Fixed Point Theory, 13(2012), No. 2, 537-546
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

BEST PROXIMITY POINT THEOREMS FOR KT-TYPES CYCLIC ORBITAL CONTRACTION MAPPINGS

ERDAL KARAPINAR*, GABRIELA PETRUŞEL**,[1] AND KENAN TAS***

*Department of Mathematics, Atilim University 06836,
Incek, Ankara, Turkey
E-mail: erdalkarapinar@yahoo.com

**Babes-Bolyai University Cluj-Napoca, Department of Business,
400174, Cluj-Napoca, Romania
E-mail: gabi.petrusel@tbs.ubbcluj.ro

***Department of Mathematics and Computer Science,
Cankaya University 06530, Yuzuncuyil, Ankara, Turkey
E-mail: kenan@cankaya.edu.tr

Abstract. In this manuscript, three new KT-types cyclic orbital contractions are defined and some related best proximity point theorems are given. Also, the notion of KT-type cyclic orbital Meir-Keeler contraction is defined and some fixed point theorems for this class of mappings are proved. The results of this manuscript generalize some theorems, on the same subject, of several authors, such as Kirk-Srinavasan-Veeramani, Eldered-Veeramani and Karpagam-Agrawal.
Key Words and Phrases: Cyclic contraction, best proximity points, KTi-types cyclic orbital contractions, cyclic orbital Meir-Keeler contraction.
2010 Mathematics Subject Classification: 47H10, 46T99, 54H25.

1. Introduction and preliminaries

Let A and B be two non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a mapping. A self-mapping T is called a *cyclic map* if $T(A) \subseteq B$ and $T(B) \subseteq A$. We denote the set of all fixed points of T $\{x^* \in A \cup B : x^* = Tx^*\}$, by $Fix(T)$.

A point $x \in A \cup B$ is called a *best proximity point of* T if $d(x, Tx) = d(A, B)$ where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. It is clear that a fixed point $z \in A \cup B$ of a cyclic map T is a best proximity point of T if the sets A and B have a non-empty intersection.

The notions of cyclic contraction and best proximity points were introduced and studied by Kirk-Srinavasan-Veeramani in [6]. Recently, many authors have focused on these topics, (see for instance [2, 3, 8, 1, 9, 4, 10, 5, 7] and the reference therein).

In 2003, Kirk-Srinavasan-Veeramani [6] proved the following fixed point theorem as a generalization of Banach contraction principle:

[1]Corresponding author.

Theorem 1.1. *Let A and B be two non-empty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a map satisfying $T(A) \subseteq B$ and $T(B) \subseteq A$ and there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

In this manuscript, new cyclic orbital contractions called KT-types are defined and some related best proximity point theorems are given about them. Also, the notion of KT-type cyclic orbital Meir-Keeler contraction is defined and some fixed point theorems are proved.

2. KT-types cyclic orbital contractions

Definition 2.1. (See [4]) Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a *cyclic orbital contraction* if for some $x \in A$ there exists a $k_x \in (0, 1)$ such that

$$d(T^{2n}x, Ty) \leq k_x d(T^{2n-1}x, y) \tag{2.1}$$

for all $n \in \mathbb{N}$ and $y \in A$.

We generalize the definition above as follows:

Definition 2.2. Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic map. If there is $x \in A$ and there exists a $k_x \in (0, \frac{1}{2})$ such that either

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, T^{2n-2}x) + d(Ty, y)] \text{ or} \tag{2.2}$$

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(Ty, y)] \text{ or} \tag{2.3}$$

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n-2}x, Ty)] \tag{2.4}$$

holds for all $n \in \mathbb{N}$ and $y \in A$, then T is said to be a cyclic orbital contraction of type KT_1, KT_2, KT_3, respectively.

Theorem 2.3. *(See [4]) Let A and B be two non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then $A \cap B$ is non-empty and T has a unique fixed point.*

Inspired by Theorem 2.3 we will prove now the following theorem.

Theorem 2.4. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a KT_1-type cyclic orbital contraction. Then $A \cap B$ is non-empty and T has a unique fixed point.*

Proof. Assume that there exists $x \in A$ satisfying (2.2). Then taking x instead of y, we have

$$d(T^2x, Tx) \leq k_x[d(Tx, x) + d(Tx, x)]$$

and so

$$d(T^2x, Tx) \leq t_x d(Tx, x), \text{ where } t_x = 2k_x \in (0, 1).$$

Similarly we have

$$d(T^3x, T^2x) = d(T^2(\underbrace{Tx}_{u}), T(\underbrace{Tx}_{u})) \leq t_u d(Tu, u) \leq (t_u)(t_x)d(Tx, x),$$

where $t_u \in (0,1)$ and

$$d(T^4x, T^3x) = d(T^3(\underbrace{Tx}_{u}), T^2(\underbrace{Tx}_{u})) \leq (t_u)(t_u)d(Tu,u) \leq (t_u)^2(t_x)d(Tx,x).$$

Therefore for any $n \in \mathbb{N}$, we have

$$d(T^{n+1}x, T^nx) \leq (t_u)^{n-1}(t_x)d(Tx,x),\ t_x \in (0,1), n \in \mathbb{N}.$$

Consequently,

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^nx) \leq \sum_{n=1}^{\infty}(t_u)^{n-1}(t_x).d(Tx,x) < \infty,\ t_x \in (0,1), n \in \mathbb{N}.$$

Thus, $\{T^nx\}$ is a Cauchy sequence. Hence, there exists a $z \in A \cup B$ such that $T^nx \to z$. Notice that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B. Both sequences tend to the same limit z. Since A and B are closed, we conclude $z \in A \cap B$. Hence, $A \cap B \neq \emptyset$.
We claim that $Tz = z$. Since $d(T^{2n}x, Tz) \leq k_x.[d(T^{2n-1}x, T^{2n-2}x) + d(Tz, z)]$, then taking the limit we obtain $d(z, Tz) \leq k_x.[d(z,z) + d(Tz,z)]$. Using that $k_x \in (0, \frac{1}{2})$, we get $d(Tz, z) = 0$ and thus $Tz = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cup B$ such that $z \neq w$ and $Tw = w$. Since T is a cyclic map, we get $w \in A \cap B$. So, $d(z,w) = d(Tz,Tw) = d(T(Tz),Tw) = d(T^2z,Tw) \leq k_x[d(Tz,z) + d(Tw,w)] = 0$, which concludes that $z = w$. Hence z is the unique fixed point of T. □

We will present now a data dependence result for the fixed points of a KT_1-type cyclic orbital contraction. Usually, the data dependence phenomena holds if for two "very closed" operators, the fixed points of it are not "too far" one from the other. In our case, a term depending on the "special" point $x \in X$ appears.

Theorem 2.5. *Let A and B be non-empty closed subsets of a complete metric space (X,d) and $T, S : A \cup B \to A \cup B$ such that:*

(i) T is a KT_1-type cyclic orbital contraction (with constant k_x^*);

(ii) there exists $x_S^* \in Fix(S) \cap A$;

(iii) there exist $\eta > 0$ such that $d(Tx, Sx) \leq \eta$, for each $x \in A \cap B$.

Then $d(x_T^*, x_S^*) \leq 2k_x^* d(Tx,x) + (1 + k_x^*)\eta$, where x_T^* is the unique fixed point of T.

Proof. By Theorem 2.4 we know that $Fix(T) = \{x_T^*\}$. Since there exists $x \in A$ satisfying (2.2), we have:

$$\begin{aligned} d(x_T^*, x_S^*) &\leq d(T^2x, x_T^*) + d(T^2x, x_S^*) = d(T^2x, Tx_T^*) + d(T^2x, x_S^*) \\ &\leq d(T^2x, Tx_T^*) + d(T^2x, Tx_S^*) + d(Tx_S^*, S(x_S^*)) \\ &\leq k_x^*\left(d(Tx,x) + d(Tx_T^*, x_T^*)\right) + k_x^*\left(d(Tx,x) + d(Tx_S^*, x_S^*)\right) + \eta \\ &= 2k_x^* d(Tx,x) + (1 + k_x^*)\eta. \end{aligned}$$

Thus

$$d(x_T^*, x_S^*) \leq 2k_x^* d(Tx,x) + (1 + k_x^*)\eta.$$

□

Notice that, if $x \in X$ is a fixed point for T, then we obtain the classical data dependence of the fixed points, i.e.,

$$d(x_T^*, x_S^*) \leq (1 + k_x^*)\eta.$$

We present now a well-posedness result for the fixed point problem related to a KT_1-type cyclic orbital contraction. Again, a term involving $d(Tx, x)$ (where $x \in X$ is the special point from the definition of a KT_1 mapping) appears.

Theorem 2.6. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ such that:*

(i) *T is a KT_1-type cyclic orbital contraction (with some constant k_x^*, where $x \in X$);*

(ii) *there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $d(x_n, Tx_n) \to 0$ as $n \to +\infty$.*

Then $d(x_n, x_T^*) \to 2k_x^* d(Tx, x)$ as $n \to +\infty$ (where x_T^* is the unique fixed point of T).

Proof. Let $\{x_n\} \subset A$ be such that $d(x_n, Tx_n) \to 0$ as $n \to +\infty$. Consider again the point $x \in A$ satisfying (2.2). Then we have:

$$\begin{aligned} d(x_n, x_T^*) &\leq d(x_n, Tx_n) + d(Tx_n, T^2x) + d(T^2x, x_T^*) \\ &= d(x_n, Tx_n) + d(T^2x, Tx_n) + d(T^2x, Tx_T^*) \\ &\leq d(x_n, Tx_n) + k_x^* \left(d(Tx, x) + d(Tx_n, x_n)\right) + k_x^* \left(d(Tx, x) + d(Tx_T^*, x_T^*)\right) \\ &= (1 + k_x^*) d(x_n, Tx_n) + 2k_x^* d(Tx, x) \to 2k_x^* d(Tx, x) \text{ as } n \to +\infty. \end{aligned}$$

□

Note that, as before, if $x \in X$ is a fixed point for T, then we obtain the usual well-posedness results for the fixed point of T, i.e., $d(x_n, x_T^*) \to 0$ as $n \to +\infty$.

Remark 2.7. If $T : A \cup B \to A \cup B$ is a KT_2-*type cyclic orbital contraction* or a KT_3-*type cyclic orbital contraction*, then similar existence, uniqueness, data dependence and well-posedness results for the fixed point problem can be analogously obtained.

Let us illustrate the application of Theorem 2.4.

Example 2.8. Consider the usual metric space $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. Let $A = [-1, 0]$ and $B = [0, 1]$ be subsets of $\mathbb{R}$. Define

$$Tx = \begin{cases} -\frac{x}{2} & \text{if } x \in A \\ -x & \text{if } x \in B \end{cases}$$

Then, it is clear that $T(A) \subseteq B$ and $T(B) \subseteq A$. On the other hand, $T^{2n}x = -\frac{x}{2^n}$ and $T^{2n-1}x = \frac{x}{2^n}$, for every $x \in A$.
Therefore, for every $y \in [0, 1], Ty = -y$. Thus, $d(T^{2n}x, Ty) = |-\frac{x}{2^n} + y|$ and $d(T^{2n-1}x, y) = |\frac{x}{2^{n-1}} - y| = d(T^{2n}x, Ty)$. There is no $k_x \in (0, 1)$ in such a way that T is a cyclic orbital contraction and thus Theorem 2.3 applies. On the other hand, we have $d(T^{2n-1}x, x) = |\frac{x}{2^n} + x|, d(T^{2n-1}x, y) = |\frac{x}{2^n} + y|, d(Ty, y) = 2|y|$, therefore the KT_i-type cyclic orbital contraction conditions ($i = 1, 2, 3$) are satisfied for some $x \in A$ and $k_x \in (0, \frac{1}{2})$. Therefore by Theorem 2.4, T has a unique fixed point, which is $x = 0$.

Corollary 2.9. *Let T be a self map on a complete metric space (X, d). If for some $x \in X$, there exists a $k_x \in (0, \frac{1}{2})$ satisfying one of the following conditions*

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, T^{2n-2}x) + d(Ty, y)], n \in I\!N; y \in X \tag{2.5}$$

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(Ty, y)], n \in I\!N; y \in X \tag{2.6}$$

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n-1}x, y) + d(T^{2n-2}x, Ty)]; n \in I\!N; y \in X \tag{2.7}$$

then, T has a unique fixed point.

Remark 2.10. Notice that the statement (2.1) in Definition 2.1 could not be generalized to the following condition:

$$d(T^{2n}x, Ty) \leq k_x[d(T^{2n}x, y) + d(Ty, x)]; n \in \mathbb{N}; y \in A \tag{2.8}$$

since both $T^{2n}x$ and y lies in A, the statement (2.8) fails to be cyclic. To avoid such cases, throughout this manuscript we define and use the notion of "opposite parity" as follows:

$p, q \in \mathbb{N}$ are opposite parity if either $T^p x \in A$, $T^q x \in B$ or $T^p x \in B$, $T^q x \in A$ holds.

3. Cyclic Meir-Keeler Contractions

Definition 3.1. (See [4]) Let (X, d) be a metric space, and A and B be non-empty subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that, for some $x \in A$, and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d(T^{2n-1}x, y) < d(A, B) + \varepsilon + \delta \ \text{ implies } \ d(T^{2n}x, Ty) < d(A, B) + \varepsilon, n \in \mathbb{N}, y \in A. \tag{3.1}$$

Then T is said to be a cyclic orbital Meir-Keeler contraction.

Definition 3.2. Let (X, d) be a metric space, and A and B be two non-empty subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that, for some $x \in A$, and for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} K(T^{2n-1}x, y) &< d(A, B) + \varepsilon + \delta \\ \text{implies } \ d(T^{2n}x, Ty) &< d(A, B) + \varepsilon, n \in \mathbb{N}, y \in A \end{aligned} \tag{3.2}$$

where $K(T^{2n-1}x, y) = \frac{1}{2}[d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$. Then T is said to be a KT_1-type cyclic orbital Meir-Keeler contraction.

Proposition 3.3. *Let A and B be two non-empty and closed subsets of a metric space X. Suppose $T : A \cup B \to A \cup B$ is a KT_1-type cyclic orbital Meir-Keeler contraction. If $x \in A$ satisfies condition (3.2) then $d(T^{n+1}x, T^n x) \to d(A, B)$, as $n \to \infty$.*

Proof. Suppose T is KT_1-type cyclic orbital Meir-Keeler contraction. Take $x \in A$ for which (3.2) is satisfied. Since either n or $n+1$ is even, then we have $\frac{1}{2}[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] \geq d(A, B)$. Consider the case

$$\frac{1}{2}[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] = d(A, B).$$

Then due to (3.2) we have

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon$$

which is equivalent to

$$d(T^{n+1}x, T^n x) < \frac{1}{2}[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] + \varepsilon.$$

Thus we have

$$d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x), \quad \text{as} \;\; \varepsilon \to 0.$$

Now, consider the other case:

$$\frac{1}{2}[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] > d(A, B).$$

Set $\varepsilon_1 = \frac{1}{2}[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] - d(A, B) > 0$. Due to (3.2), for this ε_1, there exists a δ such that

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon_1 = \frac{1}{2}[d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)].$$

Hence, $d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x)$ for all $n \in \mathbb{N}$. Let $d_n = d(T^{n+1}x, T^n x)$. Clearly $\{d_n\}$ is a non-increasing sequence which is bounded below by $d(A, B)$. Therefore $\{d_n\}$ converges to some d with $d \geq d(A, B)$. We assert that $d = d(A, B)$. Suppose not, that is, $d > d(A, B)$. Set $\varepsilon = d - d(A, B) > 0$. Thus, there exists a $\delta > 0$ which satisfies (3.2). Regarding $\{d(T^{n+1}x, T^n x)\} \to d$, there exist a $n_0 \in \mathbb{N}$ such that

$$d \leq \frac{1}{2}[d(T^{n+2}x, T^{n+1}x) + d(T^{n+1}x, T^n x)] < s + \delta = \varepsilon + d(A, B) + \delta, \quad \forall n \geq n_0.$$

Thus,

$$d(T^{n+2}x, T^{n+1}x) < d(A, B) + \varepsilon = s, \; \forall n \geq n_0$$

which is a contradiction. Hence $d = d(A, B)$. □

Proposition 3.4. *Let A and B be two non-empty and closed subsets of a metric space X and $T : A \cup B \to A \cup B$ is a KT_1-type cyclic orbital Meir-Keeler contraction. Suppose $d(A, B) = 0$. Then, for each $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and $\delta > 0$ such that*

$$d(T^p x, T^q x) < \varepsilon + \delta \quad \textit{implies that} \;\; d(T^{p+1}x, T^{q+1}x) < \varepsilon \tag{3.3}$$

with $p, q \geq n_1$.

Proof. Take $x \in X$ for which (3.2) is satisfied. Since T is a KT_1-type cyclic orbital Meir-Keeler contraction, for a given $\varepsilon > 0$, there exists $\delta > 0$ satisfies (3.2). That is,

$$\begin{array}{l} \frac{1}{2}[d(T^{2n}x, T^{2n-1}x) + d(Ty, y)] < \varepsilon + \delta \\ \text{implies} \;\; d(T^{2n}x, Ty) < \varepsilon, n \in \mathbb{N}, y \in A \end{array} \tag{3.4}$$

Regarding $d(A, B) = 0$ and Proposition 3.3, one can choose $n_1 \in \mathbb{N}$ in a way that

$$d(T^n x, T^{n+1}x) < \frac{\varepsilon + \delta}{2}, \quad \text{for each} \;\; n \geq n_1. \tag{3.5}$$

We shall show that $d(T^p x, T^q x) < \varepsilon + \delta$ implies that $d(T^{p+1}x, T^{q+1}x) < \varepsilon$. Fix $n \geq n_1$. Take $p, q \in \mathbb{N}$ which are opposite parity with $p, q \geq n_1$. Suppose that $d(T^p x, T^q x) < \varepsilon + \delta$. Without loss of generality we may assume $T^p x \in A$ and $T^q x \in B$ with $p = 2n$ and $q = 2m - 1$. Otherwise, revise the indices respectively.

Thus we have $d(T^px, T^qx) = d(T^{2n}x, T^{2m-1}x) < \varepsilon + \delta$, for $m \geq n$. Then, regarding (3.5) we get

$$\frac{1}{2}[d(T^{2m}x, T^{2m-1}x) + d(T^{2n+1}x, T^{2n}x)] \leq \frac{\varepsilon + \delta}{2} + \frac{\varepsilon + \delta}{2} < \varepsilon + \delta. \tag{3.6}$$

Consider (3.4) under the assumption $y = T^{2n}x$, the inequality (3.6) yields that

$$d(T^{2n+1}x, T^{2m}x) = d(T^{p+1}x, T^{q+1}x) < \varepsilon.$$

Thus, we observe that for a given $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and a $\delta > 0$ such that

$$d(T^px, T^qx) < \varepsilon + \delta \;\text{ implies that }\; d(T^{p+1}x, T^{q+1}x) < \varepsilon \tag{3.7}$$

where p and q are opposite parity, with $p, q \geq n_1$. □

Lemma 3.5. *Let X be a complete metric space, A and B non-empty, closed subsets of X such that $d(A, B) = 0$. Suppose $T : A \cup B \to A \cup B$ be a KT_1-type cyclic orbital Meir-Keeler contraction and $d(A, B) = 0$. Then*

$$d(T^{2n}x, Ty) < K(T^{2n-1}x, y) \;\; \textit{if} \;\; T^{2n-1}x \neq y. \tag{3.8}$$

Proof. To get (3.8), it is sufficient to show that (3.2) is equivalent to the following condition: For each $\varepsilon > 0$ there exists δ such that

$$\begin{array}{c} \varepsilon \leq K(T^{2n-1}x, y) < \varepsilon + \delta \\ \text{implies } \; d(T^{2n}x, Ty) < \varepsilon, n \in \mathbb{N}, y \in A \end{array} \tag{3.9}$$

where $K(T^{2n-1}x, y) = \frac{1}{2}[d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$. It is clear that (3.2) implies (3.9). For the converse, suppose (3.9) holds. Fix $T^{2n-1}x, y \in A \cup B$ and $\varepsilon > 0$. If $K(T^{2n-1}x, y) < \varepsilon$, since (3.9) we have $d(T^{2n}x, Ty) \leq K(T^{2n-1}x, y)$ and consequently $d(T^{2n}x, Ty) < \varepsilon$. If $K(T^{2n-1}x, y) \geq \varepsilon$, then immediately (3.2) holds. Thus, (3.9) and (3.2) are equivalent under the condition $d(A, B) = 0$. Now, we show that if (3.9) holds then we have $d(T^{2n}x, Ty) \leq K(T^{2n-1}x, y)$. If $K(T^{2n-1}x, y) = 0$ then $T^{2n-1}x = T^{2n}x$ and $Ty = y$. Hence, $d(T^{2n}x, Ty) \leq K(T^{2n-1}x, y)$. Suppose $K(T^{2n-1}x, y) \neq 0$ and fix $\varepsilon \leq K(T^{2n-1}x, y)$. Choose a $\delta > 0$ such that (3.9) holds. Notice that if $K(T^{2n-1}x, y) \leq d(T^{2n}x, Ty)$, we get a contradiction with (3.9). □

Theorem 3.6. *Let X be a complete metric space, A and B non-empty, closed subsets of X such that $d(A, B) = 0$. Suppose $T : A \cup B \to A \cup B$ be a KT_1-type cyclic orbital Meir-Keeler contraction. Then, there exists a fixed point, say $z \in A \cap B$, such that for each $x \in A$ satisfying (3.2), the sequence $\{T^{2n}x\}$ converges to z.*

Proof. Take $x \in A$. We will show that $\{T^nx\}$ is a Cauchy sequence. Suppose not. Then there exists an $\varepsilon > 0$ and a subsequence $\{T^{n(i)}\}$ of $\{T^nx\}$ with

$$d(T^{n(i)}x, T^{n(i+1)}x) > 2\varepsilon. \tag{3.10}$$

Since T is a KT_1-type cyclic orbital Meir-Keeler contraction, for this ε, there exists $\delta > 0$ such that

$$K(T^{2n-1}x, y) < \varepsilon + \delta \;\text{ implies that }\; d(T^{2n}x, Ty) < \varepsilon \tag{3.11}$$

where $K(T^{2n-1}x, y) = \frac{1}{2}[d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$. Set $r = \min\{\varepsilon, \delta\}$ and $d_m = d(T^m x, T^{m+1}x)$. Due to Proposition 3.3, one can choose $n_0 \in \mathbb{N}$ such that

$$d_m = d(T^m x, T^{m+1}x) < \frac{r}{4}, \quad \text{for} \quad m \geq n_0. \tag{3.12}$$

Let $n(i) \geq N$. Suppose $d(T^{n(i)}x, T^{n(i+1)-1}x) \leq \varepsilon + \frac{r}{2}$. Then the triangle inequality implies that

$$\begin{aligned} d(T^{n(i)}x, T^{n(i+1)}x) & \leq d(T^{n(i+1)}x, T^{n(i)-1}x) + d(T^{n(i+1)-1}x, T^{n(i+1)}x) \\ & \varepsilon + \tfrac{r}{2} + d_{n(i+1)-1} < 2\varepsilon \end{aligned} \tag{3.13}$$

which contradict the assumption (3.10). Thus, there are values of k with $n(i) \leq k \leq n(i+1)$ such that $d(T^{n(i)}, T^k x) > \varepsilon + \frac{r}{2}$ where k and $n(i)$ are opposite parity. Assume that $d(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2}$. Then

$$d_{n(i)} = d(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2} > r + \frac{r}{2} > \frac{r}{4}$$

which is a contradiction with (3.12). Hence, there are values of k with $n(i) \leq k \leq n(i+1)$ such that $d(T^{n(i)}, T^k x) < \varepsilon + \frac{r}{2}$ where k and $n(i)$ are opposite parity. Choose smallest integer k with $k \geq n(i)$ such that $d(T^{n(i)}x, T^k x) \geq \varepsilon + \frac{r}{2}$. Therefore,

$$d(T^{n(i)}x, T^{k-1}x) < \varepsilon + \frac{r}{2}. \tag{3.14}$$

Thus,

$$d(T^{n(i)}x, T^k x) \leq d(T^{n(i)}x, T^{k-1}x) + d(T^{k-1}x, T^k x) < \varepsilon + \frac{r}{2} + \frac{r}{4} = \varepsilon + \frac{3r}{4}. \tag{3.15}$$

Then there exists an integer k satisfying $n(i) \leq k \leq n(i+1)$ such that

$$\varepsilon + \frac{r}{2} \leq d(T^{n(i)}x, T^k x) < \varepsilon + \frac{3r}{4}. \tag{3.16}$$

Due to the facts

$$d(T^{n(i)}x, T^{n(i)+1}x) = d_{n(i)} < \frac{r}{4} < \varepsilon + r$$

$$d(T^k, T^{k+1}x) = d_k < \frac{r}{4} < \varepsilon + r$$

we have

$$\begin{aligned} K(T^{n(i)}x, T^k x) &= \tfrac{1}{2}[d(T^{n(i)}x, T^{n(i)+1}x) + d(T^{k+1}x, T^k x)] \\ &\leq \tfrac{1}{2}[\varepsilon + r + \varepsilon + r] = \varepsilon + r \end{aligned} \tag{3.17}$$

which implies $d(T^{n(i)+1}, T^{k+1}x) < \varepsilon$. But,

$$\begin{aligned} d(T^{n(i)+1}x, T^{k+1}x) &\geq d(T^{n(i)}x, T^k x) - d(T^{n(i)}x, T^{n(i)+1}x) - d(T^k x, T^{k+1}x) \\ &> \varepsilon + \tfrac{r}{2} - \tfrac{r}{4} - \tfrac{r}{4} = \varepsilon \end{aligned}$$

which contradicts the above inequality.

Hence $\{T^n x\}$ is a Cauchy sequence. Thus $\{T^n x\}$ converges to some $z \in A \cup B$. So, both $\{T^{2n}x\}$ and $\{T^{2n-1}x\}$ tends to same point $z \in A \cup B$. Since $\{T^{2n-1}x\}$ is a sequence in B, it converges to $z \in B$. Analogously, $\{T^{2n}x\}$ is a sequence in A which converges to $z \in A$. Taking into account that both A and B are closed, we get $z \in A \cap B$.

Let us show $Tz = z$. Taking into account of Lemma 3.5, we have

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(T^{2n}x, Tz) < K(T^{2n-1}x, z). \\ &= \lim_{n\to\infty} \tfrac{1}{2}[d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \end{aligned}$$

which implies that

$$d(Tz, z) < \frac{1}{2} d(Tz, z).$$

This is a contradiction and hence $Tz = z$.

Lastly, we show z is the unique fixed point of T. Suppose not, so there exists a point $w \in A \cap B$ such that $z \neq w$ and $Tw = w$. Hence, by Lemma 3.5, we get

$$\begin{aligned} d(w, z) &= d(Tw, z) = \lim_{n\to\infty} d(T^{2n}x, Tw) < \lim_{n\to\infty} K(T^{2n-1}x, w) \\ &\leq \lim_{n\to\infty} \tfrac{1}{2}[d(T^{2n}x, T^{2n-1}x) + d(Tw, w)] \\ &\leq \tfrac{1}{2}[d(z, z) + d(Tw, w)] = \tfrac{1}{3} d(z, w) \end{aligned}$$

which is a contradiction. Hence, $z = w$. □

Remark 3.7. One can easily state and prove similar KT_2-type and KT_3-type cyclic orbital Meir-Keeler contraction theorems.

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Received: October 12, 2011; Accepted: December 11, 2011.