathcal{P}_F,$$

therefore,

$$deg\big(i - C_n - \Pi_n \mathbb{P}_n \mathcal{P}_F, B_{\mathbb{R}^n}(0,R)\big) = deg\big(-\Pi_n \mathbb{P}_n \mathcal{P}_F, B_{\mathbb{R}^n}(0,R)\big).$$

From (3.7) it follows that the multifields $\Pi_n \mathbb{P}_n \mathcal{P}_F$ and $Pr_n \partial^* V$ are homotopic on $B_{\mathbb{R}^n}(0,R)$, and then

$$deg\big(-\Pi_n \mathbb{P}_n \mathcal{P}_F, B_{\mathbb{R}^n}(0,R)\big) = deg\big(-Pr_n \partial^* V, B_{\mathbb{R}^n}(0,R)\big) = (-1)^n \gamma_n.$$

From $ind\ V \neq 0$ it follows that there exists a sequence $\{n_k\}$, $n_k \geq n_0$, such that $\gamma_{n_k} \neq 0$, and then $\omega_{n_k} \neq 0$. So, there is a sequence $\{x^{(k)}\}$, $x^{(k)} \in B^{(n_k)}_C(0,R)$, such that $Ax^{(k)} \in \mathbb{P}_{n_k} \mathcal{P}_F(x^{(k)})$ for all k. By virtue of $A-$approximation solvability of inclusion (3.2) we obtain that inclusion (3.1) has a $T-$periodic solution. $\square$

Generalizing the results of [19], let us present some sufficient conditions for $A-$approximation solvability of inclusion (3.2).

For a Banach space Y, let us denote by $\mathcal{BC}(Y)$ the Banach space of all bounded continuous functions $x\colon (-\infty, 0] \to Y$.

Theorem 11. *Let a Hilbert space H be compactly embedded in a Banach space Y. Assume that the multimap $\widetilde{F}\colon I \times \mathcal{BC}(Y) \to P(Y)$ satisfies the following conditions:*

($\widetilde{F}$) *for a.e. $t \in I$ the multimap $\widetilde{F}(t,\cdot)\colon \mathcal{BC}(Y) \to P(Y)$ is upper semicontinuous. In addition assume that the restriction $\widetilde{F}_{|I \times \mathcal{BC}(H)}$ takes values in $Kv(H)$ and the multimap $F = \widetilde{F}_{|I \times \mathcal{BC}(H)}\colon I \times \mathcal{BC}(H) \to Kv(H)$ satisfies conditions $(F1), (F3)$. Then inclusion (3.2) is $A-$approximation solvable.*

Proof. Assume that there are sequences $\{n_k\}$ and $\{x^{(k)}\}$, $x_k \in C_T(I, H_{n_k})$, such that

$$\sup_k \|x^{(k)}\|_C < +\infty \quad \text{and} \quad Ax^{(k)} \in \mathbb{P}_{n_k} \mathcal{P}_F(x^{(k)}).$$

From $(F3)$ it follows that the set $\mathcal{P}_F(\{x^{(k)}\}_{k=1}^{\infty})$, and hence the set $A(\{x^{(k)}\}_{k=1}^{\infty})$, is bounded in $L_2(I,H)$. Then the set $\{x^{(k)}\}_{k=1}^{\infty}$ is bounded in $W^{1,2}_T(I,H)$, and so it is weakly compact. W.l.o.g. assume that $x^{(k)} \overset{W}{\rightharpoonup} x^{(0)} \in W_T^{1,2}(I,H)$. Therefore, $Ax^{(k)} \overset{L}{\rightharpoonup} Ax^{(0)}$. From the fact that H is compactly embedded in Y it follows that the space $W_T^{1,2}(I,H)$ is compactly embedded in $C_T(I,Y)$, and hence,

$$x^{(k)} \overset{C_T(I,Y)}{\longrightarrow} x^{(0)}, \quad \text{and} \quad \widetilde{x}^{(k)} \overset{\mathcal{BC}(Y)}{\longrightarrow} \widetilde{x}^{(0)}.$$

Therefore, for every $s \in I$

$$\widetilde{x}_s^{(k)} \overset{\mathcal{BC}(Y)}{\longrightarrow} \widetilde{x}_s^{(0)}. \tag{3.8}$$

Now let $f^{(k)} \in \mathcal{P}_F(x^{(k)})$ be such that $Ax^{(k)} = \mathbb{P}_{n_k} f^{(k)}$. The set $\{f^{(k)}\}_{k=1}^{\infty}$ is bounded in $L_2(I,H)$, so it is weakly compact in this space. W.l.o.g. assume that

$$f^{(k)} \overset{L}{\rightharpoonup} f^{(0)} \in L_2(I,H).$$

Let us show that $\mathbb{P}_{n_k} f^{(k)} \overset{L}{\rightharpoonup} f^{(0)}$. For this, at first we demonstrate that

$$\lim_{n\to\infty} \mathbb{P}_n f^{(0)} = f^{(0)}.$$

It fact, since

$$L_2(I,H) = \overline{\bigcup_{n=1}^{\infty} L_2(I,H_n)},$$

there are sequences $\{\hat{n}_m\}_{m=1}^{\infty} \subset \mathbb{N}$ and $\{\hat{f}^{(m)}\}_{m=1}^{\infty}$, $\hat{f}^{(m)} \in L_2(I,H_{\hat{n}_m})$ such that $\hat{f}^{(m)} \to f^{(0)}$ in $L_2(I,H)$.

We have

$$\|\mathbb{P}_{\hat{n}_m} f^{(0)} - f^{(0)}\|_2 \le \|\mathbb{P}_{\hat{n}_m} f^{(0)} - \mathbb{P}_{\hat{n}_m} \hat{f}^{(m)}\|_2 + \|\mathbb{P}_{\hat{n}_m} \hat{f}^{(m)} - f^{(0)}\|_2 \le$$
$$\le 2\|\hat{f}^{(m)} - f^{(0)}\|_2 \to 0$$

as $m \to \infty$. Further, for all $n > \hat{n}_m$

$$\|\mathbb{P}_n f^{(0)} - \mathbb{P}_{\hat{n}_m} f^{(0)}\|_2 = \|\mathbb{P}_n f^{(0)} - \mathbb{P}_n(\mathbb{P}_{\hat{n}_m} f^{(0)})\|_2 \le \|f^{(0)} - \mathbb{P}_{\hat{n}_m} f^{(0)}\|_2,$$

hence,

$$\|\mathbb{P}_n f^{(0)} - f^{(0)}\|_2 \le \|\mathbb{P}_n f^{(0)} - \mathbb{P}_{\hat{n}_m} f^{(0)}\|_2 + \|\mathbb{P}_{\hat{n}_m} f^{(0)} - f^{(0)}\|_2$$
$$\le 2\|f^{(0)} - \mathbb{P}_{\hat{n}_m} f^{(0)}\|_2.$$

So,

$$\lim_{n\to\infty} \mathbb{P}_n f^{(0)} = f^{(0)}.$$

Now for every $g \in L_2(I,H)$ we obtain

$$\langle \mathbb{P}_{n_k} f^{(k)} - f^{(0)}, g \rangle_L = \langle \mathbb{P}_{n_k} f^{(k)} - \mathbb{P}_{n_k} f^{(0)}, g \rangle_L + \langle \mathbb{P}_{n_k} f^{(0)} - f^{(0)}, g \rangle_L =$$
$$= \langle f^{(k)} - f^{(0)}, \mathbb{P}_{n_k} g \rangle_L + \langle \mathbb{P}_{n_k} f^{(0)} - f^{(0)}, g \rangle_L =$$
$$= \langle f^{(k)} - f^{(0)}, g \rangle_L + \langle f^{(k)} - f^{(0)}, \mathbb{P}_{n_k} g - g \rangle_L + \langle \mathbb{P}_{n_k} f^{(0)} - f^{(0)}, g \rangle_L.$$

Thus

$$\lim_{k\to\infty} \langle \mathbb{P}_{n_k} f_k - f_0, g \rangle_L = 0.$$

On the other hand, $\mathbb{P}_{n_k} f^{(k)} = Ax^{(k)} \xrightarrow{L} Ax^{(0)}$. So $Ax^{(0)} = f^{(0)}$, and hence, $f^{(k)} \xrightarrow{L} Ax^{(0)}$. By virtue of the Mazur's Lemma (see, e.g., [6] p. 16) there is a sequence of convex combinations $\{\overline{f}^{(m)}\}$,

$$\overline{f}^{(m)} = \sum_{k=m}^{\infty} \lambda_{mk} f^{(k)},\ \lambda_{mk} \geq 0 \ \text{ and } \sum_{k=m}^{\infty} \lambda_{mk} = 1,$$

which converges to $Ax^{(0)}$ on average. Applying Theorem 38 [[20], Chapter IV], we assume w.l.o.g that $\{\overline{f}^{(m)}\}$ converges to Ax^0 for a.e. $t \in I$. Since the embedding $H \hookrightarrow Y$ is compact, we have $\overline{f}^{(m)}(t) \xrightarrow{Y} Ax^{(0)}(t)$ for a.e. $t \in I$.

From (3.8) and $(\widetilde{F})$ it follows that for a.e. $t \in I$ and for a given $\varepsilon > 0$ there is an integer $i_0 = i_0(\varepsilon, t)$ such that

$$\widetilde{F}(t, \widetilde{x}_t^{(i)}) \subset O_\varepsilon^Y\Big(\widetilde{F}\big(t, \widetilde{x}_t^{(0)}\big)\Big) \ \text{ for all } i \geq i_0,$$

where O_ε^Y denotes the $\varepsilon-$neighborhood of a set in Y. Since $x^{(i)}(t) \in H$ for all i, we obtain

$$F(t, \widetilde{x}_t^{(i)}) \subset O_\varepsilon^Y\Big(F\big(t, \widetilde{x}_t^{(0)}\big)\Big) \ \text{ for all } i \geq i_0,$$

Then $f^{(i)}(t) \in O_\varepsilon^Y\Big(F\big(t, \widetilde{x}_t^{(0)}\big)\Big)$ for all $i \geq i_0$, and by virtue of the convexity of the set $O_\varepsilon^Y\Big(F\big(t, \widetilde{x}_t^{(0)}\big)\Big)$ we have

$$\overline{f}^{(m)}(t) \in O_\varepsilon^Y\Big(F\big(t, \widetilde{x}_t^{(0)}\big)\Big), \text{ for all } m \geq i_0.$$

Therefore, $Ax^{(0)}(t) \in F(t, \widetilde{x}_t^{(0)})$ for a.e. $t \in I$, and so

$$Ax^{(0)} \in \mathcal{P}_F(x^{(0)}).$$

□

Theorem 12. *Let a multimap $F\colon I \times \mathcal{BC}(H) \to Kv(H)$ satisfy conditions $(F1)$ and $(F3)$. Then inclusion (3.2) is $A-$approximation solvable in each of the following cases:*

(1*i*) *for a.e. $t \in I$ the multimap $F(t, \cdot)\colon \mathcal{BC}(H) \to Kv(H)$ is weakly upper semicontinuous in the following sense: for every sequence $\{\psi^{(n)}\} \in \mathcal{BC}(H)$, $\psi^{(n)} \xrightarrow{\mathcal{BC}(H)} \psi^{(0)} \in \mathcal{BC}(H)$, and for every $\varepsilon > 0$ there is an integer $N(\varepsilon, t) > 0$ such that*

$$F(t, \psi^{(n)}) \subset O_\varepsilon\big(F(t, \psi^{(0)})\big)$$

for all $n > N(\varepsilon, t)$;

(2*i*) *the multimap F satisfies condition $(F2)$ and there is an integer $q_0 > 0$ such that for each $n \geq q_0$ the restriction of $F(t, \cdot)$ on $\mathcal{BC}(H_n)$ takes values in $Kv(H_n)$ for a.e. $t \in I$.*

Proof. Assume that there are sequences $\{n_k\} \subset \mathbb{N}$ and $\{x^{(k)}\}$, $x^{(k)} \in C_T(I, H_{n_k})$, such that

$$\sup_k \|x^{(k)}\|_C < +\infty \ \text{ and } \ Ax^{(k)} \in \mathbb{P}_{n_k} \mathcal{P}_F(x^{(k)}).$$

Let condition $(1i)$ holds true. Then the multioperator $\mathcal{P}_F$ is well-defined. Similarly to the proof of Theorem 11, from $x^{(k)} \overset{W}{\rightharpoonup} x^{(0)}$ it follows that $x_t^{(k)} \overset{\mathcal{BC}(H)}{\rightharpoonup} x_t^{(0)}$, for every $t \in I$. And hence, from condition $(1i)$ we obtain that for a.e. $t \in I$

$$F(t, \widetilde{x}_t^{(i)}) \subset O_\varepsilon\Big(F\big(t, \widetilde{x}_t^{(0)}\big)\Big) \text{ for all } i \geq N(t, \varepsilon).$$

Hence we again have $Ax^{(0)} \in \mathcal{P}_F(x^{(0)})$.

Now let condition $(2i)$ holds true. Then for each $n \geq q_0$ we obtain

$$\mathbb{P}_n \mathcal{P}_F(x) = \mathcal{P}_F(x),$$

for all $x \in C_T(I, H_n)$. It is clear that for all k such that $n_k \geq q_0$ the following relation holds: $Ax^{(k)} \in \mathcal{P}_F(x^{(k)})$. □

4. Existence of periodic solutions for a gradient functional differential inclusion

For $h > 0$, consider the spaces of real-valued functions $H = W^{1,2}[0, h]$ and $Y = L_2[0, h]$. It is clear that H is compactly embedded in Y. Let the functional $V \colon Y \to \mathbb{R}$ be defined as

$$V(y) = \frac{1}{2}|y_1| + \sum_1^\infty y_k^2, \quad y = (y_1, y_2, \cdots),$$

where y_i, $i = 1, 2, \cdots$, are the Fourier's coefficients of y. It is clear that

$$\partial^* V(y) = \partial V_1(y_1) \times \{2y_2\} \times \{2y_3\} \times \cdots,$$

where

$$\partial V_1(y_1) = \begin{cases} 2y_1 + \frac{1}{2} & \text{if } y_1 > 0, \\ [-\frac{1}{2}, \frac{1}{2}] & \text{if } y_1 = 0, \\ 2y_1 - \frac{1}{2} & \text{if } y_1 < 0, \end{cases}$$

and the multimap $\partial^* V \colon Y \to Kv(Y)$ is upper semicontinuous. Moreover, the restriction $\partial^* V_{|H}$ takes values in $Kv(H)$ and

$$\|\partial^* V(y)\|_H \leq 2\|y\|_H + \frac{1}{2}, \text{ for all } y = (y_1, y_2, \cdots) \in H. \tag{4.1}$$

Consider the following functional differential inclusion

$$x'(t) \in \partial^* V(x(t)) + G(t, x_t), \text{ for a.e. } t \in I, \tag{4.2}$$

where $G \colon \mathbb{R} \times \mathcal{BC}(Y) \to P(Y)$ is a multimap.

Assume that the following conditions hold:

- (G_T) G is $T-$periodic with respect to the first argument;
- $(G1)$ for a.e. $t \in I$ multimap $G(t, \cdot) \colon \mathcal{BC}(Y) \to P(Y)$ is upper semicontinuous;
- $(G2)$ the restriction $G_{|_{I \times \mathcal{BC}(H)}}$ takes values in $Kv(H)$;
- $(G3)$ for each $\psi \in \mathcal{BC}(H)$ the multifunction $G(\cdot, \psi) \colon I \to Kv(H)$ has a measurable selection;

$(G4)$ there exists $C > 0$ such that

$$\|G(s, \widetilde{\psi}_s)\|_H \leq C(1 + \|\psi\|_2),$$

for a.e. $s \in I$ and all $\psi \in C_T(I, H)$.

Theorem 13. *Let conditions* (G_T) *and* $(G1) - (G4)$ *hold. In addition, assume that*

$$C\sqrt{T} < 2.$$

Then inclusion (4.2) *has a* $T-$*periodic solution* $x \in W_T^{1,2}(I, H)$.

Proof. Set $\widetilde{F} \colon \mathbb{R} \times \mathcal{BC}(Y) \to P(Y)$,

$$\widetilde{F}(t, \psi) = \partial^* V(\psi(0)) + G(t, \psi).$$

It is clear that multimap $\widetilde{F}$ is $T-$periodic with respect to the first argument and satisfies condition $(\widetilde{F})$ of Theorem 11.
Consider $F = \widetilde{F}_{|_{I \times \mathcal{BC}(H)}}$. It is easy to see that the multimap F takes values in $Kv(H)$ and satisfies condition $(F1)$. Notice that from condition $(F3)$ it follows that for every $r > 0$ and $x \in C_T(I, H)$ such that $\|x\|_2 \leq r$, there exists $M_r > 0$ such that $\|f\|_2 \leq M_r$ for all $f \in \mathcal{P}_F(x)$. From (4.1) and $(G4)$ we see that the multimap F satisfies condition $(F3)$. The application of Theorem 11 implies that inclusion (3.2) is $A-$approximation solvable.

It is clear that the functional V is projectively homogeneous. Let us show that it is a guiding function for inclusion (4.2). In fact, let $x \in W_T^{1,2}(I, H)$ and take an arbitrary $f \in \mathcal{P}_F(x)$. Then there are a function $g \in \mathcal{P}_G(x)$ and a selection $\upsilon(s) \in \partial^* V(x(s))$ such that

$$f(s) = \upsilon(s) + g(s) \text{ for a.e. } t \in I,$$

where

$$\mathcal{P}_G(x) = \{g \in L_2(I, H) \colon g(s) \in G(s, \widetilde{x}_s) \text{ for a.e. } s \in I\}$$

Notice that for every $s \in I$ the values $u = \upsilon(s)$ and $\omega = g(s)$ are functions in H and

$$\Big\langle \upsilon(s), f(s) \Big\rangle_H = \big\langle u, u + \omega \big\rangle_H =$$
$$= \int_0^h \big(u^2(\tau) + u'^2(\tau)\big) d\tau + \int_0^h \big(u(\tau)\omega(\tau) + u'(\tau)\omega'(\tau)\big) d\tau \geq$$
$$\geq \|u\|_H^2 - \|u\|_H \|\omega\|_H.$$

Therefore

$$\int_0^T \Big\langle \upsilon(s), f(s) \Big\rangle_H ds = \int_0^T \Big\langle \upsilon(s), \upsilon(s) + g(s) \Big\rangle_H ds \geq$$
$$\geq \int_0^T \Big(\|\upsilon(s)\|_H^2 - \|g(s)\|_H \|\upsilon(s)\|_H \Big) ds \geq$$
$$\geq \|\upsilon\|_2^2 - \int_0^T \|\upsilon(s)\|_H \; C(1 + \|x\|_2) ds.$$

From (4.1) it follows that

$$\int_0^T \Big\langle \upsilon(s), f(s) \Big\rangle_H ds \geq \|\upsilon\|_2^2 - C(1+\|x\|_2)\int_0^T (2\|x(s)\|_H + \frac{1}{2})ds \geq$$
$$\geq \|\upsilon\|_2^2 - 2C\sqrt{T}\|x\|_2^2 - (2C\sqrt{T} + \frac{TC}{2})\|x\|_2 - \frac{TC}{2}.$$

Now let us mention that for every selection $\upsilon(s) \in \partial^* V(x(s))$ there is a number $\varepsilon \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\upsilon(s) = (2x_1(s) + \varepsilon, 2x_2(s), \cdots, 2x_n(s), \cdots), \quad s \in I,$$

where $x(s) = (x_1(s), x_2(s), \cdots, x_n(s), \cdots)$, $s \in I$.
Therefore

$$\|\upsilon\|_2^2 = \int_0^T \|\upsilon(s)\|_H^2 ds = 4\int_0^T \|x(s)\|_H^2 ds + 4\varepsilon \int_0^T x_1(s)ds + \varepsilon^2 T \geq$$
$$\geq 4\|x\|_2^2 - 2\int_0^T \|x(s)\|_H ds \geq 4\|x\|_2^2 - 2\sqrt{T}\|x\|_2.$$

Hence we obtain

$$\int_0^T \Big\langle \upsilon(s), f(s) \Big\rangle_H ds \geq (4 - 2C\sqrt{T})\|x\|_2^2 - (2\sqrt{T} + 2C\sqrt{T} + \frac{TC}{2})\|x\|_2 - \frac{TC}{2} > 0$$

provided $\|x\|_2$ is sufficiently large. So

$$\overline{\lim}_{m\to\infty}\ sign\left(\int_0^T \sum_{k=1}^m \upsilon_k(s) f_k(s)ds\right) = 1.$$

Thus, V is a guiding function for inclusion (4.2). It is clear that $ind V \neq 0$. So, applying Theorem 8, we conclude that inclusion (4.2) has a $T-$periodic solution $x \in W_T^{1,2}(I, H)$. □

References

[1] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden, 1976.

[2] Y.G. Borisovich, B.D. Gelman, A.D. Myshkis, V.V. Obukhovskii, *Introduction to the Theory of Multivalued Maps and Differential Inclusions*, Librokom, Moscow, 2011 (in Russian).

[3] F.H. Clarke, *Optimization and Nonsmooth Analysis,* Second edition, Classics in Applied Mathematics 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] V.F. Dem'yanov, L.V. Vasil'ev, *Nondifferentiable Optimization.* Nauka, Moscow 1981 (in Russian); English translation: Translation Series in Mathematics and Engineering. Optimization Software, Inc., Publications Division, New York, 1985.

[6] I. Ekeland, R. Temam, *Convex Analysis and Variation Problems*, North Holland, Amsterdam, 1979.

[7] A. Fonda, *Guiding functions and periodic solutions to functional differential equations*, Proc. Amer. Math. Soc., **99**(1987), no. 1, 79-85.

[8] R.E. Gaines, J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Mathematics, no. 568, Springer-Verlag, Berlin-New York, 1977.

[9] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings.* Second edition. Topological Fixed Point Theory and Its Applications, 4. Springer, Dordrecht, 2006.

[10] J.K. Hale, J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial. Ekvac., **21**(1978), no. 1, 11-41.

[11] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay.* Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin–Heidelberg–New York, 1991.

[12] M. Kamenskii, V. Obukhovskii, P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter Series in Nonlinear Analysis and Applications 7, Walter de Gruyter, Berlin-New York 2001.

[13] S.V. Kornev, V.V. Obukhovskii, *On non-smooth multivalent guiding functions*, Differ. Uravn., **39**(2003), no. 11, 1497-1502 (in Russian); English translation: Differ. Equ., **39**(2003), no. 11, 1578-1584.

[14] S. Kornev, V. Obukhovskii, *On some developments of the method of integral guiding functions*, Funct. Differ. Equ., **12**(2005), no. 3-4, 303-310.

[15] S.V. Kornev, V.V. Obukhovskii, *Nonsmooth guiding potentials in problems of forced oscillations*, Avtomat. i Telemekh., 2007, no. 1, 3-10 (in Russian); English translation: Automation Remote Control, **68**(2007), no. 1, 1-8.

[16] M.A. Krasnosel'skii, *The Operator of Translation Along the Trajectories of Differential Equations*, Nauka, Moscow 1966 (in Russian); English translation: Translations of Mathematical Monographs 19, Amer. Math. Soc., Providence, R.I. 1968.

[17] M.A. Krasnosel'skii, A.I. Perov, *On a certain priciple of existence of bounded, periodic and almost periodic solutions of systems of ordinary differential equations*, Dokl. Akad. Nauk SSSR, **123**(1958), no. 2, 235-238 (in Russian).

[18] M.A. Krasnosel'skii, P.P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Nauka, Moscow 1975; English translation: Grundlehren der Mathematischen Wissenschaften 263, Springer-Verlag, Berlin, 1984.

[19] Nguyen Van Loi, *Method of guiding functions for differential inclusions in a Hilbert space*, Differ. Uravn., **46**(2010), no. 10, 1433-1443 (in Russian); English translation: Differ. Equ., **46**(2010), no. 10, 1438-1447.

[20] L. Schwartz, *Cours d'Analyse*, 1, Second edition. Hermann, Paris, 1981.

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