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GENERALIZED QUASICONTRACTIONS IN ORBITALLY COMPLETE ABSTRACT METRIC SPACES

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Abstract. Some fixed point results for generalized quasicontractions in abstract metric spaces obtained recently are extended to the case of a pair of mappings. Also, it is shown that in some cases regularity and/or normality condition for the underlying cone can be dropped. Examples are given to illustrate the results.

Key Words and Phrases: normal cone; regular cone; abstract metric space; common fixed point; quasicontraction; orbit.
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1. Introduction

K-metric and K-normed spaces were introduced in the mid-20th century (see, e.g., [3]–[18]) by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. L.G. Huang and X. Zhang [4] re-introduced such spaces under the name of cone metric spaces, but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. Thus, nonnormal cones can be used as well, paying attention to the fact that Sandwich theorem and continuity of the metric may not hold.

Quasicontractions in metric spaces were first used in [5] and [6] in order to obtain fixed point and common fixed point results. When abstract metric spaces are concerned, they appear, e.g., in [8, 13], and, in generalized version (but in the special case when $S = I_X$) in [17]. In the last mentioned paper several fixed point results were obtained under the assumption that the positive cone P is regular or, at least, normal.

The aim of this paper is to extend these results and to obtain common fixed points theorems for pairs of mappings under appropriate generalized commutativity conditions. Also, we show that regularity and/or normality condition may be dropped in some cases. Examples are given to illustrate the results.

We note that it was shown in the recent papers [9]–[12] that some fixed point results for mappings satisfying linear contractive conditions in abstract metric spaces can be directly obtained from their metric counterparts. The results of the present paper do not fall into this category, since some of them are new even in the context of metric spaces and they are concerned with nonlinear contractive conditions.

2. Preliminaries

We need the following definitions and results, consistent with [19, 21, 4, 7].

Let E be a real Banach space with the zero vector θ. A subset P of E is called a *cone* if: (a) P is closed, non-empty and $P \neq \{\theta\}$; (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$; (c) $P \cap (-P) = \{\theta\}$.

Given a cone P, we define the partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ stands for the interior of P and use $x \prec y$ for $x \preceq y$ and $x \neq y$. If $\operatorname{int} P \neq \emptyset$, then P is called a *solid cone* [19].

The cone P in E is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$ (the minimal such constant K is called the normal constant of P). Equivalently, the cone P is normal if

$$(\forall n)\ x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1}$$

For details see [7].

The cone P in E is called *regular* if every increasing sequence in E which is bounded from above is convergent. Equivalently, the cone P is regular if every decreasing sequence in E which is bounded from below is convergent. Every regular cone is normal [7], but the converse is not true.

Example 2.1. [19] 1° Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0\}$. Consider sequences $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1) that P is a nonnormal cone.

2° Let $E = C_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty$ and P be as in the previous example. Then P is a normal cone, but it is not regular. Indeed, let $x_n(t) = -t^n$; then the sequence (x_n) is increasing and bounded from above but $\|x_n\| = 1$ for all n, so $\lim_{n\to\infty} x_n$ does not exist.

Definition 2.2. [21, 4] Let X be a non-empty set and E a Banach space with a cone P. Suppose that a mapping $d : X \times X \to E$ satisfies:

(d$_1$) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d$_2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d$_3$) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The function d is called an *abstract metric* and (X, d) is called an *abstract metric space* (or a K-metric space [21], or a cone metric space [4]); we shall use the first mentioned term.

For examples of abstract metric spaces and definitions of basic notions, in particular convergent and Cauchy sequences, completeness etc., we refer to [21, 4]. The following remark will be useful.

Remark 2.3. 1° If $u \preceq v$ and $v \ll w$, then $u \ll w$.

2° If $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.

3° If $c \in \operatorname{int} P$, $\theta \preceq x_n$ and $x_n \to \theta$, then there exists a positive integer n_0 such that $x_n \ll c$ for all $n > n_0$. (Note that the converse is not true if P is nonnormal. Indeed, in Example 2.1.1°, $x_n \not\to \theta$, but $x_n \ll c$ for n sufficiently large.)

If (T, S) is a pair of self-maps on the abstract metric space (X, d) then its well known properties, such as R-weak-commutativity [15], can be introduced in the same way as in metric spaces. The only difference is that we use vectors instead of numbers. Thus, the pair (T, S) is said to be *R-weakly commuting* if there exists a real number $R > 0$ such that $d(TSx, STx) \preceq Rd(Tx, Sx)$ for all $x \in X$.

3. Generalized quasicontractions and common fixed points

Let (T, S) be a pair of self-maps on an abstract metric space (X, d) such that $T(X) \subset S(X)$. For arbitrary $x_0 \in X$ there exists $x_1 \in X$ such that $Sx_1 = Tx_0$. Having chosen $x_{n-1} \in X$, choose $x_n \in X$ such that $Sx_n = Tx_{n-1}$.

Definition 3.1. (**1**) The sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$, where $y_0 = Sx_0$, $y_n = Tx_{n-1} = Sx_n$ for $n \geq 1$, are called *Jungck sequences of the pair* (T, S)*, with the initial point* x_0 (of first, resp. second order) [11]. The space (X, d) is called (T, S)*-orbitally complete* if, for each x_0, every Cauchy subsequence of arbitrary Jungck sequence (y_n) of second order has a limit in X.

(**2**) For arbitrary points $x, y \in X$ denote

$$C^{T,S}(x, y) = \{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}.$$

The mapping T is called a *generalized S-quasicontraction* if there exists $x_0 \in X$ such that for arbitrary Jungck sequences (x_n), (y_n) of the pair (T, S) with the initial point x_0 and for arbitrary $z \in Z = \{y_0, x_0, y_1, x_1, y_2, x_2, \dots\}$ there exists $u \in C^{T,S}(x_0, z)$ such that $d(Tx_0, Tz) \preceq \varphi(z) \cdot u$ holds, where $\varphi : Z \to [0, 1)$ is a function such that $\sup_{z \in Z} \varphi(z) = \mu < 1$.

Obviously, each S-quasicontraction in the sense of [8] is a generalized S-quasicontraction if contraction coefficient $\lambda \leq \frac{1}{2}$. Adapting [17, Example 3.1] we show in the next example that the converse is not true.

Example 3.2. Let $X = [0, 1)$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$, let $\alpha > 0$ be fixed and let $d(x, y) = (|x - y|, \alpha|x - y|)$ for $x, y \in X$. (X, d) is a normal abstract metric space with $K = 1$. Consider mappings $T, S : X \to X$, $Tx = \frac{1}{2}x^2$, $x \in X$ and $S = I_X$. Taking $x_0 = \frac{1}{8}$, $\varphi(z) = \frac{1}{2}(\frac{1}{2} + z)$ and $\lambda = \frac{3}{4}$ it is easy to show that T is a generalized S-quasicontraction. We show that T is not an S-quasicontraction. Let $x, y \in X$; then $d(Tx, Ty) = \frac{1}{2}(|x^2 - y^2|, \alpha|x^2 - y^2|)$. Consider the set $C^{T,S}(x, y) = \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ and suppose that for arbitrary $x, y \in X$ there exists $u \in C^{T,S}(x, y)$ such that $d(Tx, Ty) \preceq \lambda u$ for some fixed $\lambda < 1$. It is easy to show that in each of the five possible cases a contradiction is obtained. E.g., if

$u = d(x, y) = (|x-y|, \alpha|x-y|)$, then one obtains $\frac{1}{2}|x+y| \leq \lambda$ which is impossible for fixed $\lambda < 1$ when $x, y \to 1$.

Adapting the proof of the first part of [17, Theorem 3.1] we obtain

Lemma 3.3. *Let (T, S) be a pair of self-maps of an abstract metric space (X, d) with the normal constant K such that $T(X) \subset S(X)$. Let T be a generalized S-quasi-contraction where x_0, (x_n), (y_n), and φ are as in Definition 3.1 with the function φ satisfying the condition $\sup_{z\in Z} \varphi(z) \leq \lambda/2K$ ($\lambda \in [0, 1)$ is fixed, so that $\lambda K < 1$). Then (y_n) is a Cauchy sequence in (X, d).*

Proof. We shall use the following notation ($k \geq 0$).

$$\begin{aligned}
O(y_k; n) &= \{y_k, y_{k+1}, \dots, y_{k+n}\},\\
\delta_n(y_k) &= \operatorname{diam} O(y_k; n) = \sup\{\, \|d(u, v)\| : u, v \in O(y_k; n)\,\},\\
O(y_k; \infty) &= \{y_k, y_{k+1}, \dots\}, \qquad \delta(y_k) = \operatorname{diam} O(y_k; \infty),
\end{aligned}$$

Let us prove first that

$$\delta_{n-1}(y_1) \leq \lambda\delta_n(y_0). \tag{2}$$

Indeed, for some $1 \leq j < k \leq n$, we have

$$\delta_{n-1}(y_1) = \operatorname{diam}\{y_1, \dots, y_n\} = \|d(y_j, y_k)\| = \|d(Tx_{j-1}, Tx_{k-1})\|.$$

But

$$\begin{aligned}
d(Tx_{j-1}, Tx_{k-1}) &\preceq d(Tx_0, Tx_{j-1}) + d(Tx_0, Tx_{k-1})\\
&\preceq \varphi(x_{j-1}) \cdot u_{j-1} + \varphi(x_{k-1}) \cdot v_{k-1},
\end{aligned}$$

where

$$\begin{aligned}
u_{j-1} \in \{&d(Sx_0, Sx_{j-1}), d(Sx_0, Tx_0), d(Sx_{j-1}, Tx_{j-1}),\\
&d(Sx_0, Tx_{j-1}), d(Sx_{j-1}, Tx_0)\}\\
= \{&d(y_0, y_{j-1}), d(y_0, y_1), d(y_{j-1}, y_j), d(y_0, y_j), d(y_{j-1}, y_1)\},
\end{aligned}$$

so that $\|u_{j-1}\| \leq \operatorname{diam} O(y_0; n)$ and similarly for $\|v_{k-1}\|$. It follows that

$$\begin{aligned}
\delta_{n-1}(y_1) &\leq K\left(\varphi(x_{j-1}) \cdot \|u_{j-1}\| + \varphi(x_{k-1}) \cdot \|v_{k-1}\|\right)\\
&\leq K \cdot \frac{\lambda}{2K} \cdot 2 \operatorname{diam} O(y_0; n) = \lambda\delta_n(y_0),
\end{aligned}$$

and (2) is proved. It follows that

$$\delta_n(y_0) = \|d(y_0, y_k)\| \quad \text{for some } k \leq n. \tag{3}$$

Indeed, suppose that $\delta_n(y_0) = \|d(y_j, y_k)\|$ for some $1 \leq j < k \leq n$. Then it follows from (2) that

$$\delta_n(y_0) \leq \operatorname{diam}\{y_1, \dots, y_n\} = \delta_{n-1}(y_1) \leq \lambda\delta_n(y_0) < \delta_n(y_0),$$

a contradiction.

Now we prove that the sequence $(\delta_n(y_0))$ is bounded:

$$\delta_n(y_0) \leq \frac{K}{1-\lambda K}\|d(y_0, Tx_0)\|. \tag{4}$$

Let k be as in (3). Then $d(y_0, y_k) \preceq d(y_0, Tx_0) + d(Tx_0, y_k)$, so that

$$\begin{aligned}\delta_n(y_0) &= \|d(y_0, y_k)\| \le K \cdot \|d(y_0, Tx_0)\| + K \cdot \|d(Tx_0, y_k)\| \\ &= K\left(\|d(y_0, Tx_0)\| + \|d(y_1, y_k)\|\right),\end{aligned}$$

and since $\|d(y_1, y_k)\| \le \delta_{n-1}(y_1) \le \lambda\delta_n(y_0)$, we obtain that $\delta_n(y_0) \le K\cdot\|d(y_0, Tx_0)\| + K \cdot \lambda\delta_n(y_0)$, wherefrom relation (4) follows.

Since obviously $\delta_n(y_0) \le \delta_{n+1}(y_0)$, we conclude that there exists $\delta(y_0) = \lim_{n\to\infty} \delta_n(y_0)$ and

$$\delta(y_0) \le \frac{K}{1-\lambda K}\|d(y_0, Tx_0)\|. \tag{5}$$

Consider now $\beta_n(y_0) = \operatorname{diam} O(y_{n+1};\infty)$ where $O(y_{n+1};\infty) = \{y_{n+1}, y_{n+2}, \dots\}$. Obviously the sequence $(\beta_n(y_0))$ is not increasing and there exists $\beta(y_0) := \lim\limits_{n\to\infty} \beta_n(y_0)$. Let us prove that $\beta(y_0) = 0$. Passing to the limit in relation (2) we obtain that

$$\delta(y_1) \le \lambda\delta(y_0) \quad \text{and similarly} \quad \delta(y_{k+1}) \le \lambda\delta(y_k) \quad \text{for } k \ge 1. \tag{6}$$

It follows that

$$\beta_{n+1}(y_0) = \operatorname{diam} O(y_{n+2};\infty) \le \lambda \operatorname{diam} O(y_{n+1};\infty) = \lambda\beta_n(y_0).$$

Passing to the limit we get that $\beta(y_0) \le \lambda\beta(y_0)$ which is only possible (since $\lambda < 1$) when $\beta(y_0) = 0$.

Thus, $\beta(y_0) = 0$ which implies that the Jungck sequence (y_n) of second order, where $y_n = Tx_{n-1} = Sx_n$, is a Cauchy sequence.

Theorem 3.4. *Let (X, d) be a (T, S)-orbitally complete normal abstract metric space for some pair of self-maps (T, S) such that:*

1° *T is a generalized S-quasicontraction (notation is as in Lemma 3.3);*

2° *(T, S) is R-weakly commuting;*

3° *S is continuous.*

Let (x_n) and (y_n) be Jungck sequences of first and second order with the initial point x_0 and let $\lim_{n\to\infty} y_n = \bar{x}$. Suppose that for each element $z \in Z$ there exists $v \in C^{T,S}(\bar{x}, z)$ such that $d(T\bar{x}, Tz) \preceq \varphi(z)\cdot v$. Then:

(i) $T\bar{x} = S\bar{x} = \bar{x}$.

(ii) $\|d(y_n, \bar{x})\| \le \dfrac{\lambda^n K}{1-\lambda K}\|d(y_0, Tx_0)\|$.

(iii) If $\bigcap_{n=0}^{\infty} T^n(X)$ is a singleton, then $\bar{x}$ is the unique common fixed point of T and S.

Note that if $Y \subset X$, then $T(Y) = \bigcup_{y\in Y} Ty$, and in particular, $T(X) = \bigcup_{x\in X} Tx$. Further, $T^2(X) = T(T(X)) = T(\bigcup_{x\in X} Tx) = \bigcup_{x\in X}\bigcup_{y\in Tx} Ty$, and $T^n(X)$ is defined by induction ($T^0(X) = X$).

Proof. According to Lemma 3.3 the sequence (y_n) is a Cauchy sequence, and since the space (X, d) is (T, S)-orbitally complete, it has a limit. So, introduction of the point $\bar{x}$ in the formulation of the theorem is correct.

(i) We shall prove first that $T\bar{x} = S\bar{x}$. Since $Tx_n \to \bar{x}$ and $Sx_n \to \bar{x}$ and since S is continuous, $STx_n \to S\bar{x}$. By the triangle inequality we have that

$$d(T\bar{x}, S\bar{x}) \preceq d(T\bar{x}, TSx_n) + d(TSx_n, STx_n) + d(STx_n, S\bar{x}) \tag{7}$$

For the first term on the right-hand side of (7) we have that $d(T\bar{x}, TSx_n) \preceq \varphi(Sx_n) \cdot v$ where

$$v \in \{d(S\bar{x}, S^2x_n), d(T\bar{x}, S\bar{x}), d(TSx_n, S^2x_n), d(S\bar{x}, TSx_n), d(S^2x_n, T\bar{x})\}.$$

The elements of the last set tend to zero or to $d(S\bar{x}, T\bar{x})$; since $\sup \varphi(Sx_n) \leq \mu < 1$, using Remark 1.2.2° it follows that $d(T\bar{x}, TSx_n) \to \theta$. For the second term in (7) we obtain using assumption 2° that $d(TSx_n, STx_n) \preceq Rd(Tx_n, Sx_n) \to \theta$. Finally, the third term also tends to zero because of $STx_n \to S\bar{x}$. Using property (1) (and thus normality of the cone) we conclude that $d(T\bar{x}, S\bar{x}) = \theta$, i.e. $T\bar{x} = S\bar{x}$.

Now we prove that $T\bar{x} = \bar{x}$. For each n it is $d(T\bar{x}, Tx_{n+1}) \preceq \varphi(x_{n+1}) \cdot v$, where

$$v \in \{d(S\bar{x}, Sx_{n+1}), d(S\bar{x}, T\bar{x}), d(Sx_{n+1}, Tx_{n+1}), d(S\bar{x}, Tx_{n+1}), d(Sx_{n+1}, T\bar{x})\}.$$

Elements of the last set tend, respectively, to $d(S\bar{x}, \bar{x}) = d(T\bar{x}, \bar{x})$, θ, θ, $d(T\bar{x}, \bar{x})$ and $d(\bar{x}, T\bar{x})$. Hence, passing to the limit (and again using boundedness of φ and property (1)), we obtain that $d(T\bar{x}, \bar{x}) = \theta$, i.e. $T\bar{x} = \bar{x} = S\bar{x}$.

(ii) Using inequalities (6) we obtain that $\delta(y_n) \leq \lambda\delta(y_{n-1}) \leq \cdots \leq \lambda^n\delta(y_0)$. Taking into account relation (5), we get that

$$\delta(y_n) \leq \frac{\lambda^n K}{1 - \lambda K}\|d(y_0, Tx_0)\|. \tag{8}$$

Since $Tx_n = y_{n+1}$ we have also $\delta(Tx_n) \leq \frac{\lambda^{n+1}K}{1-\lambda K}\|d(y_0, Tx_0)\| \leq \frac{\lambda^n K}{1-\lambda K}\|d(y_0, Tx_0)\|$ (because of $\lambda < 1$). Hence, both $\delta(Tx_n)$ and $\delta(Sx_n)$ have the same upper bound $\frac{\lambda^n K}{1-\lambda K}\|d(y_0, Tx_0)\|$.

Let $m > n$. Then, by (8),

$$\|d(y_n, y_m)\| \leq \delta(y_n) \leq \frac{\lambda^n K}{1 - \lambda K}\|d(y_0, Tx_0)\|.$$

Passing to the limit when $m \to \infty$ we obtain that

$$\|d(Sx_n, \bar{x})\| = \|d(y_n, \bar{x})\| \leq \frac{\lambda^n K}{1 - \lambda K}\|d(y_0, Tx_0)\|.$$

as stated in (ii). Similarly, one obtains that

$$\|d(Tx_n, \bar{x})\| \leq \frac{\lambda^n K}{1 - \lambda K}\|d(y_0, Tx_0)\|.$$

(iii) Let $\bigcap_{n=0}^{\infty} T^n(X) = \{w\}$. Let v be any common fixed point for T and S, i.e., $v = Tv = Sv$. It follows that also $v = T^n v = S^n v$ for $n = 0, 1, 2, \ldots$ and hence $v \in T^n(X)$ for each n. Thus, $v = w$ and the fixed common point $\bar{x}$ of T and S is unique.

The theorem is proved.

Theorem 3.1 of [17] is obtained as a special case of our Theorem 3.4 putting $S = I_X$. Note that condition that the cone P is regular was here avoided, but it was just stated and not used in [17], either.

On the other hand, Theorem 3.4 is obviously a generalization of [8, Theorem 2.1]. Example 3.2 shows that this generalization is proper.

A similar example can be constructed even for Banach contractions and in the setting of metric spaces. Let (X, d) be a metric space and T, S be two self-maps on X. We shall call T a *Banach S-contraction* if $d(Tx, Ty) \leq \lambda d(Sx, Sy)$ holds for some $\lambda \in (0,1)$ and all $x, y \in X$ and a *generalized Banach S-contraction* if there exists $x_0 \in X$ such that $d(Tx_0, Ty) \leq \varphi(y) d(Sx_0, Sy)$ holds for all $y \in Z$, where $\sup_{y\in Z} \varphi(y) \leq \frac{1}{2}\lambda$ for some $\lambda < 1$. We shall show that there exists a generalized Banach S-contraction which is not a Banach S-contraction.

Example 3.5. Let $X = [0,1)$, d be the Euclidean metric, and let $T, S : X \to X$ be defined by $Tx = \frac{1}{2}x^3$, $Sx = \frac{1}{2}x^2$. Obviously, $TX \subset SX$ and S is continuous. Then $d(Tx, Ty) \leq \lambda d(Sx, Sy)$ does not hold for all $x, y \in X$ and any fixed $\lambda \in (0,1)$, Indeed, otherwise we would have $\frac{1}{2}|x^3 - y^3| \leq \frac{1}{2}\lambda|x^2 - y^2|$, i.e., $\frac{x^2+xy+y^2}{x+y} \leq \lambda$, which is impossible for fixed $\lambda \in (0,1)$ when $x, y \to 1$. Hence, T is not a Banach S-contraction.

To show that T is a generalized Banach S-contraction, take $x_0 = \frac{1}{4}$. Then $d(Sx_0, Sy) = |\frac{1}{32} - \frac{1}{2}y^2| = \frac{1}{2}|\frac{1}{4} - y| \cdot |\frac{1}{4} + y|$ and

$$\begin{aligned} d(Tx_0, Ty) &= \left|\frac{1}{128} - \frac{1}{2}y^3\right| = \frac{1}{2}\left|\frac{1}{4} - y\right|\left(\frac{1}{16} + \frac{1}{4}y + y^2\right) \\ &= \frac{1 + 4y + 16y^2}{8(1+4y)} \cdot d(Sx_0, Sy). \end{aligned}$$

Hence, taking $\varphi(y) = \frac{1+4y+16y^2}{8(1+4y)}$, since $\max_{y\in Z} \varphi(y) = \varphi(\frac{1}{4}) = \frac{3}{16} \leq \frac{\lambda}{2}$ for any $\lambda \in \left[\frac{3}{8}, 1\right)$, we conclude that T is a generalized Banach S-contraction.

The next example illustrates the case when conditions of Theorem 3.4 are fulfilled.

Example 3.6. Let $X = [0,1)$, $E = C_{\mathbb{R}}[0,1]$ and $P = \{f : f(t) \geq 0\}$. This cone is normal with constant $K = 1$, but it is not regular (see Example 2.1.2°). An abstract metric d on X is defined by $d(x,y) = |x - y| \cdot \phi$, where $\phi \in P$ is an arbitrary function (e.g., $\phi(t) = 2^t$). Consider mappings $T, S : X \to X$ given by $Tx = \frac{1}{3}x^2$, $Sx = x^2$, $x \in X$. Then one can easily check:

1° For $x_0 = \frac{1}{3}$ and $z \in Z$ (see Definition 3.1),

$$d(Tx_0, Tz) = \left|\frac{1}{27} - \frac{1}{3}z^2\right| \cdot \phi = \frac{1}{3}\left|\frac{1}{9} - z^2\right| \cdot \phi = \frac{1}{3}d(Sx_0, Sz),$$

so T is a generalized S-quasicontraction with $\varphi(z) \equiv \frac{1}{3} \leq \lambda/2$ for any $\lambda \in \left[\frac{2}{3}, 1\right)$ and $u = d(Sx_0, Sz) \in C^{T,S}(x_0, z)$.

2° $d(TSx, STx) = |\frac{1}{3}x^4 - \frac{1}{9}x^4| \cdot \phi = \frac{2}{9}x^4 \cdot \phi \preceq R \cdot \frac{2}{3}x^2 \cdot \phi = |x^2 - \frac{1}{3}x^2| \cdot \phi = R \cdot d(Sx, Tx)$ for $R = 1$, which means that the pair (T, S) is R-weakly-commuting.

3° S is continuous.

Moreover, it can be easily verified that for $\bar{x} = 0$, $d(T\bar{x}, Tz) \preceq \varphi(z) \cdot v$ with $v = d(S\bar{x}, Sz) \in C^{T,S}(\bar{x}, z)$. Since X is obviously (T, S)-orbitally complete, all conditions of Theorem 3.4 are fulfilled. The point $\bar{x} = 0$ is a (unique) common fixed point for T and S.

4. p-CONTRACTIONS WITHOUT REGULARITY CONDITION

The concept of a p-contraction was in the setting of metric spaces introduced in [16] (see also [10, 14]), and then in the setting of abstract metric spaces in [17]. It was shown by examples in [16] that it is more general than the classical Banach contraction and that a p-contraction need not be continuous. Recall the following definition (in this section we use the classical notation $O(x;\infty) = \{x, Tx, T^2x, \dots\}$).

Definition 4.1. [17] *A self-map T on an abstract metric space (X, d) is called a p-contraction if there exist $x \in X$ and a number $\lambda \in [0, 1)$ such that*

$$d(Ty, T^2y) \preceq \varphi(y) \cdot d(y, Ty)$$

holds for each $y \in O(x;\infty)$, where $\varphi : O(x;\infty) \to [0, 1)$ is a function such that $\sup_{y\in O(x;\infty)} \varphi(y) = \lambda < 1$.

A fixed point result for p-contractions was proved in [17, Theorem 3.5] assuming that the positive cone P is regular. We shall prove that the first part of this result holds without even P being normal. The only assumption will be that P is solid (which ensures that the limit of any sequence is unique if it exists).

Lemma 4.2. *Let (X, d) be a T-orbitally complete abstract metric space, where the positive cone P is solid and $T : X \to X$ is a p-contraction, with the respective vector x. Then there exists $\lim_{n\to\infty} T^n x = \bar{x} \in X$.*

Proof. Let λ and φ be as in Definition 4.1. Put $T^0x = x$ and $y = T^{n-1}x \in O(x;\infty)$. Then for each $n \in \mathbb{N}$

$$d(Ty, T^2y) = d(T^nx, T^{n+1}x) \preceq \varphi(T^{n-1}x)d(T^{n-1}x, T^nx) \preceq \lambda d(T^{n-1}x, T^nx).$$

It follows that

$$d(T^nx, T^{n+1}x) \preceq \lambda d(T^{n-1}x, T^nx) \preceq \cdots \preceq \lambda^n d(x, Tx). \tag{9}$$

Now for $m > n$, using (9), one obtains

$$\begin{aligned} d(T^nx, T^mx) &\preceq d(T^nx, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \cdots + d(T^{m-1}x, T^mx) \\ &\preceq \lambda^n d(x, Tx) + \lambda^{n+1} d(x, Tx) + \cdots + \lambda^{m-1} d(x, Tx) \\ &= (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x, Tx) \\ &= \lambda^n \frac{1 - \lambda^{m-n}}{1 - \lambda} d(x, Tx) \preceq \frac{\lambda^n}{1 - \lambda} d(x, Tx). \end{aligned}$$

Since $\frac{\lambda^n}{1-\lambda} d(x, Tx) \to \theta$ in the Banach norm of E, by Remark 2.3.3°, it follows that $\frac{\lambda^n}{1-\lambda} d(x, Tx) \ll c$ for each $c \gg \theta$ and arbitrary n greater than some $n_0(c) \in \mathbb{N}$. But then by Remark 2.3.1°, we obtain that $d(T^nx, T^mx) \ll c$, which means that (T^nx) is a Cauchy sequence in (X, d). Since (X, d) is T-orbitally complete, there exists $\bar{x} \in X$ such that $T^nx \to \bar{x}$.

The rest of the proof of [17, Theorem 3.5] use only normality (and not regularity) of the cone P, so we obtain

Theorem 4.3. *Let (X, d) be a T-orbitally complete abstract metric space, where the positive cone P is normal and $T : X \to X$ is a p-contraction. Let $\lim_{n\to\infty} T^n x = \bar{x}$ (see Lemma 4.2) and suppose that the mapping $X \ni z \mapsto \|d(z, Tz)\|$ is T-orbitally lower semicontinuous at the point $\bar{x}$. Then $\bar{x}$ is a fixed point of T. This fixed point is unique if $\bigcap_{n=0}^{\infty} T^n(X)$ is a singleton, where $T^n(X) = T(T^{n-1}(X))$ for $n \in \mathbb{N}$ and $T^0(X) = X$.*

Adapting an example from [20] we show that the assumption about lower continuity is essential in the last theorem. The second part of the example shows the importance of the condition for uniqueness of the fixed point.

Example 4.4. **(a)** Let $X = [0, 1]$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$ and $d(x, y) = (|x - y|, \alpha|x - y|)$, $\alpha \geq 0$. Then (X, d) is a complete abstract metric space. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{2}x^2, & x \neq 0. \end{cases}$$

Function $I(z) = \|d(z, Tz)\|$ of the form

$$I(z) = \begin{cases} \frac{1}{2}\sqrt{1 + \alpha^2}, & z = 0 \\ \left|z - \frac{1}{2}z^2\right| \sqrt{1 + \alpha^2}, & z \neq 0. \end{cases}$$

is obviously not T-orbitally lower semicontinuous. It is easy to check that all other conditions of Theorem 4.3 are fulfilled, but the p-contraction T has no fixed points.

(b) In the same abstract metric space X as in example **(a)** consider the mapping $T_1 : X \to X$ given by

$$T_1 x = \begin{cases} 1, & x = 1 \\ \frac{1}{2}x^2, & x \neq 1. \end{cases}$$

Here, all the conditions of Theorem 4.3 are fulfilled for the point $\overline{x} = 0$ except that $\bigcap_{n=0}^{\infty} T^n(X) = \{0, 1\}$ is not a singleton. The mapping T_1 has two fixed points.

We conclude with an easy example in which the conditions of Theorem 4.3 are satisfied, but the conditions of [4, Theorem 1.1] are not.

Example 4.5. Let X, P and d be as in Example 3.6, and let $Tx = \frac{1}{2}x^2$. It is easy to check that T is a p-contraction with $\lambda = \frac{3}{4}$. On the other hand, Banach contraction condition would mean that $d(Tx, Ty) \preceq \frac{3}{4}d(x, y)$ for all $x, y \in X$. This is equivalent to

$$\left|\frac{1}{2}x^2 - \frac{1}{2}y^2\right| \cdot \phi(t) \leq \frac{3}{4}|x - y| \cdot \phi(t), \qquad t \in [0, 1],$$

i.e. $\frac{|x+y|}{2} \leq \frac{3}{4}$, and $|x + y| \leq \frac{3}{2}$. But the last inequality cannot hold for all $x, y \in [0, 1]$.

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