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THE MAJORIZATION FIXED POINT PRINCIPLE AND APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

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Abstract. In this article we consider a modification of the Kantorovich and Banach – Caccioppoli fixed point principles for operators satisfying a local Lipschitz condition. In particular, we give sharp estimates for the inner and outer radius of existence and uniqueness. Furthermore, new *a priori* and *a posteriori* error estimates for successive approximations are deduced. Our examples reflect the fact that many existence and uniqueness results for fixed points of operators in the spaces C and L_p can be strengthened by passing to local Lipschitz conditions.

Key Words and Phrases: Banach – Caccioppoli fixed point principle, Kantorovich majorant fixed point principle, local Lipschitz condition, a priori and a posteriori error estimates, equations with polynomial operators, mixed Hammerstein equations, generalized Uryson equations.
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1. Introduction

The method of successive approximations makes it possible to solve existence and uniqueness problems for fixed points of wide classes of operators. Classical results in this field, such as the Banach – Caccioppoli principle and some of its modifications and generalizations, apply to operators satisfying a Lipschitz condition with a small Lipschitz constant (contractions). However, the method works as well for other classes of operators which are not contractions. In particular, the well known Kantorovich fixed point principle [3] for differentiable operators deals with operators that, in general, are not contractions; moreover, this principle covers some cases when the Banach – Caccioppoli principle does not apply.

Recall that the Banach – Caccioppoli fixed point principle refers to operators in complete metric spaces. The Kantorovich fixed point principle deals only with operators in Banach spaces; moreover, it is applicable only to differentiable operators. In this article we consider some modification of the Kantorovich fixed point principle which also covers nondifferentiable operators. Some variants of this modification have been used by the second author in the 90ies; in [6] an "almost sharp" variant of this principle was given. The variant discussed in the present article is an essential complement: in this variant we describe exact (unimprovable) estimates of the inner and outer radius of the domain of existence of a unique fixed point of the operator under consideration. In addition, we obtain new *a priori* and *a posteriori* error estimates for successive approximations of the corresponding fixed point.

In the last part we sketch some applications of the new fixed point principle to nonlinear integral operators of different type.

2. The principle of majorized mappings

Consider the equation

$$x = Ax, \tag{2.1}$$

where A is an operator defined on a ball $B[x_0, R] = \{x : \|x - x_0\| \le R\}$ of a Banach space X $(x_0 \in X)$.

Definition 2.1 The operator A satisfies a *local Lipschitz condition* in the ball $B[x_0, R]$ with a nonnegative function $k(\cdot)$ on $[0, R]$ if

$$\|Ax_1 - Ax_2\| \le k(r)\|x_1 - x_2\|, \tag{2.2}$$

where

$$\|x_1 - x_0\| \le r, \qquad \|x_2 - x_0\| \le r, \qquad 0 < r \le R.$$

The basic part of the theorem presented below for smooth operators A is given in [3]. Here we present the theorem for both smooth and nonsmooth operators. To start with let us introduce some notation.

Let $a_+(\cdot)$ and $a_-(\cdot)$ be the functions defined by

$$a_\pm(r) = a \pm \int_0^r k(t)\,\mathrm{d}t, \qquad \text{where} \qquad a = \|Ax_0 - x_0\|. \tag{2.3}$$

In what follows we call the functions $a_\pm(r)$ *majorant functions* of the operator A.

If the function $a_+(\cdot)$ has fixed points in the interval $[0, R]$ we denote the smallest of them by r^*. Similarly, we denote by r_* the smallest fixed point of the function $a_-(\cdot)$. Finally, let

$$r^{**} = \sup_{r^* < r \le R} \{r : \ a_+(r) < r\} \tag{2.4}$$

(provided that the set under the sup sign is nonempty). We also put

$$L(x_0, r^*, r^{**}) = \begin{cases} \{x : \ r^* < \|x_0 - x\| < r^{**}\} & \text{if} \ \ a_+(R) \ge R, \\ \{x : \ r^* < \|x_0 - x\| \le r^{**}\} & \text{if} \ \ a_+(R) < R, \end{cases} \tag{2.5}$$

and

$$L[x_0, r_*, r^*] = \{x : \ r_* \le \|x_0 - x\| \le r^*\}. \tag{2.6}$$

With this notation, our main theorem reads as follows.

Theorem 2.2 *Suppose that the operator A is defined on the ball $B[x_0, R]$ of a Banach space X $(x_0 \in X)$ and satisfies the local Lipschitz condition (2.2) in the ball $B[x_0, R]$ with a nonnegative function $k(\cdot)$ on $[0, R]$. Assume that the functions $a_\pm(\cdot)$ have fixed points in the interval $[0, R]$. Then the operator A has a unique fixed point $x^* \in L[x_0, r_*, r^*]$, and this fixed point is unique in $B[x_0, r]$ for each r satisfying $r^* \le r < r^{**}$ (i.e., there are no fixed points in $B[x_0, r_*] \bigcup L[x_0, r^*, r^{**}]$).*

This theorem on majorized mappings is a modification of the method of successive approximations. It is easy to see that the hypotheses of the Banach – Caccioppoli theorem are covered by the conditions of Theorem 2.2.

Let us discuss some advantages of Theorem 2.2. First of all, Theorem 2.2 uses a local Lipschitz condition instead of existence conditions for a continuous derivative; this essentially extends the class of mappings covered by this theorem. Secondly, the method used in Theorem 2.2 is built on the analysis of a real differentiable function having fixed points in some interval. Finally, the method adopted here is convenient for a comparison between the method of majorized mappings and the Banach – Caccioppoli principle.

Figures 1 – 3 below show the relationship between the principle of majorized mappings and the Banach – Caccioppoli principle in the general setting. The notation *BC-zone, U-zone, E-zone* means the following: the set of radii r of the balls, where the Banach – Caccioppoli principle of the fixed point is applicable is denoted by (BC), where uniqueness holds by (U), and where existence holds by (E). The Banach – Caccioppoli principle is applicable in the ball $B[x_0, r]$, where the radius r should satisfy the inequality

$$r^* \le r < r_{cr}, \qquad \text{where} \qquad r_{cr} = \inf_{k(r)=1} r.$$

So according to the Banach – Caccioppoli theorem, any fixed point x^* of A lies in the ball $B[x_0, r^*]$ and is unique in each ball $B[x_0, r]$, where $r^* \le r < r_{cr}$. But with Theorem 2.2 we can get better result: any fixed point x^* of A lies in the domain $L[x_0, r_*, r^*]$ and is unique in each ball $B[x_0, r]$, where $r^* \le r < r^{**}$ for Figures 1 – 2 and $r^* \le r \le r^{**}$ for Figure 3 (thus $0 \le r < r^{**}$ is the uniqueness (U) zone for Figures 1 – 2 and $0 \le r \le r^{**}$ for Figure 3).

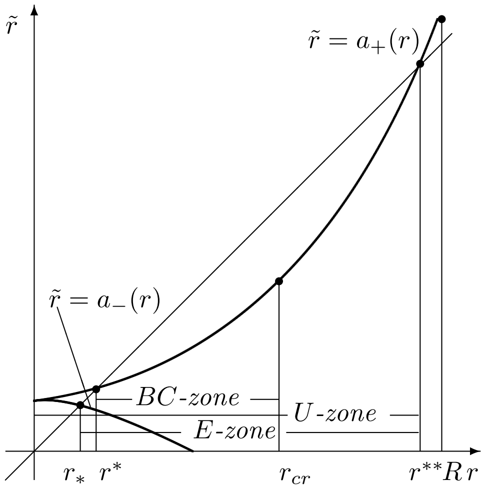

Figure 1

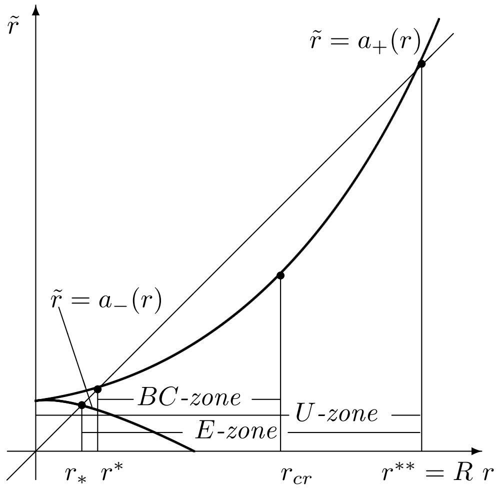

Figure 2

Figure 4 illustrates the case when the Banach – Caccioppoli theorem does not apply, but the principle of majorized mappings does. So in this case the operator A is neither contracting nor expanding in the ball $B[x_0, r]$, where $r = r^* = r_{cr} = R$.

One may find a part of the proof of Theorem 2.2 in [7]. Here we give the full proof. To this end, we first need the following

Lemma 2.3 *Suppose that the operator A is defined on the ball $B[x_0, R]$ of a Banach space X ($x_0 \in X$) and satisfies the local Lipschitz condition (2.2) in the ball $B[x_0, R]$ with a nonnegative function $k(\cdot)$ on $[0, R]$. Then the inequality*

$$\|A(x+h) - Ax\| \leq \int_r^{r+\delta} k(t)dt \qquad (\|x - x_0\| \leq r,\ \|h\| \leq \delta,\ r + \delta \leq R). \tag{2.7}$$

holds.

The assertion of the lemma follows from the obvious chain of inequalities

$$\|A(x+h) - Ax\| \leq \sum_{j=1}^{s} \left\| A\left(x + \frac{j}{s}h\right) - A\left(x + \frac{j-1}{s}h\right) \right\| \leq \sum_{j=1}^{s} k\left(r + \frac{j}{s}\delta\right)\frac{\delta}{s}$$

after passing to the limit as $s \to \infty$.

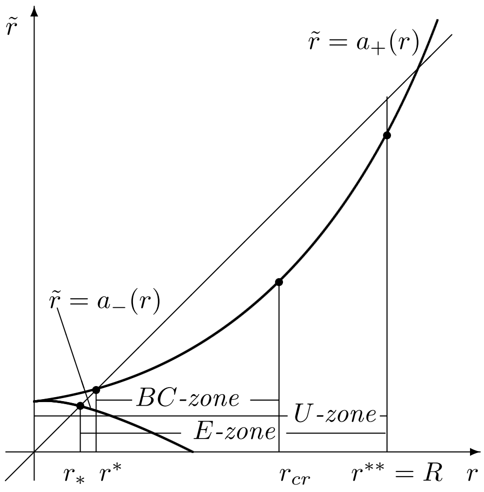

Figure 3

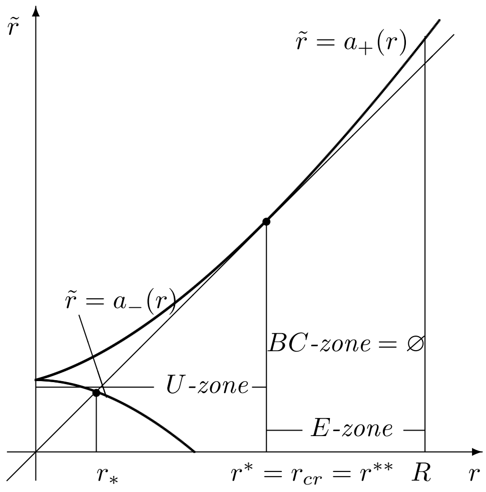

Figure 4

Proof of Theorem 1. First of all let us prove that the successive approximations

$$r_{n+1} = a_+(r_n) \qquad (r_0 = 0,\ n = 0, 1, \ldots), \tag{2.8}$$

converge. Note that (2.3) implies

$$a'_+(r) = k(r) \geq 0, \qquad r \in [0, R],$$

by virtue of (2.2). So the function a_+ does not decrease in the interval $[0, R]$ and r_n makes sense for any n. Moreover,

$$r_n \leq r^* \qquad (n = 0, 1, \ldots), \tag{2.9}$$

where r^* is the smallest root (whose existence is assumed in Theorem 2.2 of the equation

$$r = a_+(r). \tag{2.10}$$

For $n = 0$, inequality (2.9) is evident, and if it is proved for $n = k$, then from $r_k \leq r^*$ we get $a_+(r_k) \leq a_+(r^*)$ due to the monotonicity of $a_+(\cdot)$. So $r_{k+1} \leq r^*$ and by induction the inequality (2.9) is proved for any n.

Using again the monotonicity of $a_+(\cdot)$ we can prove by induction that the the sequence $\{r_n\}$ is monotonically increasing. In fact, $r_n \leq r_{n+1}$ implies $r_{n+1} = a_+(r_n) \leq a_+(r_{n+1}) = r_{n+2}$, and the inequality $0 = r_0 \leq r_1$ is obvious.

So far we have established the existence of the limit

$$r^* = \lim_{n\to\infty} r_n.$$

By (2.8) and the continuity of $a_+(\cdot)$, the limit r^* is a root of equation (2.10); moreover, r^* is the smallest root of (2.10) in $[0, R]$, by (2.9).

Let

$$x_{n+1} = Ax_n, \qquad (n = 0, 1, \dots), \tag{2.11}$$

where x_0 is the center of the ball $B[x_0, R]$. We claim that all elements in (2.11) make sense and form a convergent sequence.

For $n = 0$ we have, by (2.3),

$$\|x_1 - x_0\| = \|Ax_0 - x_0\| = a = a_+(0) = a_+(r_0) = r_1,$$

hence $x_1 \in B[x_0, R]$. Suppose that we have already proved that $x_1, x_2, ..., x_n \in B[x_0, R]$, and that

$$\|x_{k+1} - x_k\| \leq r_{k+1} - r_k \qquad (k = 0, 1, \dots, n-1). \tag{2.12}$$

Then using Lemma 2.3 we obtain

$$\|x_{n+1} - x_n\| = \|Ax_n - Ax_{n-1}\| \leq \int_{r_{n-1}}^{r_n} k(t)dt = a_+(r_n) - a_+(r_{n-1}) = r_{n+1} - r_n.$$

So (2.11) is proved for $k = n$, and the fact that $x_{n+1} \in B[x_0, R]$ is also proved since

$$\begin{aligned} \|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + ... + \|x_1 - x_0\| \leq \\ &\leq (r_{n+1} - r_n) + (r_n - r_{n-1}) + ... + (r_1 - r_0) = r_{n+1} \leq R. \end{aligned}$$

Consequently, the inclusion $x_k \in B[x_0, R]$ and the estimate (2.12) are established for all $k = 0, 1, ...,$ by induction.

From (2.12) it further follows that

$$\begin{aligned} \|x_{n+p} - x_n\| &\leq \|x_{n+p} - x_{n+p-1}\| + ... + \|x_{n+1} - x_n\| \leq \\ &\leq (r_{n+p} - r_{n+p-1}) + ... + (r_{n+1} - r_n) = r_{n+p} - r_n, \end{aligned} \tag{2.13}$$

which implies the convergence of the sequence $\{x_n\}$. Let us denote

$$x^* = \lim_{n\to\infty} x_n.$$

Passing to the limit in (2.11) and taking into account the continuity of the operator A we get

$$x^* = Ax^*,$$

which shows that x^* is a root of equation (2.1). Moreover, inequality (2.12) implies

$$\|x^* - x_n\| \le r^* - r_n \qquad (n = 0, 1, \dots),$$

which gives an estimate of the convergence speed.

Let us now prove that the operator A has no fixed point in the ball $B[x_0, r_*]$. Estimating $\|x^* - x_0\| = r_0$ from below we get

$$\|x^* - x_0\| = \|Ax^* - x_0\| \ge \|Ax_0 - x_0\| - \|Ax^* - Ax_0\|, \tag{2.14}$$

By Lemma 2.3 we have

$$\|Ax^* - Ax_0\| \le \int_0^{r_0} k(t)dt.$$

So using equality (2.3) we get from (2.14)

$$\|x^* - x_0\| \ge a - \int_0^{r_0} k(t)dt = a_-(r_0)$$

which implies

$$a_-(r_0) \le r_0. \tag{2.15}$$

It is easy to see that inequality (2.15) is valid for all $r_0 \ge r_*$, where r_* is the point of intersection of the graph of the function $\tilde{r} = a_-(r)$ and the bisectrix $\tilde{r} = r$. This immediately implies that the operator A does not have fixed points in the ball $B[x_0, r_*]$.

We conclude that $r_* \le r_0 \le r^*$, i.e., the fixed point x^* of the operator A lies in the annulus $L[x_0, r_*, r^*]$. The theorem is proved. □

Theorem 2.4 *Suppose that all conditions of Theorem 2.2 are fulfilled. Then the following holds.*

1) *The successive approximations*

$$\xi_{n+1} = A\xi_n \qquad (n = 0, 1, \dots) \tag{2.16}$$

with the initial approximation $\xi_0 \in B[x_0, r^*] \cup L(x_0, r^*, r^{**})$ *are defined for any* n *and converge to the fixed point* x^*.

2) *The estimates*

$$\|x^* - \xi_n\| \le r^* + \rho_n - 2r_n \qquad (n = 0, 1, \dots), \tag{2.17}$$

$$\|\xi_{n+1} - \xi_n\| \le \rho_{n+1} + \rho_n - 2r_n \qquad (n = 0, 1, \dots) \tag{2.18}$$

are valid, where $\{r_n\}$ *are the successive approximations from Theorem 2.2 and*

$$\rho_{n+1} = a_+(\rho_n) \qquad (n = 0, 1, \dots), \tag{2.19}$$

where the initial approximation is $\rho_0 = \|\xi_0 - x_0\|$, *and* $\rho_0 \ge r_0 = 0$.

Proof. Consider the successive approximations (2.16), with the initial approximation ξ_0 being an arbitrary element from $B[x_0, r^*] \cup L(x_0, r^*, r^{**})$). It is easy to see at Figure 5 and Figure 6 that if $\rho_0 \ge r_0$, then $\rho_n \ge r_n$ for any n. Note also that the sequence $\{\rho_n\}$ is increasing to r^* if $\rho_0 < r^*$ and is decreasing to r^* if $\rho_0 > r^*$; in the case $\rho_0 = r^*$ all terms in the sequence $\{\rho_n\}$ coincide with r^*.

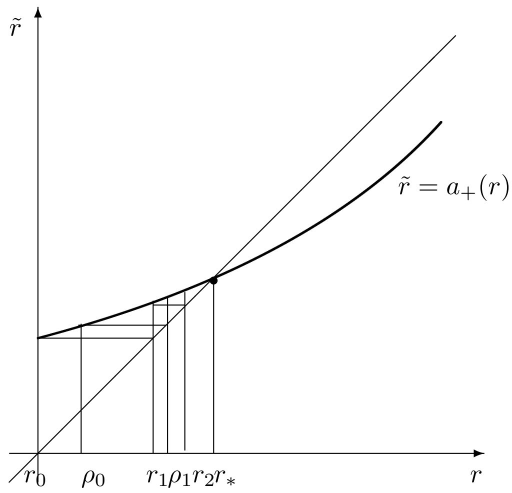

Figure 5

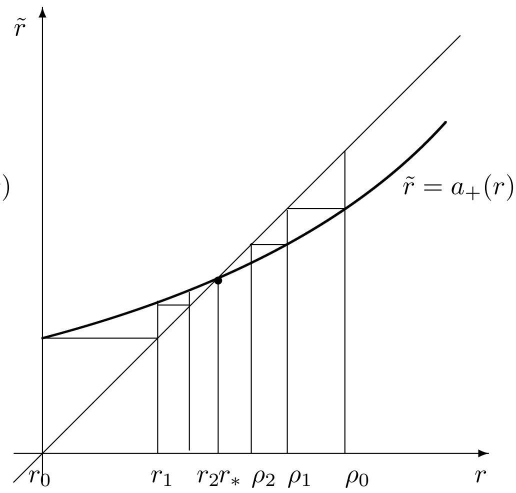

Figure 6

Literally in the same way as in the proof of Theorem 2.2 one may show that the sequence $\{\rho_n\}$ has a limit, say ρ^*. Moreover, ρ^* (the root of equation (2.10)) coincides with r^*.

Now we prove that the successive approximations sequence $\{\xi_n\}$ converges and therefore gives a root of equation (2.1). We have

$$\|\xi_1 - x_1\| = \|A\xi_0 - Ax_0\|$$

and by Lemma 2.3 we get

$$\|\xi_1 - x_1\| \le \int_{r_0}^{\rho_0} k(t)dt = a_+(\rho_0) - a_+(r_0) = \rho_1 - r_1,$$

and

$$\|\xi_1 - x_0\| \le \|\xi_1 - x_1\| + \|x_1 - x_0\| \le (\rho_1 - r_1) + (r_1 - r_0) \le \rho_1 \le R.$$

Clearly, $\xi_1 \in B[x_0, R]$.

The remaining part goes by induction. Suppose that

$$\xi_k \in B[x_0, R], \qquad \|\xi_k - x_k\| \le \rho_k - r_k \qquad (k = 0, 1, \dots, n). \tag{2.20}$$

Then $\xi_{n+1} - x_{n+1} = A\xi_n - Ax_n$. Using again Lemma 2.3 we obtain

$$\|\xi_{n+1} - x_{n+1}\| = \|A\xi_n - Ax_n\| \le \int_{r_n}^{\rho_n} k(t)dt = a_+(\rho_n) - a_+(r_n) = \rho_{n+1} - r_{n+1},$$

hence

$$\|\xi_{n+1} - x_0\| \le \|\xi_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \le$$
$$(\rho_{n+1} - r_{n+1}) + (r_{n+1} - r_0) \le \rho_{n+1} \le R$$

which shows that $\xi_{n+1} \in B[x_0, R]$. So by induction we conclude that (2.20) is valid for any k.

Since the sequences $\{r_n\}$ and $\{\rho_n\}$ have the common limit r^* it follows from (2.20) that the convergence of the sequence $\{x_n\}$ implies the convergence of the sequence $\{\xi_n\}$ with

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} x_n = x^*.$$

So we have proved that the sequence of successive approximations converges to x^* for any initial approximation $\xi_0 \in B[x_0, R]$. This also implies the uniqueness of the root of equation (2.1).

Now we prove the estimate (2.17). Since

$$\|x^* - \xi_n\| \le \|x^* - x_n\| + \|x_n - \xi_n\| \qquad (n = 0, 1, \dots),$$

by (2.13) and (2.20) we get

$$\|x^* - \xi_n\| \le \|x^* - x_n\| + \|x_n - \xi_n\| \le (r^* - r_n) + (\rho_n - r_n).$$

Moreover, from $\rho_n > r_n$ it follows that

$$\|x^* - \xi_n\| \le r^* + \rho_n - 2r_n \quad (n = 0, 1, \dots),$$

and so we have proved (2.17).

It remains to show that the estimate (2.18) is true. Using Lemma 2.3 we have

$$\begin{aligned}\|\xi_{n+1} - \xi_n\| &\le \|\xi_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \xi_n\| \\ &\le \rho_{n+1} - r_{n+1} + r_{n+1} - r_n + \rho_n - r_n = \rho_{n+1} + \rho_n - 2r_n\end{aligned}$$

which shows that (2.18) is true as well and concludes the proof. □

3. Examples

To illustrate the applicability of our abstract results, we consider now four examples.

Let X be a Banach space. Consider the Lemarié-Rieusset equation ([5], see also [7])

$$x = \eta + T(x, \dots, x), \tag{3.1}$$

where $\eta \in X$ and T is an m-linear ($m \ge 2$) continuous operator defined on X. As is well known, the operator T satisfies the Lipschitz condition

$$\|Tx_1 - Tx_2\| \le Cmr^{m-1}\|x_1 - x_2\| \qquad (x_1, x_2 \in B[x_0, r],\ 0 < r \le \infty),$$

which is (2.2) with $k(r) = Cmr^{m-1}$ and C being the norm of the m-linear operator T.

The majorant functions (3) have here the form

$$a_\pm(r) = a \pm \int_0^r k(t)\mathrm{d}t = a \pm C\int_0^r mt^{m-1}\mathrm{d}t = a \pm Cr^m, \quad a = \|\eta\|.$$

Thus, the equation

$$a + Cr^m = r \tag{3.2}$$

allows us to study existence and uniqueness conditions for equation (3.1).

It is easy to solve this equation for $m = 2$. It is also possible to find a solution for each $m = 3, 4, ...$, but in this case the solution in general form cannot be determined. From Figure 7 it is clear that the graph of the function $a_+(\cdot)$ depends on the value of $a = \|\eta\|$. So the set of roots of equation (3.2) also depends on a, and the existence condition for roots reads

$$a \le a_{cr}, \quad \text{where} \quad a_{cr} = \left(\frac{1}{Cm}\right)^{\frac{1}{m-1}} \frac{m-1}{m}. \tag{3.3}$$

Consequently, in the case when condition (3.3) is satisfied equation (3.1) has a unique root $x^* \in L[x_0, r_*, r^*]$. Moreover, the operator A has no fixed points in the set $B[0, r_*] \cup L(x_0, r^*, r^{**})$.

We point out that our reasoning gives more information than the Lemarié-Rieusset theorems do, inasmuch as the domain of existence of solutions can be described more precisely than in Lemarié-Rieusset's work.

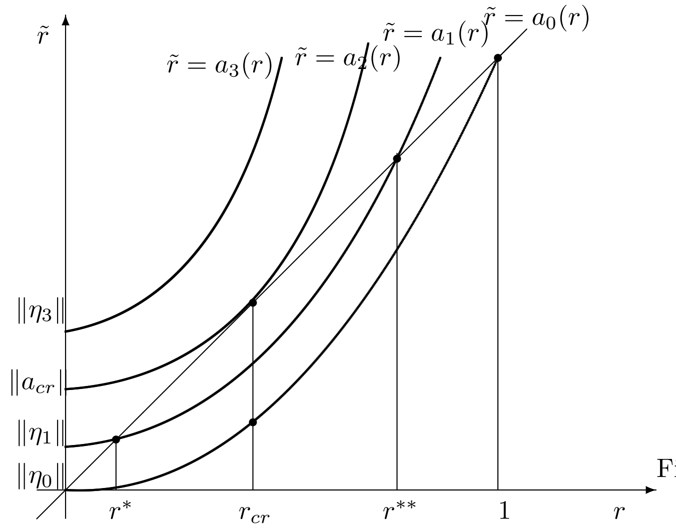

Figure 7

Our second example is concerned with the nonlinear integral equation of mixed Hammerstein type (see [8])

$$x(t) = f(t) + \lambda \sum_{j=1}^{m} \int_a^b k_j(t, s) h_j(x(s)) \, \mathrm{d}s, \tag{3.4}$$

where the kernel $k_j(t, s)$ is, for each j, a measurable function with respect to the variables $t, s \in [a, b]$, h_j is a continuous function, λ is a parameter, f is a given function, and x is an unknown function. This equation was studied in [8].

Let us consider first equation (3.4) in the space $C[a, b]$ of continuous functions on $[a, b]$. Assume that the functions h_j $(j = 1, ..., m)$ satisfy the conditions

$$|h_j(y_1) - h_j(y_2)| \le w_j(r)|y_1 - y_2| \qquad (|y_1|, |y_2| \le r, \quad 0 < r \le R, \quad w_j(r) \ge 0),$$

where the functions $w_j(r)$ are nondecreasing. Further, suppose that the kernels $k_j(t,s)$ $(j = 1, ..., m)$ define linear integral operators K_j in the space C; this means that each kernel $k_j(t,s)$ is Lebesgue integrable with respect to s in $[a,b]$ for $t \in [a,b]$,

$$\sup_{a \le t \le b} \int_a^b |k_j(t,s)| \, \mathrm{d}s < \infty,$$

and each function

$$\widetilde{k}_j(t,s) = \int_a^s k_j(t,\sigma) \, \mathrm{d}\sigma,$$

continuously depends on t in average, i.e.,

$$\lim_{t \to \tau} \int_a^b |\widetilde{k}_j(t,s) - \widetilde{k}_j(\tau,s)| \, \mathrm{d}s = 0.$$

In addition, we have

$$\|K_j\| = \sup_{a \le t \le b} \int_a^b |k_j(t,s)| \, \mathrm{d}s < \infty.$$

Under these hypotheses the operator

$$Ax(t) = f(t) + \lambda \sum_{j=1}^{m} \int_a^b k_j(t,s) h_j(x(s)) \, \mathrm{d}s \tag{3.5}$$

acts in the space C and satisfies the local Lipschitz condition (2) in the ball $B[0,R]$ with

$$k(r) = |\lambda| \sum_{j=1}^{m} \|K_j\| w_j(r). \tag{3.6}$$

The functions (2.3) read here

$$a_{\pm}(r) = |\lambda| \left(a \pm \int_0^r \sum_{j=1}^{m} w_j(t) \|K_j\| \, \mathrm{d}t \right).$$

Theorems 1 and 2 allow us to formulate solvability conditions for equation (3.4), to define the annular domain where this solution is situated, and to estimate the rate of convergence of successive approximations.

Now let us consider equation (3.4) in the space $L_p[a,b]$. At first glance one might think that the results obtained for the space $C[a,b]$ easily carry over to the space $L_p[a,b]$, but this is not true. The appropriate estimates can be obtained if the Lipschitz conditions for the nonlinearities $h_j(u)$ are of a special form. Moreover, these conditions are true only in the case when the nonlinearities $h_j(u)$ are defined for all $u \in \mathbb{R}$ and have power type growth with respect to the variables u.

Let us assume that there exist nonnegative constants (ξ, η) such that the inequality

$$|h_j(u_1) - h_j(u_2)| \le \left(\xi + \eta r^{\frac{p-q_j}{q_j}}\right)|u_1 - u_2| \qquad (|u_1|, |u_2| \le r,\ 0 < r < \infty) \tag{3.7}$$

is valid. Then each operator $H_j x(t) = h_j(x(t))$, $j = 1, ..., m$, acts from $L_p[a,b]$ into $L_{q_j}[a,b]$ and satisfies in each ball $B_r(L_p[a,b])$ the local Lipschitz condition

$$\begin{gathered}\|H_j(x_1) - H_j(x_2)\|_{L_{q_j}} \le \widetilde{h}_j(r)\|x_1 - x_2\|_{L_p} \\ (\|x_1\|_{L_p}, \|x_2\|_{L_p} \le r,\ 0 < r < \infty),\end{gathered} \tag{3.8}$$

where

$$\widetilde{h}_j(r) = \inf_{(\xi,\eta)\in T(H_j)} \left\{\xi(b-a)^{\frac{p-q_j}{pq_j}} + \eta r^{\frac{p-q_j}{q_j}}\right\} \tag{3.9}$$

and $T(H_j)$ denotes the set of pairs (ξ, η) satisfying (3.7).

In order to prove (3.9) it is sufficient to verify that

$$\widetilde{h}_j(r) \le \left\{\xi(b-a)^{\frac{p-q_j}{pq_j}} + \eta r^{\frac{p-q_j}{q_j}}\right\} \tag{3.10}$$

for arbitrary $(\xi, \eta) \in T(H_j)$. Observe that (3.7) implies

$$|h_j(x_1(s)) - h_j(x_2(s))| \le \left(\xi + \eta\big(\max\{|x_1(s)|, |x_2(s)|\}\big)^{\frac{p-q_j}{q_j}}\right)|x_1(s) - x_2(s)|,$$

and

$$\|H_j x_1 - H_j x_2\|_{L_{q_j}} \le \left(\xi(b-a)^{\frac{p-q_j}{pq_j}} + \eta\|\max\{|x_1|, |x_2|\}\|_{L_p}^{\frac{p-q_j}{q_j}}\right)\|x_1 - x_2\|_{L_p}, \tag{3.11}$$

for $x_1, x_2 \in L_p$. If $\|x_1\|_{L_p}, \|x_2\|_{L_p} \le r$ then $\|\max\{|x_1|, |x_2|\}\|_{L_p} \le 2^{\frac{1}{p}} r$, and the latter inequality implies only the estimate

$$\|H_j x_1 - H_j x_2\|_{L_{q_j}} \le \left(\xi(b-a)^{\frac{p-q_j}{pq_j}} + 2^{\frac{p-q_j}{pq_j}}\eta r^{\frac{p-q_j}{q_j}}\right)\|x_1 - x_2\|_{L_p}$$

which is worse than (3.10).

Nevertheless, (3.11) implies (3.10). To see this, let $\|x_1\|_{L_p}, \|x_2\|_{L_p} < r$ and $\delta > 0$ such that $\|x_1\|_{L_p}, \|x_2\|_{L_p} \le r - \delta$. Let N be an integer such that $2r < N\delta$, and put

$$\psi_j = \left(1 - \frac{n}{N}\right)x_1 + \frac{n}{N}x_2, \qquad n = 0, 1, \ldots, N.$$

Then

$$\|H_j x_1 - H_j x_2\|_{L_{q_j}} \le \sum_{n=1}^{N} \|H_j \psi_n - H_j \psi_{n-1}\|_{L_{q_j}}$$

and, by (3.11),

$$\|H_j x_1 - H_j x_2\|_{L_{q_j}} \le$$

$$\left(\frac{1}{N}\sum_{n=1}^{N}\left(\xi(b-a)^{\frac{p-q_j}{pq_j}} + \eta\|\max\{|\psi_{n-1}|, |\psi_n|\}\|_{L_p}^{\frac{p-q_j}{q_j}}\right)\right)\|x_1 - x_2\|_{L_p}.$$

Moreover, $\|\psi_{n-1} - \psi_n\|_{L_p} \le \dfrac{1}{N}\,\|x_1 - x_2\|_{L_p} \le 2r\dfrac{1}{N} < \delta$. Therefore,

$$\|\max\{|\psi_{n-1}|, |\psi_n|\}\|_{L_p} =$$

$$\||\psi_{n-1}| + \max\{0, |\psi_n| - |\psi_{n-1}|\}\|_{L_p} \le r - \delta + \||\psi_{n-1}| - |\psi_n|\|_{L_p} \le r,$$

hence

$$\|H_j x_1 - H_j x_2\|_{L_{q_j}} \le \left(\xi(b-a)^{\frac{p-q_j}{pq_j}} + \eta r^{\frac{p-q_j}{q_j}}\right)\|x_1 - x_2\|_{L_p}.$$

Thus, (3.10) holds true in the case when $\|x_1\|_{L_p}, \|x_2\|_{L_p} < r$. Passing to the limit proves the validity of (3.10) for all $\|x_1\|_{L_p}, \|x_2\|_{L_p} \le r$. In [1] a different proof of (3.10) is given under the hypothesis that (3.7) holds.

Further, let us assume that for each $j = 1, \dots, m$ the kernel $k_j(t,s)$ is measurable with respect to t, s and belongs to the Zaanen space $Z(q_j, p')$ ($p' = p/(p-1)$). Recall [4] that $Z(\alpha, \beta)$ is the space of measurable functions $z(t,s)$ with two variables $t, s \in [a,b]$ for which the integrals

$$\int_a^b \int_a^b z(t,s)x(s)y(t)\,\mathrm{d}s\mathrm{d}t, \qquad x \in L_\alpha, y \in L_\beta$$

exist; the norm in this space is defined by the formula

$$\|z\|_{Z(\alpha,\beta)} = \sup_{\|x\|_{L_\alpha}, \|y\|_{L_\beta} \le 1} \int_a^b \int_a^b |z(t,s)x(s)y(t)|\,\mathrm{d}s\,\mathrm{d}t. \tag{3.12}$$

Of course, this norm of a function $z(t,s)$ is nothing else but the norm of the linear integral operator Z with kernel $|z(t,s)|$, considered as an operator between the spaces L_α and $L_{\beta'}$, $\beta' = \beta/(\beta-1)$. Some methods for calculating or estimating this norm for various α and β may be found in [4].

Under these assumptions the operator (3.5) satisfies the local Lipschitz condition (2) in the ball $B_r(L_p[a,b])$ with

$$k(r) = \sum_{j=1}^m \widetilde{h}_j(r)\|k_j\|_{Z(q_j,p')},$$

where $\widetilde{h}_j(r)$ is defined in (3.9). Thus, the majorant functions of the operator A are defined here by the equations

$$a_\pm(r) = |\lambda| \left(a \pm \int_0^r \sum_{j=1}^m \widetilde{h}_j(\varrho)\|k_j\|_{Z(q_j,p')}\,\mathrm{d}\varrho\right).$$

Again, Theorems 1 and 2 allow us to formulate solvability conditions for equation (3.4), to define the annular domain where this solution is situated, and to estimate the rate of convergence of successive approximations.

As a third example, let us consider the nonlinear integral equation

$$x(t) = \int_a^b K(t, s, x(s), x(t))\,\mathrm{d}s, \tag{3.13}$$

where the function $K(t, s, u, v)$ is measurable with respect to the variables t, s and continuous with respect to the variables u, v, and x is the unknown function.

First we consider equation (3.13) in the space $C[a, b]$. To this end, we assume that the function K satisfies the condition

$$\bigl|K(t, s, u_1, v_1) - K(t, s, u_2, v_2)\bigr| \le l(t, s, r)\bigl|u_1 - u_2\bigr| + m(t, s, r)\bigl|v_1 - v_2\bigr|$$

$$\bigl(|u_1|, |u_2|, |v_1|, |v_2| \le r, \quad 0 < r \le \infty\bigr),$$

where $l(t, s, r)$ and $m(t, s, r)$ are nonnegative and nondecreasing functions on $[a, b] \times [a, b] \times [0, \infty)$.

Then the operator

$$Ax(t) = \int_a^b K\bigl(t, s, x(s), x(t)\bigr)\,\mathrm{d}s, \tag{3.14}$$

satisfies the local Lipschitz condition (2) in the ball $B[x_0, R]$ with

$$k(r) = \max_{a \le t \le b} \int_a^b \bigl(l(t, s, r) + m(t, s, r)\bigr)\,\mathrm{d}s.$$

In this case,

$$a_{\pm}(r) = a \pm \int_0^r \max_{a \le t \le b} \int_a^b \bigl(l(t, s, \varrho) + m(t, s, \varrho)\bigr)\,\mathrm{d}s\,\mathrm{d}\varrho.$$

Now let us consider equation (3.13) in the space $L_p[a, b]$. As in the previous example, the results do not carry over automatically from $C[a, b]$ to $L_p[a, b]$. Again, this works only if we consider nonlinearities satisfying a special Lipschitz condition. Moreover, we can treat only the case when the nonlinearity $K(t, s, u, v)$ is defined for all $u, v \in \mathbb{R}$ and has power growth with respect to the variables u and v.

Assume that

$$|K(t, s, u_1, v_1) - K(t, s, u_2, v_2)| \le$$

$$\left(\sum_{j=0}^{\mu} a_j(t, s) r^{\theta_j}\right) |u_1 - u_2| + \left(\sum_{k=0}^{\nu} b_k(t, s) r^{\vartheta_k}\right) |v_1 - v_2|$$

$$(|u_1|, |u_2| \le r, \;\; 0 = \theta_0 < \theta_1 < \ldots < \theta_\mu \le p - 1, \;\; 0 \le \vartheta_0 < \vartheta_1 < \ldots < \vartheta_\nu \le p),$$

where $a_j(t,s)$ belongs to the Zaanen space $Z(\frac{p}{1+\theta_j}, p')$ ($p' = p/(p-1)$), and $b_k(t,s)$ belongs to the Zaanen space $Z(\frac{p}{\vartheta_k}, p')$ ($p' = p/(p-1)$). This inequality implies that

$$|Ax_1(t) - Ax_2(t)| \leq \sum_{j=0}^{\mu} \int_a^b a_j(t,s) r(s)^{\theta_j} |x_1(s) - x_2(s)|\, ds +$$

$$\sum_{k=0}^{\nu} \int_a^b b_j(t,s) r(s)^{\vartheta_k}\, ds\, |x_1(t) - x_2(t)|,$$

where $r(s) = \sup\, \{|x_1(s)|, |x_2(s)|\}$. The same argument as in the previous example shows that

$$\|Ax_1 - Ax_2\|_{L_p} \leq \left(\sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{1+\theta_j}, p')} r^{\theta_j} + \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{\vartheta_k}, 1)} r^{\vartheta_k} \right) \|x_1 - x_2\|_{L_p}.$$

We conclude that the operator (3.14) satisfies the local Lipschitz condition (2) in the ball $B[0,R]$ with

$$k(r) = \sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{1+\theta_j}, p')} r^{\theta_j} + \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{\vartheta_k}, 1)} r^{\vartheta_k}.$$

Moreover, by means of the function $k(\cdot)$ we can define the functions

$$a_{\pm}(r) = a \pm \left(\sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{1+\theta_j}, p')} \frac{r^{1+\theta_j}}{1+\theta_j} + \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{\vartheta_k}, 1)} \frac{r^{1+\vartheta_k}}{1+\vartheta_k} \right).$$

Finally, our last example refers to the nonlinear integral equation

$$x(t) = F\left(t, x(t), \int_a^b K\big(t, s, x(s)\big)\, \mathrm{d}s\right), \tag{3.15}$$

where $F(t,u,v)$ is continuous with respect to the variables u, v for fixed t, and also continuous with respect to the variable t; as before, x is the unknown function.

The operator

$$Ax(t) = F\left(t, x(t), \int_a^b K\big(t, s, x(s)\big)\, \mathrm{d}s\right), \tag{3.16}$$

may be rewritten in the form

$$Ax = F(x, Bx), \tag{3.17}$$

where F is the superposition operator defined by $F(x,y)(t) = F(t, x(t), y(t))$, and

$$Bx(t) = \int_a^b K\big(t, s, x(s)\big) \mathrm{d}s. \tag{3.18}$$

As before, we study equation (3.15) first in the space $C[a,b]$. Assume that the function K satisfies the condition

$$|K(t,s,u)| \le n_0(t,s,r) \quad (|u| \le r),$$
$$|K(t,s,u_1) - K(t,s,u_2)| \le n(t,s,r)|u_1 - u_2| \quad (|u_1|,|u_2| \le r),$$
$$\big(|u_1|,|u_2| \le r,\ |v_1|,|v_2| \le \rho,\ 0 < r,\rho \le \infty\big)$$

where $n(t,s,r)$ and $n_0(t,s,r)$ are nonnegative functions on $[a,b]\times[a,b]\times[0,\infty)$ which are nondecreasing with respect to r and measurable with respect to t,s.

Then the operator B satisfies

$$|Bx(t)| \le \int_a^b n_0(t,s,\|x\|)\ \mathrm{d}s \quad (\|x\| \le r)$$

and

$$|Bx_1(t) - Bx_2(t)| \le \int_a^b n(t,s,r)\ \mathrm{d}s\, \|x_1 - x_2\|, \quad (\|x_1\|,\|x_2\| \le r).$$

Further assume that

$$\begin{gathered} |F(t,u_1,v_1) - F(t,u_2,v_2)| \le l(t,r,\rho)|u_1 - u_2| + m(t,r,\rho)|v_1 - v_2|, \\ |u_1|,|u_2| \le r,\ |v_1|,|v_2| \le \rho, \end{gathered} \tag{3.19}$$

where $l(t,r,\rho)$ and $m(t,r,\rho)$ are nonnegative functions on $[a,b]\times[0,\infty)\times[0,\infty)$ which are nondecreasing with respect to r, ρ and measurable with respect to t. Then the superposition operator $F(x,y)(t) = F(t,x(t),y(t))$ satisfies the inequality

$$|F(t,x_1,y_1) - F(t,x_2,y_2)| \le m(t,r,\rho)\|x_1 - x_2\| + n(t,r,\rho)\|y_1 - y_2\|$$
$$\|x_1\|,\|x_2\| \le r,\ \|y_1 - y_2\| \le \rho.$$

As a result, the operator A satisfies the Lipschitz condition

$$\|Ax_1 - Ax_2\| \le \sup_{a\le t\le b}\left(l\left(t,r,\int_a^b n_0(t,s,r)\mathrm{d}s\right) + m\left(t,r,\int_a^b n_0(t,s,r)\mathrm{d}s\right)\times \right.$$
$$\left. \times \int_a^b n(t,s,r)\mathrm{d}s\right)\|x_1 - x_2\|,$$

which is nothing else but the local Lipschitz condition (2) with

$$k(r) = \sup_{a\le t\le b}\left(l\left(t,r,\int_a^b n_0(t,s,r)\mathrm{d}s\right) + m\left(t,r,\int_a^b n_0(t,s,r)\mathrm{d}s\right)\times \right.$$
$$\left. \times \int_a^b n(t,s,r)\mathrm{d}s\right).$$

The majorant functions $a_{\pm}(r)$ for the operator A are defined here by

$$a_{\pm}(r) = a \pm \int_0^r \sup_{a\le t\le b} \Bigg(l\Big(t, \varrho, \int_a^b n_0(t,s,\varrho)\mathrm{d}s\Big) + m\Big(t, \varrho, \int_a^b n_0(t,s,\varrho)\mathrm{d}s\Big)\times \\ \times \int_a^b n(t,s,\varrho)\mathrm{d}s\Bigg)\mathrm{d}\varrho.$$

To conclude, let us consider equation (3.15) also in the space $L_p[a,b]$. Assume that

$$|K(t,s,u)| \le \sum_{j=0}^{\mu} a_j(t,s)|u|^{\theta_j} \qquad (|u|\le r, \quad 0\le \theta_0 < \theta_1 < \ldots < \theta_\mu \le p),$$

and

$$|K(t,s,u_1) - K(t,s,u_2)| \le \sum_{k=0}^{\nu} b_k(t,s) r^{\vartheta_k}\, |u_1 - u_2| \\ (|u_1|, |u_2| \le r, \quad 0 \le \vartheta_0 < \vartheta_1 < \ldots < \vartheta_\nu \le p-1),$$

where $a_j \in Z(\frac{p}{\theta_j}, q')$, $b_k \in Z(\frac{p}{1+\vartheta_k}, q')$. Then

$$\|Kx\|_{L_q} \le \sum_{j=0}^{\mu} \Big\| \int_\Omega a_j(t,s)|x(s)|^{\theta_j} ds \Big\|_{L_q} \le \sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{\theta_j},q')} r^{\theta_j}. \tag{3.20}$$

and

$$\|Kx_1 - Kx_2\|_{L_q} \le \sum_{k=0}^{\nu} \Big\| \int_a^b b_k(t,s) r^{\vartheta_k}\, |x_1(s) - x_2(s)| ds \Big\|_{L_q} \\ \le \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k},q')} r^{\vartheta_k}\, \|x_1 - x_2\|_{L_p} \tag{3.21}$$

where we use the argument from the second example in the proof of (3.21). Furthermore, assume that

$$|F(t,u_1,v_1) - F(t,u_2,v_2)| \le c|u_1 - u_2| + \big(\mu(t) + \nu\rho^{\frac{q-p}{p}}\big)|v_1 - v_2|, \\ |v_1|, |v_2| \le \rho,\ 0 < \rho < \infty,\ \mu \in L_{\frac{qp}{q-p}}. \tag{3.22}$$

Then

$$\|F(x_1,y_1) - F(x_2,y_2)\|_{L_p} \le c\|x_1 - x_2\|_{L_p} + \big(\|\mu\|_{L_{\frac{qp}{q-p}}} + \nu\rho^{\frac{q-p}{p}}\big)\|y_1 - y_2\|_{L_q}$$

where in the proof of this inequality the argument used from the second example is again applied. Furthermore,

$$\|F(x_1,y_1) - F(x_2,y_2)\|_{L_p} \le \\ c\|x_1 - x_2\|_{L_p} + \inf_{(\mu,\nu)\in T(F)} \ \big(\|\mu\|_{L_{\frac{qp}{q-p}}} + \nu\rho^{\frac{q-p}{p}}\big)\|y_1 - y_2\|_{L_q},$$

where $T(F)$ denotes the set of pairs (μ,ν) for which inequality (3.22) holds.

Summing up all these inequalities we get

$$\|Ax_1 - Ax_2\|_{L_p} \le$$

$$\left(c + \inf_{(\mu,\nu)\in T(F)} \left(\|\mu\|_{L_{\frac{qp}{q-p}}} + \nu\left(\sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{\theta_j},q')} r^{\theta_j}\right)^{\frac{q-p}{p}}\right)\times\right.$$
$$\left.\times \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k},q')} r^{\vartheta_k}\right) \|x_1 - x_2\|_{L_p}.$$

As a result, we obtain the expression

$$k(r) = c + \inf\nolimits_{(\mu,\nu)\in T(F)} \left(\|\mu\|_{L_{\frac{qp}{q-p}}} + \nu\left(\sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{\theta_j},q')} r^{\theta_j}\right)^{\frac{q-p}{p}}\right)\times$$
$$\times \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k},q')} r^{\vartheta_k}$$

for the Lipschitz constant $k(r)$ and the expression

$$a_\pm(r) = a \pm \left(cr + \int_0^r \inf_{(\mu,\nu)\in T(F)} \left(\|\mu\|_{L_{\frac{qp}{q-p}}} + \nu\left(\sum_{j=0}^{\mu} \|a_j\|_{Z(\frac{p}{\theta_j},q')} \varrho^{\theta_j}\right)^{\frac{q-p}{p}}\right.\right.\times$$
$$\left.\left.\times \sum_{k=0}^{\nu} \|b_k\|_{Z(\frac{p}{1+\vartheta_k},q')} \rho^{\vartheta_k}\right) \mathrm{d}\varrho\right).$$

for the majorant functions $a_\pm(r)$ of the operator A.

4. Conclusion

The examples considered above easily carry over to the more general setting of nonlinear operator equations with unknown functions defined on a measurable space Ω with σ-finite measure and taking values in a finite dimensional space. Our reasoning reflects the fact that different solvability and uniqueness results can be essentially strengthened by a closer scrutiny of Lipschitz conditions. Observe that in the case when $k(r)$ does not depend on r the fixed point principle for majorizing functions reduces to the Banach – Caccioppoli principle. On the other hand, an analogue of the fixed point principle with majorizing functions is not valid for operators in arbitrary complete metric spaces.

References

[1] J. Appell, P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge 1990.

[2] J.A. Ezquerro, M.A. Hernandez M.A., *Picard's iterations for integral equations of mixed Hammerstein type*, Canad. Math. Bull., **51(3)**(2008), 372-377.

[3] L.V. Kantorovich, G.P. Akilov, *Functional Analysis* [in Russian], Nauka, Moscow 1984.

[4] M.A. Krasnoselskiĭ, P.P. Zabreiko, E.I. Pustylnik, P.E. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leyden 1976.

[5] P.G. Lemarié-Rieusset, *Point fixe d'une application non contractante*, Rev. Mat. Iberoam., **22**(2006), 339-356.

[6] P.P. Zabreiko, *Some elementary fixed point theory*, in: Kilbas A.A., Rogosin S.V. (ed.), Analytic Methods of Analysis and Differential Equations, Cambridge Scientific Publishers, 2006, 255-272.

[7] P.P. Zabreiko, Yu.V. Korots, *Analysis of implicit successive approximations* [in Russian], Dokl. Belor. Nat. Acad. Sci., **6**(2009), 32-38.

[8] P.P. Zabreiko, Yu.V. Korots, *The principle of majorized mappings for mixed Hammerstein integral equations* [in Russian], Dokl. Belor. Nat. Acad. Sci., **4**(2010), 5-12.

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