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POSITIVE SOLUTIONS FOR SECOND ORDER DIFFERENTIAL SYSTEMS WITH NONLOCAL CONDITIONS

XINAN HAO*, LISHAN LIU* AND YONGHONG WU**

*School of Mathematical Sciences, Qufu Normal University
Qufu 273165, Shandong, People's Republic of China
E-mail: haoxinan2004@163.com lls@mail.qfnu.edu.cn

**Department of Mathematics and Statistics
Curtin University of Technology, Perth,
6845 WA, Australia
E-mail: yhwu@maths.curtin.edu.au

Abstract. A class of second order differential systems with nonlocal conditions

$$\begin{cases} u''(t) + f_1(t, u(t), v(t)) = 0, \quad t \in (0, 1), \\ v''(t) + f_2(t, u(t), v(t)) = 0, \quad t \in (0, 1), \\ u'(0) = v'(0) = 0, \ u(1) = \alpha u(\eta), \ v(1) = \alpha v(\eta) \end{cases}$$

is considered under some conditions concerning the first eigenvalue of the relevant linear problem. By constructing a cone $K_1 \times K_2$ which is the Cartesian product of two cones and computing the fixed point index in $K_1 \times K_2$, the existence of positive solutions for the systems is established. An example is provided to illustrate the main results.

Key Words and Phrases: Positive solution, nonlocal boundary value problem, differential systems, cone, fixed point index.
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1. Introduction

In this paper, we study the existence of positive solutions for the following second order differential systems with nonlocal conditions

$$\begin{cases} u''(t) + f_1(t, u(t), v(t)) = 0, \quad t \in (0, 1), \\ v''(t) + f_2(t, u(t), v(t)) = 0, \quad t \in (0, 1), \\ u'(0) = v'(0) = 0, \ u(1) = \alpha u(\eta), \ v(1) = \alpha v(\eta), \end{cases} \tag{1.1}$$

where $\alpha, \eta \in (0, 1)$ are given constants, $f_i \in C(I \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ $(i = 1, 2)$, in which $I = [0, 1]$, $\mathbb{R}^+ = [0, +\infty)$.

The subject of nonlocal boundary value problems (BVPs) is an important branch of differential equations. Over the past decades, great efforts have been devoted to the study of nonlinear multi-point BVPs due to their theoretical challenge and extensive real-word application. Recently, much attention has been focused on investigating the existence and multiplicity of positive solutions for nonlinear nonlocal BVPs [1,9,11,13-18,20,21]. On the other hand, many problems in applied mathematics lead to the study of systems of differential equations, for instance, in population dynamics and chemical engineering [5,8]. Recently, many authors studied systems BVPs for differential equations, for example [2,7,10,12,19,22]. In [2], the authors established some fixed point theorems for systems of two equations in Banach spaces, and then applied them to the nonlinear differential equations two point BVP

$$\begin{cases} u'' + f(t,u,v) = 0, \quad t \in [0,\pi], \\ v'' + g(t,u,v) = 0, \quad t \in [0,\pi], \\ u(0) = u(\pi) = v(0) = v(\pi) = 0, \end{cases}$$

where the nonlinearities f and g in the two equations are sublinear. In [12], a new version of Krasnosel'skii's fixed point theorem in cones is obtained for systems of operator equations. As an application, the author investigated the existence and multiplicity of positive periodic solutions of nonlinear differential system

$$\begin{cases} u_1'(t) = -a_1(t)u_1(t) + \epsilon_1 f_1(t, u_1(t), u_2(t)), \\ u_2'(t) = -a_2(t)u_2(t) + \epsilon_2 f_2(t, u_1(t), u_2(t)), \end{cases}$$

where f_1 and f_2 are posed with some kinds of monotonicity conditions. In [10, 22] the following three point BVP for a second order differential systems

$$\begin{cases} u'' + f(t,v) = 0, \quad t \in (0,1), \\ v'' + g(t,u) = 0, \quad t \in (0,1), \\ u(0) = v(0) = 0,\ u(1) = \alpha u(\eta),\ v(1) = \alpha v(\eta) \end{cases}$$

is considered, where $\eta \in (0,1),\ 0 < \alpha\eta < 1$, and the existence and multiplicity results of positive solutions were shown by using Krasnosel'skii fixed point theorem and fixed point index theory in cones. In [7], Henderson et al. studied the positive solutions for second order differential systems.

Motivated by the above works, we will consider the generic second order differential systems with nonlocal conditions. In the present paper, the existence of positive solutions is obtained by means of the fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear problem. We remark here that we deal with our problem on the Cartesian product of two cones. By constructing a cone $K_1 \times K_2$ which is the Cartesian product of two cones in the Banach space $C[0,1]$ and computing the fixed point index in $K_1 \times K_2$, the existence of positive solutions is established. Since the nonlinear term is superlinear in one equation and sublinear in the other equation, and the assumptions made involve the first eigenvalue of the relevant linear problem, it seems to be difficult to prove our results by using the fixed point theorems of cone expansion and compression as were done in [7,10,12,19]. Our methods are different from those used in [2,7,10,12,19,22], but utilize some methods in [3, 4]. Our results generalize and extend some known results.

By a positive solution of problem (1.1), we mean a pair of functions $(u,v) \in C^2(I,\mathbb{R}^+) \times C^2(I,\mathbb{R}^+)$ which satisfies (1.1) and $u(t) > 0,\ v(t) > 0$ for all $t \in I$.

We impose the following assumptions:

($\mathbf{H_1}$) $\limsup_{u\to 0} \max_{t\in I} \sup_{v\in\mathbb{R}^+} \frac{f_1(t,u,v)}{u} < \lambda_1 < \liminf_{u\to\infty} \min_{t\in I} \inf_{v\in\mathbb{R}^+} \frac{f_1(t,u,v)}{u}$.

($\mathbf{H_2}$) $\limsup_{v\to \infty} \max_{t\in I} \sup_{u\in\mathbb{R}^+} \frac{f_2(t,u,v)}{v} < \lambda_1 < \liminf_{v\to 0} \min_{t\in I} \inf_{u\in\mathbb{R}^+} \frac{f_2(t,u,v)}{v}$.

($\mathbf{H_3}$) For a fixed $t \in I$ and $u \in \mathbb{R}^+$, $\sup_{v\in\mathbb{R}^+} f_1(t,u,v) < +\infty$.

($\mathbf{H_4}$) For a fixed $t \in I$ and $v \in \mathbb{R}^+$, $\sup_{u\in\mathbb{R}^+} f_2(t,u,v) < +\infty$.

In the assumptions ($\mathbf{H_1}$) and ($\mathbf{H_2}$), λ_1 is the first eigenvalue of the linear problem

$$\begin{cases} u''(t) + \lambda u(t) = 0, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta). \end{cases}$$

Based on the knowledge of linear differential equations, we know that λ_1 is the minimal positive root of the equation $\cos\sqrt{x} = \alpha \cos \eta\sqrt{x}$ and $0 < \lambda_1 < \frac{\pi^2}{4}$.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, the result of existence of positive solutions is established. Finally, we formulate a concrete example to illustrate our result.

2. Preliminaries

Let $E = C[0,1]$, then E is a Banach space with the norm $\|u\| = \max_{t\in I} |u(t)|$. Set $P = \{u \in E : u(t) \geq 0,\ t \in I\}$.

Lemma 2.1 ([17]). *Let $\alpha, \eta \in (0,1)$, then for $h \in P$, the problem*

$$\begin{cases} u''(t) + h(t) = 0, \quad t \in (0,1), \\ u'(0) = 0,\ u(1) = \alpha u(\eta) \end{cases}$$

has a unique solution $u(t) = \int_0^1 G(t,s)h(s)ds$, where the Green's function $G(t,s)$ is defined by

$$G(t,s) = \begin{cases} \dfrac{1-\alpha\eta}{1-\alpha} - s, & 0 \leq s \leq \eta,\ 0 \leq t \leq s, \\ \dfrac{1-s}{1-\alpha}, & \eta \leq s \leq 1,\ 0 \leq t \leq s, \\ \dfrac{1-\alpha\eta}{1-\alpha} - t, & 0 \leq s \leq \eta,\ s \leq t \leq 1, \\ \dfrac{1-\alpha s}{1-\alpha} - t, & \eta \leq s \leq 1,\ s \leq t \leq 1. \end{cases}$$

It is easy to verify that $G(t,s)$ has the following properties:

(i) $G(t,s)$ is a nonnegative continuous function on $I \times I$.

(ii) $G(t,s) \leq G(0,s)$ for any $t, s \in I$.

(iii) $G(t,s) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta} G(0,s)$ for any $t, s \in I$.

Let $K = \{u \in P : \min_{t\in I} u(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|u\|\}$, then K is a cone in E. For any $\lambda \in I,\ u, v \in P$, we define the mappings $A_v(\lambda,\cdot),\ B_u(\lambda,\cdot) : P \to P$ and $T_\lambda(\cdot,\cdot) : P\times P \to P\times P$ by

$$A_v(\lambda,u)(t) = \int_0^1 G(t,s)[(1-\lambda)u^2(s) + \lambda f_1(s,u(s),v(s))]ds, \quad t \in I,$$

$$B_u(\lambda,v)(t) = \int_0^1 G(t,s)[(1-\lambda)\sqrt{v(s)} + \lambda f_2(s,u(s),v(s))]ds, \quad t \in I,$$

and

$$T_\lambda(u,v)(t) = (A_v(\lambda,u)(t),\ B_u(\lambda,v)(t)), \quad t \in I.$$

It is well known that the positive solution of problem (1.1) is equivalent to the non-zero fixed point of T_1 in $K\times K$. We first give the following result for a completely continuous operator.

Lemma 2.2 *$T_\lambda : K\times K \to K\times K$ is completely continuous.*

Proof. From the continuity of f_1, f_2 and $G(t,s)$, by standard arguments, $A_v, B_u : P \to P$ are completely continuous, and hence T_λ is completely continuous. In the following, we shall prove that $T_\lambda(K\times K) \subset K\times K$. For $(u,v) \in K\times K$ and $t \in I$, it is clear that $A_v(\lambda,u)(t) \geq 0$, and

$$\begin{aligned} A_v(\lambda,u)(t) &= \int_0^1 G(t,s)[(1-\lambda)u^2(s) + \lambda f_1(s,u(s),v(s))]ds \\ &\leq \int_0^1 G(0,s)[(1-\lambda)u^2(s) + \lambda f_1(s,u(s),v(s))]ds, \end{aligned}$$

then

$$\|A_v(\lambda,u)\| \leq \int_0^1 G(0,s)[(1-\lambda)u^2(s) + \lambda f_1(s,u(s),v(s))]ds.$$

Thus

$$\begin{aligned} A_v(\lambda,u)(t) &= \int_0^1 G(t,s)[(1-\lambda)u^2(s) + \lambda f_1(s,u(s),v(s))]ds \\ &\geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\int_0^1 G(0,s)[(1-\lambda)u^2(s) + \lambda f_1(s,u(s),v(s))]ds \\ &\geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|A_v(\lambda,u)\|, \quad t \in I, \end{aligned}$$

i.e.

$$\min_{t\in I} A_v(\lambda,u)(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|A_v(\lambda,u)\|.$$

Therefore, $A_v(\lambda,u) \subset K$. Similarly, we can show that $B_u(\lambda,v) \subset K$. Hence, $T_\lambda(K\times K) \subset K\times K$.

The following two lemmas are needed in our arguments.

Lemma 2.3 ([4]). *Let E be a Banach space and let $K_i \subset E$ $(i = 1,2)$ be a closed convex cone in E. For $r_i > 0$ $(i = 1,2)$, denote $K_{r_i} = \{u \in K_i : \|u\| < r_i\}$, $\partial K_{r_i} =$*

$\{u \in K_i : \|u\| = r_i\}$. *Suppose* $A_i : K_i \to K_i$ *is completely continuous. If* $u_i \neq A_i u_i,\ u_i \in \partial K_{r_i}$, *then*

$$i(A, K_{r_1} \times K_{r_2}, K_1 \times K_2) = i(A_1, K_{r_1}, K_1) \cdot i(A_2, K_{r_2}, K_2),$$

where $A(u,v) := (A_1 u, A_2 v),\ (u,v) \in K_1 \times K_2$.

Lemma 2.4 ([6]). *Let* $A : K \to K$ *be a completely continuous operator, and denote* $K_r = \{u \in K : \|u\| < r\},\ \partial K_r = \{u \in K : \|u\| = r\}$.

(i) If $\|Au\| > \|u\|,\ u \in \partial K_r$, *then* $i(A, K_r, K) = 0$;

(ii) If $\|Au\| < \|u\|,\ u \in \partial K_r$, *then* $i(A, K_r, K) = 1$.

3. Main results

Theorem 3.1 *Assume that conditions* $(\mathbf{H_1}) - (\mathbf{H_4})$ *hold. Then problem* (1.1) *has at least one positive solution.*

Proof. For $\lambda = 0,\ A_v(0,u)(t) = \int_0^1 G(t,s)u^2(s)ds$. Taking

$$r_0 = (\int_0^1 G(0,s)ds)^{-1} = \frac{2(1-\alpha)}{1-\alpha\eta^2},$$

then for $u \in \partial K_r,\ 0 < r < r_0$, we have

$$\|A_v(0,u)\| \leq \int_0^1 G(0,s)u^2(s)ds \leq \|u\|^2 \int_0^1 G(0,s)ds = \frac{r^2}{r_0} < r = \|u\|.$$

By Lemma 2.4, we get that $i(A_v(0,\cdot), K_r, K) = 1,\ 0 < r < r_0$. On the other hand, taking $R_0 = \left[\frac{\alpha^3(1-\eta)^3}{(1-\alpha\eta)^3}\int_0^1 G(0,s)ds\right]^{-1}$ $(> r_0)$, then for $u \in \partial K_R,\ R > R_0$, we have

$$\begin{aligned}
\|A_v(0,u)\| \geq& \frac{\alpha(1-\eta)}{1-\alpha\eta}\int_0^1 G(0,s)u^2(s)ds \\
\geq& \frac{\alpha(1-\eta)}{1-\alpha\eta}\int_0^1 G(0,s)ds\left[\frac{\alpha(1-\eta)}{1-\alpha\eta}\|u\|\right]^2 \\
=& \left[\frac{\alpha(1-\eta)}{1-\alpha\eta}\right]^3 \|u\|^2 \int_0^1 G(0,s)ds \\
=& \frac{R^2}{R_0} > R = \|u\|.
\end{aligned}$$

By Lemma 2.4, we have $i(A_v(0,\cdot), K_R, K) = 0$. Similarly, we can show that

$$\|B_u(0,v)\| \leq \sqrt{\|v\|}\int_0^1 G(0,s)ds,$$

$$\|B_u(0,v)\| \geq \left[\frac{\alpha(1-\eta)}{1-\alpha\eta}\right]^{\frac{3}{2}} \sqrt{\|v\|}\int_0^1 G(0,s)ds.$$

Choose

$$\overline{R_0} = \left(\int_0^1 G(0,s)ds\right)^2, \quad \overline{r_0} = \left[\frac{\alpha(1-\eta)}{1-\alpha\eta}\right]^3 \left(\int_0^1 G(0,s)ds\right)^2,$$

then $0<\overline{r_0}<\overline{R_0}<+\infty$. For $\overline{r}\in(0,\overline{r_0}),\ \overline{R}>\overline{R_0}$, we get that

$$\|B_u(0,v)\|<\|v\|,\quad v\in\partial K_{\overline{R}},$$
$$\|B_u(0,v)\|>\|v\|,\quad v\in\partial K_{\overline{r}}.$$

Thus, by Lemma 2.4, for $\overline{r}\in(0,\overline{r_0}),\ \overline{R}>\overline{R_0}$, we have

$$i(B_u(0,\cdot),K_{\overline{r}},K)=0,\quad i(B_u(0,\cdot),K_{\overline{R}},K)=1.$$

The additivity of fixed point index and Lemma 2.3 yield that

$$\begin{aligned}&i(T_0,(K_R\setminus\overline{K_r})\times(K_{\overline{R}}\setminus\overline{K_{\overline{r}}}),K\times K)\\ =&i(A_v(0,\cdot),K_R\setminus\overline{K_r},K)\cdot i(B_u(0,\cdot),K_{\overline{R}}\setminus\overline{K_{\overline{r}}},K)=-1.\end{aligned}$$

In the following, we shall show that

$$i(T_\lambda,(K_{R_1}\setminus\overline{K_{r_1}})\times(K_{R_2}\setminus\overline{K_{r_2}}),K\times K)=i(T_0,(K_{R_1}\setminus\overline{K_{r_1}})\times(K_{R_2}\setminus\overline{K_{r_2}}),K\times K),$$

here $r_1\in(0,r_0),\ R_1>R_0,\ r_2\in(0,\overline{r_0}),\ R_2>\overline{R_0}$ will be determined later. By the homotopy invariance of fixed point index, we need only to prove that

$$(u,v)\neq T_\lambda(u,v),\quad (u,v)\in\partial\left[(K_{R_1}\setminus\overline{K_{r_1}})\times(K_{R_2}\setminus\overline{K_{r_2}})\right].\tag{3.1}$$

We divide the proof into four steps.

Step 1. By the first inequality of $(\mathbf{H_1})$, there exist $0<\varepsilon_1<\lambda_1$ and $0<r_1<\min\{r_0,\ \lambda_1-\varepsilon_1\}$ such that

$$f_1(t,u,v)\le(\lambda_1-\varepsilon_1)u,\quad t\in I,\ 0<u\le r_1,\ v\in\mathbb{R}^+.\tag{3.2}$$

We shall prove that $(u,v)\neq T_\lambda(u,v)$ for all $\lambda\in I$ and $(u,v)\in\partial K_{r_1}\times K$. In fact, if there exist $\lambda_0\in I$ and $(u_0,v_0)\in\partial K_{r_1}\times K$ such that $(u_0,v_0)=T_{\lambda_0}(u_0,v_0)$, then by the definition of operator T_{λ_0}, (u_0,v_0) satisfies

$$\begin{cases}-u_0''(t)=(1-\lambda_0)u_0^2(t)+\lambda_0 f_1(t,u_0,v_0),\quad t\in I,\\ u_0'(0)=0,\ u_0(1)=\alpha u_0(\eta).\end{cases}\tag{3.3}$$

By (3.2) and (3.3), we have

$$-u_0''(t)\le(1-\lambda_0)u_0^2(t)+\lambda_0(\lambda_1-\varepsilon_1)u_0(t)<(\lambda_1-\varepsilon_1)u_0(t),\quad t\in I.\tag{3.4}$$

Multiplying inequality (3.4) by $(\tan\sqrt{\lambda_1}-\tan\sqrt{\lambda_1}\eta)\cos\sqrt{\lambda_1}t$ and integrating over $[0,\eta]$, and multiplying inequality (3.4) by $(\tan\sqrt{\lambda_1}-\tan\sqrt{\lambda_1}t)\cos\sqrt{\lambda_1}t$ and integrating over $[\eta,1]$, respectively, and then using the additivity of integration, we have

$$\begin{aligned}&-\tan\sqrt{\lambda_1}\int_0^1u_0''(t)\cos\sqrt{\lambda_1}tdt+\tan\sqrt{\lambda_1}\eta\int_0^\eta u_0''(t)\cos\sqrt{\lambda_1}tdt\\ &\quad+\int_\eta^1u_0''(t)\sin\sqrt{\lambda_1}tdt\\ \le&(\lambda_1-\varepsilon_1)\tan\sqrt{\lambda_1}\int_0^1u_0(t)\cos\sqrt{\lambda_1}tdt\\ &\quad-(\lambda_1-\varepsilon_1)\tan\sqrt{\lambda_1}\eta\int_0^\eta u_0(t)\cos\sqrt{\lambda_1}tdt-(\lambda_1-\varepsilon_1)\int_\eta^1u_0(t)\sin\sqrt{\lambda_1}tdt.\end{aligned}$$

Using integration by parts in the left side and the fact that $\cos\sqrt{\lambda_1} = \alpha\cos\eta\sqrt{\lambda_1}$, through some tedious calculations we have

$$\begin{aligned}&\tan\sqrt{\lambda_1}\int_0^1 u_0(t)\cos\sqrt{\lambda_1}tdt - \tan\sqrt{\lambda_1}\eta\int_0^\eta u_0(t)\cos\sqrt{\lambda_1}tdt\\ &-\int_\eta^1 u_0(t)\sin\sqrt{\lambda_1}tdt \le 0,\end{aligned} \tag{3.5}$$

hence

$$\begin{aligned}&(\tan\sqrt{\lambda_1} - \tan\sqrt{\lambda_1}\eta)\int_0^\eta u_0(t)\cos\sqrt{\lambda_1}tdt\\ &\le \int_\eta^1 u_0(t)\sin\sqrt{\lambda_1}tdt - \tan\sqrt{\lambda_1}\int_\eta^1 u_0(t)\cos\sqrt{\lambda_1}tdt \le 0.\end{aligned}$$

But

$$(\tan\sqrt{\lambda_1} - \tan\sqrt{\lambda_1}\eta)\int_0^\eta u_0(t)\cos\sqrt{\lambda_1}tdt > 0,$$

which is a contradiction.
Step 2. By the second inequality of $(\mathbf{H_1})$, there exist $\varepsilon_2 > 0$ and $H > 0$ such that

$$f_1(t,u,v) \ge (\lambda_1+\varepsilon_2)u, \quad t\in I,\ u > H,\ v\in\mathbb{R}^+.$$

Let $C_1 = (\lambda_1+\varepsilon_2)^2 + \sup_{t\in I, u\in[0,H], v\in\mathbb{R}^+}|f_1(t,u,v) - (\lambda_1+\varepsilon_2)u|$, by $(\mathbf{H_3})$ we know $C_1 < +\infty$. Thus it is easy to see that

$$f_1(t,u,v) \ge (\lambda_1+\varepsilon_2)u - C_1, \quad t\in I,\ u\in\mathbb{R}^+,\ v\in\mathbb{R}^+, \tag{3.6}$$

$$u^2 \ge (\lambda_1+\varepsilon_2)u - (\lambda_1+\varepsilon_2)^2 \ge (\lambda_1+\varepsilon_2)u - C_1,\ u\in\mathbb{R}^+. \tag{3.7}$$

If there exist $\lambda_0\in I$ and $(u_0,v_0)\in K\times K$ such that $(u_0,v_0) = T_{\lambda_0}(u_0,v_0)$, then (u_0,v_0) satisfies (3.3). By (3.3), (3.6) and (3.7), we have

$$-u_0''(t) \ge (\lambda_1+\varepsilon_2)u_0(t) - C_1, \quad t\in I.$$

Following the procedure used in (3.5), we have

$$\begin{aligned}&\tan\sqrt{\lambda_1}\int_0^1 u_0(t)\cos\sqrt{\lambda_1}tdt - \tan\sqrt{\lambda_1}\eta\int_0^\eta u_0(t)\cos\sqrt{\lambda_1}tdt\\ &\qquad\qquad -\int_\eta^1 u_0(t)\sin\sqrt{\lambda_1}tdt \le \frac{C_1(1-\alpha)}{\varepsilon_2\sqrt{\lambda_1}\alpha\cos\sqrt{\lambda_1}\eta},\end{aligned}$$

consequently

$$(\tan\sqrt{\lambda_1} - \tan\sqrt{\lambda_1}\eta)\int_0^\eta u_0(t)\cos\sqrt{\lambda_1}tdt \le \frac{C_1(1-\alpha)}{\varepsilon_2\sqrt{\lambda_1}\alpha\cos\sqrt{\lambda_1}\eta},$$

and

$$\frac{\alpha(1-\eta)}{1-\alpha\eta}(\tan\sqrt{\lambda_1} - \tan\sqrt{\lambda_1}\eta)\|u_0\|\int_0^\eta \cos\sqrt{\lambda_1}tdt \le \frac{C_1(1-\alpha)}{\varepsilon_2\sqrt{\lambda_1}\alpha\cos\sqrt{\lambda_1}\eta}.$$

So,

$$\|u_0\| \le \frac{C_1(1-\alpha)(1-\alpha\eta)}{\varepsilon_2\sqrt{\lambda_1}\alpha(1-\eta)\tan\sqrt{\lambda_1}\eta\sin\sqrt{\lambda_1}(1-\eta)}. \tag{3.8}$$

If we choose $\overline{R} = \frac{C_1(1-\alpha)(1-\alpha\eta)}{\varepsilon_2\sqrt{\lambda_1}\alpha(1-\eta)\tan\sqrt{\lambda_1}\eta\sin\sqrt{\lambda_1}(1-\eta)} + 1$ and $R_1 \geq \max\{R_0,\ \overline{R}\}$, then $(u,v) \neq T_\lambda(u,v)$ for $t \in I$ and $(u,v) \in \partial K_{R_1} \times K$.

Step 3. By the second inequality of $(\mathbf{H_2})$, there exist $\varepsilon_3 > 0$ and $0 < h < \frac{1}{(\lambda_1+\varepsilon_3)^2}$ such that

$$f_2(t,u,v) \geq (\lambda_1 + \varepsilon_3)v, \quad t \in I,\ 0 \leq v \leq h,\ u \in \mathbb{R}^+. \tag{3.9}$$

By $0 \leq v \leq h < \frac{1}{(\lambda_1+\varepsilon_3)^2}$, we have

$$\sqrt{v} \geq (\lambda_1 + \varepsilon_3)v, \quad 0 \leq v \leq h. \tag{3.10}$$

Choosing $0 < r_2 < \min\{\overline{r_0},\ h\}$, by (3.9), (3.10), and proceeding as in Step 1, we can get that $(u,v) \neq T_\lambda(u,v)$ for $t \in I$ and $(u,v) \in K \times \partial K_{r_2}$.

Step 4. By the first inequality of $(\mathbf{H_2})$, there exist $0 < \varepsilon_4 < \lambda_1$ and $M > 0$ such that

$$f_2(t,u,v) \leq (\lambda_1 - \varepsilon_4)v, \quad t \in I,\ v > M,\ u \in \mathbb{R}^+.$$

Taking $C_2 = \frac{1}{4(\lambda_1-\varepsilon_4)} + \sup_{t\in I, v\in[0,M], u\in\mathbb{R}^+} f_2(t,u,v)$, by $(\mathbf{H_4})$ we know $C_2 < +\infty$. Then it is easy to see that

$$f_2(t,u,v) \leq (\lambda_1 - \varepsilon_4)v + C_2, \quad t \in I,\ u \in R^+,\ v \in \mathbb{R}^+, \tag{3.11}$$

$$\sqrt{v} \leq (\lambda_1 - \varepsilon_4)v + C_2, \quad v \in \mathbb{R}^+. \tag{3.12}$$

As for the arguments in Step 2, if $(u_0,v_0) = T_{\lambda_0}(u_0,v_0)$ for $\lambda_0 \in I$ and $(u_0,v_0) \in K \times K$, we can show that

$$\|v_0\| \leq \frac{C_2(1-\alpha)(1-\alpha\eta)}{\varepsilon_4\sqrt{\lambda_1}\alpha(1-\eta)\tan\sqrt{\lambda_1}\eta\sin\sqrt{\lambda_1}(1-\eta)} := \widetilde{R}.$$

Choose $R_2 > \max\{\overline{R_0},\ \widetilde{R}\}$, then we have $(u,v) \neq T_\lambda(u,v)$ for $\lambda \in I$ and $(u,v) \in K \times \partial K_{R_2}$.

Combining the above four Steps, we have proved that (3.1) is valid, then

$$\begin{aligned} &i(T_1, (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}}), K \times K) \\ =&i(T_0, (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}}), K \times K) = -1. \end{aligned}$$

Therefore, T_1 has a fixed point $(u^*,v^*) \in (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}})$. Obviously, (u^*,v^*) is a positive solution of problem (1.1). This completes the proof.

Remark 3.2 *Because f_1 is superlinear and f_2 is sublinear, it is difficult to construct a single cone in product space and use the norm-type expansion and compression theorem. Since f_1 and f_2 in the two equations have different features, the solution operator corresponding to one of the equations has the properties of cone expansion and the other equation has the properties of cone compression.*

Remark 3.3 *Through similar but more complicated arguments, we could obtain a similar result for the following m-point BVP of second order differential systems*

$$\begin{cases} u''(t) + f(t,u(t),v(t)) = 0, \quad t \in (0,1), \\ v''(t) + g(t,u(t),v(t)) = 0, \quad t \in (0,1), \\ u'(0) = v'(0) = 0,\ u(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\eta_i),\ v(1) = \displaystyle\sum_{i=1}^{m-2} \beta_i v(\eta_i), \end{cases}$$

where $\alpha_i, \eta_i \in (0,1)$ $(i = 1, 2, \cdots, m-2)$ *are given constants,* $f, g \in C(I \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

4. An example

Consider the following three point BVP

$$\begin{cases} u''(t) + (1+t^2)u^2 + u^3 \dfrac{2+\sin v}{1+v} = 0, \quad t \in (0,1), \\ v''(t) + (2-t)v^{\frac{1}{3}} + v^{\frac{1}{2}} \dfrac{3+\sin u}{1+u} = 0, \quad t \in (0,1), \\ u'(0) = v'(0) = 0, \ u(1) = \dfrac{2}{3} u\left(\dfrac{1}{2}\right), \ v(1) = \dfrac{2}{3} v\left(\dfrac{1}{2}\right). \end{cases} \tag{4.1}$$

Problem (4.1) can be regarded as a problem of the form (1.1) with $\alpha = \frac{2}{3}$, $\eta = \frac{1}{2}$,

$$f_1(t,u,v) = (1+t^2)u^2 + u^3 \frac{2+\sin v}{1+v}, \quad f_2(t,u,v) = (2-t)v^{\frac{1}{3}} + v^{\frac{1}{2}} \frac{3+\sin u}{1+u}.$$

It is easy to verify that the conditions $(\mathbf{H_3})$ and $(\mathbf{H_4})$ hold. In addition,

$$\begin{aligned} 0 \le \limsup_{u\to 0} \max_{t\in I} \sup_{v\in\mathbb{R}^+} \frac{f_1(t,u,v)}{u} &\le \limsup_{u\to 0}(2u + 3u^2) = 0, \\ \liminf_{u\to\infty} \min_{t\in I} \inf_{v\in\mathbb{R}^+} \frac{f_1(t,u,v)}{u} &\ge \liminf_{u\to\infty} u = +\infty, \\ 0 \le \limsup_{v\to\infty} \max_{t\in I} \sup_{u\in\mathbb{R}^+} \frac{f_2(t,u,v)}{v} &\le \limsup_{v\to\infty}(2v^{-\frac{2}{3}} + 3v^{-\frac{1}{2}}) = 0, \\ \liminf_{v\to 0} \min_{t\in I} \inf_{u\in\mathbb{R}^+} \frac{f_2(t,u,v)}{v} &\ge \limsup_{v\to 0} v^{-\frac{2}{3}} = +\infty, \end{aligned}$$

then conditions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied. Hence, problem (4.1) has at least one positive solution by Theorem 3.1.

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