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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF A NONLINEAR q-DIFFERENCE EQUATION

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Abstract. This paper is concerned with a boundary value problem of the nonlinear q-difference equation $-D_q^2 u(t) = f(t, u(t))$, with some boundary conditions. Under certain conditions on f, the existence of positive solutions is obtained by applying a fixed point theorem in cones.
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1. Introduction

In the past few years the existence of positive solutions of nonlinear boundary value problems for differential equations, difference equations, fractional differential equations, as well as boundary value problems on time scale have been studied extensively, see for example [2, 5, 6, 8, 9, 12, 13, 19] and the references therein. These results depend on fixed point theorems on cones. A cone is a closed convex set K of a Banach space X such that $\lambda K \subset K$ for all $\lambda \geqslant 0$ and $K \bigcap (-K) = \{0\}$.

One of the interesting results, with differential operator, was obtained in [13]. The authors deal with the following nonlinear boundary value problem

$$u'' + f(t, u(t)) = 0, \qquad 0 \leqslant t \leqslant 1, \tag{1.1}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) &= 0, \\ \gamma u(1) + \delta u'(1) &= 0, \end{aligned} \tag{1.2}$$

where $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$, $\alpha, \beta, \gamma, \delta \geqslant 0$, and $f \in C([0,1] \times [0,\infty); [0,\infty))$. Let $k(t,s)$ be the Green's function of $u'' = 0$ with (1.2) and

$$\zeta = \min\left\{\frac{\gamma + 4\delta}{4(\gamma+\delta)}, \frac{\alpha + 4\beta}{4(\alpha+\beta)}\right\}.$$

Then the following result is obtained, [13].

Theorem 1.1. *Assume that there exist two distinct positive constants* λ, η *such that*

$$f(t,u) \leqslant \lambda \left(\int_0^1 k(s,s)\, ds\right)^{-1}, \quad (t,u) \in [0,1] \times [0,\lambda],$$

and

$$f(t,u) \geqslant \eta \left(\int_{1/4}^{3/4} k(1/2,s)\, ds\right)^{-1}, \quad (t,u) \in [1/4,3/4] \times [\zeta\,\eta, \eta].$$

Then $(1.1)-(1.2)$ *has at least one positive solution* u *such that* $\|u\|$ *lies between* λ *and* η.

Recently existence of positive solutions of a nonlinear q-difference equation is obtained in [7]. This paper considered the second order q-difference equation

$$-D_q^2 u(t) = a(t)\, g(u(t)), \quad 0 \leqslant t \leqslant 1, \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} \alpha u(0) - \beta D_q u(0) = 0, \\ \gamma u(1) + \delta D_q u(1) = 0, \end{aligned} \tag{1.4}$$

where $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$, $\alpha, \beta, \gamma, \delta \geqslant 0$, $a(\cdot)$, $g(\cdot)$ are assumed to be nonnegative continuous functions for $t \in [0,1]$, $u \in [0,\infty)$. Let

$$g_0 := \lim_{u\to 0} \frac{g(u)}{u}, \qquad g_\infty := \lim_{u\to\infty} \frac{g(u)}{u}.$$

Then the following result was obtained.

Theorem 1.2. *The problem* $(1.3)-(1.4)$ *has at least one positive solution* $u \in C[0,1]$, *if* $g_0 = 0$ *and* $g_\infty = \infty$, *or* $g_0 = \infty$ *and* $g_\infty = 0$.

We aim to generalize the nonlinear term in (1.3) and put less restrictive conditions on this term, so that the above result will be a special case. In fact we will give a q-analog of the results obtained in [13].

In the following section we give a brief account on q-calculus and then we state the fixed-point theorem of Krasnoselskii. In Section 3 we state the boundary value problems with the solution via Green's function. This Green's function plays the major task in getting the positive solution. In Section 4 we give some applications which guarantee the existence of positive solutions when g_0, g_∞ take values different from zero and infinity.

2. Preliminaries

Let $0 < q < 1$, we say that a set A of real numbers is q-geometric if for every $x \in A$, $qx \in A$. Let f be a real or complex valued function defined on a q-geometric set A. The q-difference operator is defined by, cf. [14],

$$D_q f(x) := \frac{f(x) - f(qx)}{x(1-q)}, \quad x \neq 0. \tag{2.1}$$

If $0 \in A$, the q-derivative at zero is defined by, [1],

$$D_q f(0) := \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A, \tag{2.2}$$

if the limit exists and does not depend on x. The q-integration is defined by F. H. Jackson, cf. [14], via

$$\int_0^x f(t) d_q t := x(1-q) \sum_{n=0}^{\infty} q^n f(xq^n), \quad x \in A, \tag{2.3}$$

provided that the series converges. In general,

$$\int_a^b f(t) d_q t := \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad a, b \in A. \tag{2.4}$$

The reader may easily computes for $\alpha > -1$, that

$$\int_0^x t^\alpha d_q t = \frac{(1-q) x^{\alpha+1}}{1 - q^{\alpha+1}}, \tag{2.5}$$

which is clearly gives the classical case as $q \to 1^-$. Clearly q-derivative exists for any function (provided that $D_q f(0)$ exists if zero is in its domain) and the q-integral exists for any bounded function, $x \in \mathbb{R}$. Furthermore, one can easily show that the q-integral (2.3) exists on $[0, a]$ if for $\alpha < 1$, $x^\alpha f(x)$ is bounded on $(0, a]$, cf. [15, p. 68].
The following theorem is a version of the fundamental theorem of q-calculus.

Theorem 2.1. *Let $f \in C[0, 1]$, $a \in [0, 1]$ and*

$$H(x) = \int_a^x f(t)\, d_q t, \quad x \in [0, 1], \tag{2.6}$$

then $H \in C[0, 1]$, and

$$D_q H(x) = f(x), \quad x \in [0, 1]. \tag{2.7}$$

Also,

$$\int_c^b D_q f(t)\, d_q t = f(b) - f(c), \quad c, b \in [0, 1]. \tag{2.8}$$

Our main result depend on a fixed point theorem due to Krasnoselskii, see [10, 16, 17, 18]. This theorem is applied for a completely continuous operator. Such an operator is defined as follows.

Definition 2.2. If Z and Y are Banach spaces and B is a subset of Z, then an operator $F : B \to Y$ is completely continuous if it is continuous and maps bounded subset of B into relatively compact subset of Y, [4, p. 55].

In some literature a completely continuous operator may be referred as a compact operator, see for example [3, p. 89] or [11, p. 221].

The following form is a modified version of Krasnoselskii's fixed point theorem and due to Guo, [10, p. 94];

Theorem 2.3. *Assume that K is a cone in a Banach space X and Ω_1, Ω_2 are two bounded open subsets such that $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let*

$$F : K \cap \big(\overline{\Omega}_2 \backslash \Omega_1\big) \longrightarrow K$$

be completely continuous and let that one of the conditions

(1) $\|Fx\| \leqslant \|x\|, \quad \forall x \in K \cap \partial\Omega_1,$ *and* $\|Fx\| \geqslant \|x\|, \quad \forall x \in K \cap \partial\Omega_2,$
(2) $\|Fx\| \geqslant \|x\|, \quad \forall x \in K \cap \partial\Omega_1,$ *and* $\|Fx\| \leqslant \|x\|, \quad \forall x \in K \cap \partial\Omega_2,$

is satisfied. Then F has at least one fixed point in $K \cap \big(\overline{\Omega}_2 \backslash \Omega_1\big)$.

3. Main Results

Consider the nonlinear second order q-difference equation

$$-D_q^2 u(t) = f(t, u(t)), \quad 0 \leqslant t \leqslant 1, \tag{3.1}$$

with the boundary conditions (1.4). We assume that the function $f : [0, q^{-1}] \times [0, \infty) \to [0, \infty)$ is continuous. The reason of extending f on $[1, q^{-1}] \times [0, \infty)$ will be explained in Remak 3.2 below.

Lemma 3.1.

(1) *Any solution of* (3.1) *and* (1.4) *is given by*

$$u(t) = \int_0^1 G(t, qs)\, f(s, u(s))\, d_q s, \tag{3.2}$$

where $G(t, qs)$ is the Green's function of $-D_q^2 u(t) = 0$ with (1.4)*, which is*

$$G(t, qs) = \frac{1}{\rho} \begin{cases} \big(\gamma + \delta - \gamma t\big)\big(\beta + \alpha qs\big), & 0 \leqslant qs \leqslant t \leqslant 1, \\ \\ \big(\gamma + \delta - \gamma qs\big)\big(\beta + \alpha t\big), & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases} \tag{3.3}$$

(2) *Let*

$$\sigma = \min\left\{ \frac{\gamma + 4\delta}{4(\gamma + \delta)}, \frac{\alpha + 4\beta}{4(\alpha q + \beta)} \right\}, \quad \Big(\sigma \in (0, 1]\Big). \tag{3.4}$$

Then

$$G(t, qs) \geqslant \sigma\, G(qs, qs), \qquad 1/4 \leqslant t \leqslant 3/4, \quad 0 \leqslant qs \leqslant 1. \tag{3.5}$$

Also

$$G(t, qs) \leqslant G(qs, qs), \qquad 0 \leqslant t, qs \leqslant 1. \tag{3.6}$$

Proof. Using variation of parameters method, the solution of (3.1) has the form

$$u(t) = c_1 + c_2\, t - \int_0^t \big[t - qs\big] f(s, u(s))\, d_q s. \tag{3.7}$$

Substituting in (1.4), we get

$$c_1 = \frac{\beta}{\rho} \int_0^1 [(\gamma+\delta) - \gamma qs]\, f(s,u(s))\, d_q s, \quad c_2 = \frac{\alpha}{\rho} \int_0^1 [(\gamma+\delta) - \gamma qs]\, f(s,u(s))\, d_q s.$$

Substituting in (3.7) we obtain

$$\begin{aligned} u(t) &= \frac{\beta}{\rho} \int_0^1 [(\gamma+\delta) - \gamma qs]\, f(s,u(s))\, d_q s + t\frac{\alpha}{\rho} \int_0^1 [(\gamma+\delta) - \gamma qs]\, f(s,u(s))\, d_q s \\ &\quad - \int_0^t \big[t - qs\big] f(s,u(s))\, d_q s \\ &= \frac{\beta}{\rho} \int_0^1 [(\gamma+\delta) - \gamma qs]\, f(s,u(s))\, d_q s + t\frac{\alpha}{\rho} \int_0^1 [(\gamma+\delta) - \gamma qs]\, f(s,u(s))\, d_q s \\ &\quad - \int_0^{q^{-1}t} \big[t - qs\big] f(s,u(s))\, d_q s, \end{aligned} \tag{3.8}$$

giving the form (3.2). The proof of the second part is simple and is given in [7]. □

Remark 3.2. The shift in (3.8) because of the integrand is zero at $s = q^{-1}t$. This contributes only one term (which is zero) in (2.3). This does not change the value of the function $u(\cdot)$, but we gain the symmetry of the Green's function, that is

$$G(t, qs) = G(qs, t); \qquad \text{for all } t,\ qs \in [0,1].$$

This shift explains the needing of extending the definition of f on $[1, q^{-1}] \times [0, \infty)$. Continuity of f on $[1, q^{-1}] \times [0, \infty)$ will be needed in Lemma 3.3 below.

On $X = C[0,1]$ define the operator

$$\big(Fu\big)(t) = \int_0^1 G(t, qs)\, f(s, u(s))\, d_q s, \qquad u \in X. \tag{3.9}$$

Thus by Theorem 2.1, $F : X \to X$.

Lemma 3.3. *Let U be a bounded subset of X. If we restrict F on U, then F is completely continuous.*

Proof. Let $\Phi(t, s, u(s)) = G(t, qs)\, f(s, u(s))$. First we show that (3.9) is continuous on X. For $u_0 \in X$ let $r = \|u_0\| + 1$. The function Φ is uniformly continuous on the compact set $D = [0,1] \times [0, q^{-1}] \times [0, r]$. Then for $\epsilon > 0$ there exists $\delta > 0$ such that $|\Phi(t,s,u) - \Phi(t,s,w)| < \epsilon$ when $|u - w| < \delta$. Thus

$$\begin{aligned} \|Fu - Fu_0\| &= \max_{t\in[0,1]} \left| \int_0^1 \Phi(t,s,u(s))\, d_q s - \int_0^1 \Phi(t,s,u_0(s))\, d_q s \right| \\ &\leqslant \max_{t\in[0,1]} \int_0^1 |\Phi(t,s,u(s)) - \Phi(t,s,u_0(s))|\, d_q s < \epsilon. \end{aligned}$$

which means that F is continuous at u_0. Since U is bounded we can assume that there is an $r > 0$ such that $\|u\| \leqslant r$, for any $u \in U$. So for any $u \in U$, we have

$$\|Fu\| = \max_{t\in[0,1]} \int_0^1 \Phi(t,s,u(s))\, d_q s \leqslant M,$$

where $M = \max_{(t,s,u)\in D} \Phi(t,s,u(s))$. Thus the set $F(U)$ is bounded. Since Φ is uniformly continuous on D, then we can find $\delta > 0$ such that $|\Phi(t,s,u(s)) - \Phi(\tau,s,u(s))| < \epsilon$, for $|t-\tau| < \delta$. Therefore

$$\begin{aligned}\big|\big(Fu\big)(t) - \big(Fu\big)(\tau)\big| &= \left|\int_0^1 \Phi(t,s,u(s))\, d_q s - \int_0^1 \Phi(\tau,s,u(s))\, d_q s\right| \\ &\leqslant \int_0^1 \big|\Phi(t,s,u(s)) - \Phi(\tau,s,u(s))\big|\, d_q s < \epsilon,\end{aligned}$$

for any $u \in U$. Hence the set $F(U)$ is relatively compact by Arzela-Ascoli Theorem. Therefore F is completely continuous. □

Define a cone K in X by

$$K = \left\{u \in X : u(t) \geqslant 0,\ \min_{1/4\leqslant t\leqslant 3/4} u(t) \geqslant \sigma\|u\|\right\}, \tag{3.10}$$

where σ is defined in (3.4).

Lemma 3.4. *F maps K into K.*

Proof. For $u \in K$, $F(u) \geqslant 0$. Using Lemma 3.1, we get

$$\begin{aligned}\min_{1/4\leqslant t\leqslant 3/4} \big(Fu\big)(t) &= \min_{1/4\leqslant t\leqslant 3/4} \int_0^1 G(t,qs)\, f(s,u(s))\, d_q s \\ &\geqslant \sigma \int_0^1 G(qs,qs)\, f(s,u(s))\, d_q s \\ &\geqslant \sigma \max_{0\leqslant t\leqslant 1} \int_0^1 G(t,qs)\, f(s,u(s))\, d_q s \\ &= \sigma\, \|Fu\|.\end{aligned}$$

□

Let A and B be the positive numbers defined by

$$A := \left(\int_0^1 G(qs,qs)\, d_q s\right)^{-1}, \qquad B := \left(\int_{1/4}^{3/4} G(1/2,qs)\, d_q s\right)^{-1}. \tag{3.11}$$

Using (3.3) and (2.5) we get

$$\frac{1}{A} = \frac{1}{\rho}\left((\gamma+\delta)\beta + \frac{q\big[\alpha(\gamma+\delta)-\beta\gamma\big]}{1+q} - \frac{\gamma\alpha q^2}{1+q+q^2}\right)$$

$$\frac{1}{B} = \begin{cases} \dfrac{\frac{\gamma}{2}+\delta}{2\rho}\left(\beta+\dfrac{q\alpha}{(1+q)}\right), & \text{if } 0<q\leqslant\dfrac{2}{3}, \\[2ex] \dfrac{\frac{\gamma}{2}+\delta}{\rho}\left(\dfrac{\beta(2-q)}{4q}+\dfrac{\alpha(4-q^2)}{16q(1+q)}\right) \\ \quad +\dfrac{\beta+\frac{\alpha}{2}}{\rho}\left(\dfrac{(\gamma+\delta)(3q-2)}{4q}-\dfrac{\gamma(9q^2-4)}{16q(1+q)}\right), & \text{if } \dfrac{2}{3}<q<1. \end{cases}$$

Theorem 3.5. *Assume that there are two distinct positive numbers m, M such that*

$$f(t,u) \leqslant m\,A, \qquad (t,u)\in[0,1]\times[0,m], \tag{3.12}$$

$$f(t,u) \geqslant M\,B, \quad (t,u)\in[1/4,3/4]\times[\sigma M, M]. \tag{3.13}$$

Then the boundary value problem (3.1) *and* (1.4) *has at least one positive solution u such that $\|u\|$ between m and M.*

Proof. We will give the proof when $m<M$. The proof of the other case is omitted because it is being similar. Let $\Omega_1=\{u\in K:\ \|u\|<m\}$. Thus for $t\in[0,1]$ and $u\in\partial\Omega_1$, we have

$$\begin{aligned}\big(Fu\big)(t) &= \int_0^1 G(t,qs)\,f(s,u(s))\,d_qs \\ &\leqslant \int_0^1 G(qs,qs)\,f(s,u(s))\,d_qs \\ &\leqslant m\,A\int_0^1 G(qs,qs)\,d_qs = m = \|u\|,\end{aligned}$$

where we used (3.6),(3.11) and (3.12). Thus,

$$\|Fu\|\leqslant\|u\|, \qquad u\in K\cap\partial\Omega_1.$$

Let $\Omega_2=\{u\in K:\ \|u\|<M\}$. For $u\in\partial\Omega_2$, $t\in[1/4,3/4]$, and from (3.10), we get

$$M=\|u\|\geqslant u(t)\geqslant \min_{[1/4,3/4]} u(t)\geqslant\sigma\|u\|=\sigma M.$$

Hence using (3.13) we obtain

$$\begin{aligned}\big(Fu\big)(\tfrac{1}{2}) &= \int_0^1 G(1/2,qs)\,f(s,u(s))\,d_qs \\ &\geqslant \int_{1/4}^{3/4} G(1/2,qs)\,f(s,u(s))\,d_qs \\ &\geqslant M\,B\int_{1/4}^{3/4} G(1/2,qs)\,d_qs = M = \|u\|.\end{aligned}$$

Thus,

$$\|Fu\| \geqslant \|u\|, \qquad u \in K \cap \partial\Omega_2.$$

Therefore by Theorem 2.3, F has a fixed point u in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, with $m < \|u\| < M$. □

4. Applications

Let

$$\begin{aligned} C_0 : &= \lim_{u\to 0^+} \sup_{t\in[0,1]} \frac{f(t,u)}{u}, \quad c_0 := \lim_{u\to 0^+} \inf_{t\in[0,1]} \frac{f(t,u)}{u}, \\ C_\infty : &= \lim_{u\to \infty} \sup_{t\in[0,1]} \frac{f(t,u)}{u}, \quad c_\infty := \lim_{u\to \infty} \inf_{t\in[0,1]} \frac{f(t,u)}{u}, \end{aligned}$$

Application 4.1. Assume that one of the following hypotheses hold

(a) $C_0 \in [0, A)$ and $c_\infty \in (\frac{B}{\sigma}, \infty]$,

(b) $c_0 \in (\frac{B}{\sigma}, \infty]$ and $C_\infty \in [0, A)$.

Then the problem (3.1) and (1.4) has at least one positive solution.

Proof. Suppose that (a) holds, then for $\epsilon = A - C_0$ there exists $\delta > 0$ (δ can be chosen arbitrarily small) such that

$$\sup_{t\in[0,1]} \frac{f(t,u)}{u} \leqslant \epsilon + C_0 = A, \quad u \in [0,\delta].$$

Thus

$$f(t,u) \leqslant A\, u \leqslant A\, \delta, \quad u \in [0,\delta],$$

which is the condition (3.12) in Theorem 3.5. For the case $c_\infty < \infty$, we have for $\epsilon = c_\infty - \frac{B}{\sigma}$, there exists $M > 0$ (M can be chosen arbitrarily large) such that

$$\inf_{t\in[0,1]} \frac{f(t,u)}{u} \geq -\epsilon + c_\infty = \frac{B}{\sigma}, \quad \frac{u}{\sigma} \geqslant M.$$

Hence,

$$f(t,u) \geqslant \frac{B}{\sigma}\, u \geqslant \frac{B}{\sigma}\, \sigma\, M = B\, M, \quad (t,u) \in [0,1] \times [\sigma\, M, \infty).$$

Consequently it is satisfied on $[1/4, 3/4] \times [\sigma\, M, M]$. If $c_\infty = \infty$, then one can easily show that the previous case holds. Thus condition (3.13) is satisfied and the result follows.

Assume that (b) is true, then for $\epsilon = c_0 - \frac{B}{\sigma}$, $c_0 < \infty$, there exists $M' > 0$ (M' can be chosen arbitrarily small) such that

$$\inf_{t\in[0,1]} \frac{f(t,u)}{u} \geq -\epsilon + c_0 = \frac{B}{\sigma}, \quad u \in [0, M'].$$

Hence,

$$f(t,u) \geqslant \frac{B}{\sigma}\, u \geqslant \frac{B}{\sigma}\, \sigma\, M' = B\, M', \quad (t,u) \in [0,1] \times [\sigma\, M', M'].$$

Consequently it is satisfied on $[1/4, 3/4] \times [\sigma M', M']$. For the case $c_0 = \infty$ the same argument holds. Thus condition (3.13) is satisfied. Also, since $C_\infty \in [0, A)$, then for $\epsilon = A - C_\infty$, there exists $\ell > 0$ such that

$$\sup_{t\in[0,1]} \frac{f(t,u)}{u} \leqslant \epsilon + C_\infty = A, \quad u \in [\ell, \infty). \tag{4.1}$$

We have the two cases:

(1) Assume that $\sup_{t\in[0,1]} f(t,u)$ is bounded, then there is $L > 0$ (L can be chosen arbitrarily large) such that

$$f(t,u) \leqslant L, \qquad (t,u) \in [0,1] \times [0,\infty).$$

Thus for $m = \frac{L}{A}$ (m can be chosen arbitrarily large), we get

$$f(t,u) \leqslant Am, \qquad (t,u) \in [0,1] \times [0,m].$$

(2) Assume that $\sup_{t\in[0,1]} f(t,u)$ is not bounded, hence there is $m \geqslant \ell$ and $t^* \in [0,1]$ such that

$$f(t,u) \leqslant f(t^*, m), \qquad (t,u) \in [0,1] \times [0,m].$$

By (4.1) we obtain

$$f(t,u) \leqslant f(t^*, m) \leqslant Am, \qquad (t,u) \in [0,1] \times [0,m].$$

i.e. the condition (3.12) is satisfied and the result follows. □

Application 4.2. Assume that both of the following hypotheses hold

(c) c_∞, $c_0 \in (\frac{B}{\sigma}, \infty]$,

(d) there exists a positive number k such that

$$f(t,u) \leqslant A\,k, \quad (t,u) \in [0,1] \times [0,k].$$

Then (3.1) and (1.4) has at least two solutions u_1, u_2 such that

$$0 < \|u_1\| < k < \|u_2\|.$$

Proof. Because of $c_\infty \in (\frac{B}{\sigma}, \infty]$, then as in Application 4.1, there exists M_1 (which can be taken arbitrarily large so that $M_1 > k$) such that

$$f(t,u) \geqslant B\,M_1, \quad (t,u) \in [1/4, 3/4] \times [\sigma\,M_1, M_1].$$

Also since $c_0 \in (\frac{B}{\sigma}, \infty]$, then as in Application 4.1, there is $M_2 > 0$ (which can chosen arbitrarily small so that $M_2 < k$) such that

$$f(t,u) \geqslant B\,M_2, \quad (t,u) \in [1/4, 3/4] \times [\sigma\,M_2, M_2].$$

Therefore, by Theorem 3.5, there exist two solutions u_1, u_2 such that

$$M_2 < \|u_1\| < k < \|u_2\| < M_1.$$

□

The following Application can be proved in a similar way to the above ones.

Application 4.3. Assume that both of the following hypotheses hold

(e) C_0, $C_\infty \in [0, A)$,

(f) there exists a positive number p such that

$$f(t,u) \geqslant B\,p, \quad (t,u) \in [1/4, 3/4] \times [\sigma\, p, p].$$

Then (3.1) and (1.4) has at least two solutions u_1, u_2 such that

$$0 < \|u_1\| < p < \|u_2\|.$$

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References

[1] M.H. Abu-Risha, M.H. Annaby, M.E. Ismail, Z.S. Mansour, *Linear q-difference equations,* Zeit. Anal. Anwend., **26**(2007), 481–494.

[2] R. Agarwal, D. O'Regan, V. Lakshmikantham, *Twin nonnegative solutions for higher-order boundary value problems,* Nonlinear Anal., **43**(2001), 61–73.

[3] A. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.

[4] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[5] P. Eloe, Y. Raffoul, D. Reid and K. Yin, *Positive solutions of nonlinear functinal difference equations*, Int. J. Comput. Math., **42**(2001), 639–646.

[6] M. El-Shahed, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, Abstr. Appl. Anal., **2007**, art. ID 10368, 8 pages.

[7] M. El-Shahed and H.A. Hassan, *Positive solutions of q-difference equation*, Proc. Amer. Math. Soc., **138**(2010), 1733-1738.

[8] L.H. Erbe and H. Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., **120**(1994), 743–748.

[9] J. Davis, J. Henderson and P. Wong, *General lidstone problems: Multiplicity and symmetry of solutions*, J. Math. Anal. Appl., **251**(2000), 527–548.

[10] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones,* Acadamic Press, San Diego, 1988.

[11] V. Hutson, J. Pym and M. Cloud, *Applications of Functional Analysis and Operator Theory*, Volume 200 in Mathematics in Science and Engineering, Elsevier B.V., Amsterdam, 2005.

[12] F. Li, and G. Han, *Generalization for Amann's and Leggett-Williams' three solution theorems and applications*, J. Math. Anal. Appl., **298**(2004), 638–654.

[13] W. Lian, F. Wong and C. Yeh, *On the existence of positive solutions of nonlinear second order differential equations*, Proc. Amer. Math. Soc., **124**(1996), 1117–1126.

[14] F.H. Jackson, *On q-definite integrals*, Quart. J. Pure and Appl. Math., **41**(1910), 193–203.

[15] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2002.

[16] M.A. Krasnoselskii, *Positive Solutions of Operator Equations,* Noordhoff, Groningen, 1964.

[17] M.K. Kwong, *On Krasnoselskii's cone fixed point theorem*, Fixed Point Theory Appl., art. ID 164537, 18 pages, (2008).

[18] D. O'Regan and R. Precup, *Compressionexpansion fixed point theorem in two norms and applications*, J. Math. Anal. Appl., **309**(2005), 383–391.

[19] Y. Sang, and H. Su, *Several sufficient of solvability for a nonlinear higher order three-point boundary value problem on time scale*, Appl. Math. Comput., **190**(2007), 566–575.

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