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EXISTENCE OF SOLUTIONS OF A NONLINEAR THIRD ORDER BOUNDARY VALUE PROBLEM

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Abstract. This paper deals with a third-order three-point boundary value problem. Applying the Banach contraction principle and Leray Schauder nonlinear alternative, we establish the existence of solutions for the considered problem.
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1. Introduction

In this work we investigate the existence of solutions for the following third-order three point boundary value problem (BVP):

$$(P_1)\ \begin{cases} u'''(t) + f(t, u(t)) = 0,\ 0 < t < 1 \\ u(0) = \alpha u(1), u'(1) = \beta u'(\eta), u'(0) = 0, \end{cases}$$

where $\eta \in (0,1)$, α, $\beta \in \mathbb{R}$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function. We mainly use the Banach contraction principle and Leray Schauder nonlinear alternative to prove the existence and uniqueness results. For this, we formulated the boundary value problem (P_1) as fixed point problem. We also study the compactness of solutions set.

Third-order problems have been intensively studied recently by Graef and Yang [6], Guo et al [9], Hopkins and Kosmatov [10], and Sun [13]. Applying Krasnoselskii and Leggett and Williams fixed point theorems, Anderson in [3] considered the three-point boundary value problem for the same equation equation in the case $t_1 < t < t_2$ and the three point conditions $u(t_1) = u'(t_2) = 0$, $\gamma u(t_3) + \delta u''(t_3) = 0$. Excellent surveys on theoretical results can be found in Agarwal [1] and R Ma [12]. More results can be found in [2,4,7,8,11].

2. Existence and Uniqueness results

Let $E = C([0,1],\mathbb{R})$, with the norm $||y|| = \max\limits_{t\in[0,1]} |y(t)|$. We assume that

$$\zeta = (1-\alpha)(1-\beta\eta) \neq 0.$$

Now we start by solving an auxiliary problem.

Lemma 2.1. *Let $y \in L^1([0,1],\mathbb{R})$. The function*

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds - \frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta-s)\, y(s)\, ds \tag{2.1}$$
$$+\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha)+\alpha\beta\eta(1-s)+\alpha s\right) y(s)ds$$

is the unique solution of the BVP

$$(P_2)\ \begin{cases} u'''(t) + y(t) = 0, 0<t<1 \\ u(0) = \alpha u(1), u'(1) = \beta u'(\eta), u'(0) = 0. \end{cases}$$

Proof. Rewriting the differential equation as $u'''(t) = -y(t)$ and integrating three times, we obtain $u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds + At^2 + Bt + C$, the constants A, B and C are given by the three point boundary conditions.

To solve the BVP (P_1) we make the following hypothesis:
(i) $t \to f(t,x)$ is measurable for all $x \in \mathbb{R}$.
(ii) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$.

Theorem 2.2. *Assume that there exists a nonnegative function $k \in L^1([0,1],\mathbb{R}_+)$ such that*

$$|f(t,x) - f(t,y)| \leq k(t)\,|x-y|, \forall x,y \in \mathbb{R}, t\in[0,1]. \tag{2.2}$$

and

$$A = \int_0^1 \left[(|\zeta| + |\alpha\beta|)(1-s)^2 + (1+2|\alpha|)(|\beta|+1)(1-s)\right] k(s)ds < 2|\zeta|,$$

then the BVP (P_1) has a unique solution u in E.

Proof. We transform the boundary value problem (P_1) to a fixed point problem, define the integral operator $T : E \to E$ by

$$Tu(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(s,u(s))\, ds \tag{2.3}$$
$$-\frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta-s) f(s,u(s))\, ds$$
$$+\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha)+\alpha\beta\eta(1-s)+\alpha s\right) f(s,u(s))\, ds,$$

We shall prove that T is a contraction. Let $u, v \in E$, then

$$|Tu(t) - Tv(t)| \leq \frac{1}{2}\int_0^1 (1-s)^2\,|f(s,u(s)) - f(s,v(s))|\, ds \tag{2.4}$$

$$+\frac{1}{2}\left|\frac{\beta}{\zeta}\right|(1+2|\alpha|)\int_0^1(1-s)|f(s,u(s))-f(s,v(s))|\,ds$$

$$+\frac{1}{2|\zeta|}\int_0^1(1-s)(1+2|\alpha|+|\alpha\beta|(1-s))|f(s,u(s))-f(s,v(s))|\,ds$$

Using (2.2) we obtain

$$|Tu(t)-Tv(t)|\leq\frac{1}{2}\int_0^1(1-s)^2k(s)|u(s)-v(s)|\,ds \tag{2.5}$$

$$+\frac{1}{2}\left|\frac{\beta}{\zeta}\right|(1+2|\alpha|)\int_0^1(1-s)k(s)|u(s)-v(s)|\,ds$$

$$+\frac{1}{2|\zeta|}\int_0^1(1-s)(1+2|\alpha|+|\alpha\beta|(1-s))k(s)|u(s)-v(s)|\,ds$$

$$\leq\frac{1}{2|\zeta|}\int_0^1\Big[(|\zeta|+|\alpha\beta|)(1-s)^2+(1+2|\alpha|)(|\beta|+1)(1-s)\Big]k(s)|u(s)-v(s)|\,ds$$

taking the supremum it yields $\|Tu-Tv\|<\|u-v\|$. Consequently T is a contraction, so, it has a unique fixed point which is the unique solution of the BVP (P_1).

Now we give some existence results for the BVP (P_1).

Theorem 2.3. *Assume that $f(t,0)\neq 0$ and there exist nonnegative functions $k,h\in L^1([0,1],\mathbb{R}_+)$ such that*

$$|f(t,x)|\leq k(t)|x|+h(t),\ (t,x)\in[0,1]\times\mathbb{R}, \tag{2.6}$$

$$\left(1+\eta\frac{|\alpha\beta|}{2|\zeta|}\right)\int_0^1(1-s)^2k(s)\,ds+\frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta(\eta-s)k(s)\,ds \tag{2.7}$$

$$+\frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1(1-s)k(s)\,ds<1$$

Then the BVP (P_1) has at least one nontrivial solution $u^*\in E$.

To prove this theorem, we apply Leray Schauder nonlinear alternative:

Lemma 2.4. [5]. *Let F be a Banach space and Ωa bounded open subset of F, $0\in\Omega$. $T:\overline{\Omega}\to F$ be a completely continuous operator. Then, either there exists $x\in\partial\Omega$, $\lambda>1$ such that $T(x)=\lambda x$, or there exists a fixed point $x^*\in\overline{\Omega}$.*

Proof. First let us define the open bounded $\Omega\subset E$. Set

$$M=\left(1+\frac{|1+2\alpha|}{2|\zeta|}\right)\int_0^1(1-s)^2k(s)ds+\frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta(\eta-s)k(s)ds$$

$$+\frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1(1-s)k(s)ds$$

and

$$N=\left(1+\frac{|1+2\alpha|}{2|\zeta|}\right)\int_0^1(1-s)^2h(s)ds+\frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta(\eta-s)h(s)ds$$

$$+\frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1(1-s)h(s)ds.$$

By hypothesis (2.7) we know that $M < 1$. Since $f(t,0) \neq 0$, then there exists an interval $[\sigma, \tau] \subset [0,1]$ such that $\min_{\sigma \leq t \leq r} |f(t,0)| > 0$. Since $h(t) \geq |f(t,0)|$, $\forall t \in [0,1]$, then $N > 0$. Let $m = \dfrac{N}{1-M}$, then bounded open set Ω is defined by $\Omega = \{u \in C[0,1] : \|u\| < m\}$.

The proof of T completely continuous operator on Ω, will be done in some steps.

(i) T is continuous.

Indeed, let (u_n) be a sequence that converges to u in E. Then

$$|Tu_n(t) - Tu(t)| \leq \frac{1}{2}\int_0^1 (1-s)^2 |f(s,u_n(s)) - f(s,u(s))|\, ds \tag{2.8}$$

$$+\frac{1}{2}\left|\frac{\beta}{\zeta}\right|(1+2|\alpha|)\int_0^1 (1-s)|f(s,u_n(s)) - f(s,u(s))|\, ds$$

$$+\frac{1}{2|\zeta|}\int_0^1 (1-s)(1+2|\alpha| + |\alpha\beta|(1-s))|f(s,u_n(s)) - f(s,u(s))|\, ds$$

$$\leq \left(1 + \frac{(|\beta|+1)(1+2|\alpha|) + |\alpha\beta|}{|\zeta|}\right)\|f(.,u_n(.)) - f(.,u(.))\|_{L_1}$$

which implies $\|Tu_n - Tu\| \to 0$, as $n \to \infty$.

(ii)Let $B_r = \{u \in E; \|u\| \leq r\}$ be a bounded subset. We prove that $T(\Omega \cap B_r)$ relatively compact:

a)For some $u \in \Omega \cap B_r$ and using (2.6) we have

$$\|Tu\| \leq M\|u\| + N \leq Mr + N,$$

yielding that $T(\Omega \cap B_r)$ is uniformly bounded.

b) $T(\Omega \cap B_r)$ is equicontinuous. Indeed for all $t_1, t_2 \in [0,1]$, $u \in \Omega$, we have by applying (2.6)

$$\|Tu(t_1) - Tu(t_2)\| \leq M\|u(t_1) - u(t_2)\|,$$

when $t_1 \to t_2$, then $\|Tu(t_1) - Tu(t_2)\|$ tends to 0, consequently $T(\Omega \cap B_r)$ is equicontinuous. From Arzela -Ascoli Theorem we deduce that T is completely continuous operator.

Now we can able to apply Leray Schauder nonlinear alternative for $T : \overline{\Omega} \to E$. Assume that $u \in \partial\Omega$, $\lambda > 1$ such $Tu = \lambda u$, then

$$\lambda m = \lambda ||u|| = ||Tu|| = \max_{0\leq t\leq 1} |(Tu)(t)| \leq$$

$$||u||\left[\left(1+\eta\frac{|\alpha\beta|}{2|\zeta|}\right)\int_0^1 (1-s)^2 k(s)\, ds + \frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta (\eta - s)k(s)\, ds\right.$$

$$\left.+\frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1 (1-s)\, k(s)ds\right] +$$

$$\left(1+\eta\frac{|\alpha\beta|}{2|\zeta|}\right)\int_0^1 (1-s)^2 h(s)ds + \frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta (\eta - s)\, h(s)ds$$

$$+\frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1 (1-s)\, h(s)ds = M\|u\| + N.$$

From this we obtain $\lambda \leq M + \frac{N}{m} = 1$, this contradicts the fact that $\lambda > 1$. By Lemma 4 we conclude that the operator T has a fixed point $u^* \in \overline{\Omega}$ and then the BVP (P_1) has a nontrivial solution $u^* \in E$.

Theorem 2.5. *The set of solutions of the BVP (P_1) is compact.*

Proof. Let $\Sigma = \{u \in E; u \text{ solution of BVP } (P_1)\}$, let us show, by using Arzela-Ascoli Theorem (Any subset of E is compact if and only if it is bounded, closed and equicontinuous), that Σ is compact.

(i) Let $(u_n)_{n\geq 1}$ be a sequence in Σ, then

$$u_n(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(s, u_n(s))\, ds \tag{2.10}$$
$$-\frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta - s) f(s, u_n(s))\, ds$$
$$+\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha)\right) + \alpha\beta\eta\left((1-s)+\alpha s\right) f(s, u_n(s))\, ds.$$

Using the same reasoning as in the proof of Theorem 2.3, we prove that Σ is bounded and equicontinuous. Now we prove that Σ is closed. From the condition (2.6) we have

$$|f(t, u_n)| \leq k(t)|u_n| + h(t) \leq k(t)\, m + h(t) = g_m(t). \tag{2.11}$$

The Lebesgue dominated convergence Theorem and the assumption(ii) on f guaranty that

$$u(t) = \lim u_n(t) =$$
$$-\frac{1}{2}\int_0^t (t-s)^2 f(s, u(s))\, ds - \frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta - s) f(s, u(s))\, ds$$
$$+\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha) + \alpha\beta\eta(1-s) + \alpha s\right) f(s, u(s))\, ds, \forall t \in [0,1]$$

hence $u \in \Sigma$ and consequently Σ is compact.

Example 2.6. Consider the three point BVP

$$\begin{cases} u''' + 2\frac{\sqrt{3}u^3}{3+u^4}\sqrt{t} + te^{-t} = 0, \quad 0 < t < 1 \\ u(0) = -2u(1), u'(1) = 3u'\left(\frac{1}{2}\right), u'(0) = 0 \end{cases} \tag{2.12}$$

We have $\alpha = -2$, $\beta = 3, \eta = \frac{1}{2}$, $\zeta = \frac{3}{2}$, $f(t,x) = 2\frac{\sqrt{3}u^3}{3+u^4}\sqrt{t} + te^{-t}$ and, $|f(t,x)| \leq k(t)|x| + h(t)$, where $k(t) = \sqrt{t}$, $h(t) = te^{-t}, k,\ h \in L_1([0,1], \mathbb{R}_+)$. Using Theorem 2.3, it yields

$$M = \frac{4}{3}\int_0^1 (1-s)^2\sqrt{s}ds + \frac{5}{3}\int_0^1 (1-s)\sqrt{s}ds + 5\int_0^\eta (\eta - s)\sqrt{s}ds$$
$$= 0,88286 < 1$$

Then BVP (2.12) has at least one nontrivial solution u^* in E.

Example 2.7. Consider the three point BVP

$$\begin{cases} u''' + \frac{tu}{\sqrt{3}\sqrt{t^2+1}} - e^t + \cos t^2 = 0, 0 < t < 1, \\ u(0) = \frac{1}{3}u(1), u'(1) = -\frac{1}{2}u'\left(\frac{1}{4}\right), u'(0) = 0 \end{cases} \tag{2.13}$$

where $\alpha = \frac{1}{3}, \beta = -\frac{1}{2}$, $\eta = \frac{1}{4}$, $|\zeta| = \frac{3}{4}$. Applying Theorem 2.2, we get

$$|f(t,x) - f(t,y)| \le k(t)\,|x-y|\,, \forall x, y \in \mathbb{R}, t \in [0,1]\,.$$

where $k(t) = \frac{t}{\sqrt{3}\sqrt{t^2+1}}$. By simple calculus we get

$$\begin{aligned} A &= \int_0^1 \frac{11}{12}(1-s)^2 \frac{s}{\sqrt{3}\sqrt{s^2+1}} + \frac{5}{2}(1-s)\frac{s}{\sqrt{3}\sqrt{s^2+1}} ds \\ &= 0.25 < 3/2 \end{aligned}$$

then BVP (2.13) has a unique solution in E.

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