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THE AZIMI HAGLER SPACE HAS THE WEAK FIXED POINT PROPERTY

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Abstract. We prove that the Azimi Hagler space fails Opial's property and GLD but has GGLD and thus wfpp.

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In [1] Azimi and Hagler defined a space which is hereditarily l_1 and is not a Schur space. This paper shows some properties of this space related to the fixed point property, arriving finally at the result that it has the weak fixed point property.

We start by recalling some known concepts: Let $(X, \|\cdot\|)$ be a Banach space and $C \subset X$. A mapping $T : C \to X$ is non expansive, if for every $x, y \in C$, one has $\|Tx - Ty\| \leq \|x - y\|$. X is said to satisfy the (weak) fixed point property, if every mapping $T : C \to C$ where C is a non empty bounded convex (non empty weakly compact) subset of X, has a fixed point.

The Azimi Hagler is defined as follows:

Definition 1. *Let $\{F_1, ..., F_n, ...\}$ be a collection of finite intervals contained in $\mathbb{N}$; we say that this sequence is admissible if $\max F_i < \min F_{i+1}$ for $i = 1, 2, 3, ...$ Let $\{\alpha_i\} \subset \mathbb{R}$ satisfying*

(1) $\alpha_1 = 1$ *and* $\alpha_{i+1} < \alpha_i$ *for* $i = 1, 2, ...$
(2) $\lim_i \alpha_i = 0$.
(3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

For $x = (t_1, t_2, t_3, ...) \in c_{00}$ let

$$\|x\| = \max \sum_{i=1}^{n} \alpha_i |\langle x, F_i \rangle| ,$$

where the max is taken over all n and admissible sequences $\{F_1, ..., F_n\}$ and for any set $F \subset \mathbb{N}$, $\langle x, F \rangle = \sum_{j \in F} t_j$.

The Azimi Hagler space X is the completion of $(c_{00}, \|\cdot\|)$ where c_{00} is the vector space of sequences with only a finite number of non zero entries.

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Recall that a Banach space X is asymptotically isometric to l_1 if it has a basis $\{x_n\}$ and there exists a null sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ such that for every sequence $\{t_n\}$ in l_1,

$$\sum_n (1-\varepsilon_n)|t_n| \leq \left\| \sum t_n x_n \right\| \leq \sum (1+\varepsilon_n)|t_n|$$

and X is called hereditarily (asymptotically isometric to) l_1, if every subspace contains in turn a subspace that is isomorphic (asymptotically isometric) to l_1. Finally, X is called a Schur space if every weakly convergent sequence in X converges in norm.

Lemmas 1, 2 and theorem 1 were proved in [1] and will be useful for our purposes.

Lemma 1. *The canonical sequence $\{e_i\}$ is a bimonotone Schauder basis for X, the sequence $\{e_{2n} - e_{2n-1}\}$ is weakly null and* $\left\| \sum_{i=1}^{k} (e_{2i-1} - e_{2i}) \right\| = \sum_{i=1}^{2k} \alpha_i$.

Lemma 2. *For $x \in X$ let $s(x) = \max |\langle x, G \rangle|$, where the max is taken over all intervals in $\mathbb{N}$. Suppose $\{u_i\}$ is a block basic sequence of $\{e_i\}$ with $\|u_i\| = 1$ and support contained in the interval G_i with $G_i \cap G_{i+1} = \emptyset$ for $i = 1, 2, \ldots$; then if $\lim_i s(u_i) = 0$, there is a subsequence $\{v_k\}$ of $\{u_i\}$ equivalent to the usual basis of l_1.*

Theorem 1. *X is hereditarily l_1 and fails the Schur property.*

From the above theorem, we get the following result.

Corollary 1. *X does not have the fixed point property (FPP).*

Proof. The proof in [1] of theorem 1 can be easily modified to show that X is actually hereditarily asymptotically isometric to l_1 and this by a result of Dowling et al. [2], implies that X does not have the fixed point property. □

Now we will analyze in Azimi Hagler's space several properties that imply the weak fixed point property (wfpp) and finally arrive to the conclusion that X indeed possesses this property. Since X is not reflexive we only study properties that don't imply reflexivity.

Recall the following definitions.

Definition 2. *Let Y be a Banach space.*

(1) *Y has the Opial property if for every weakly null sequence $\{x_n\} \subset Y$ and for every $x \in Y$, $x \neq 0$*

$$\limsup_n \|x_n\| < \limsup_n \|x_n - x\| .$$

(2) *Let Y have a Schauder basis and let $\{P_n\}$ be the sequence of natural projections. Y has the Gossez-Lami Dozo property (GLD)* [7]*, if for every $\varepsilon > 0$ there exists $r > 0$ such that for every $x \in Y$ and $n \in \mathbb{N}$ we have*

$$\|P_n x\| = 1 \text{ and } \|(I - P_n)x\| \geq \varepsilon \text{ imply } \|x\| \geq 1 + r.$$

(3) *Y has the generalized Gossez-Lami Dozo property (GGLD)* [8]*, if for every weakly null subsequence $\{y_n\}$ such that $\lim_n \|y_n\| = 1$ we have that*

$$\limsup_m \limsup_n \|y_n - y_m\| > 1.$$

(4) *A bounded sequence* $\{y_n\} \subset Y$ *is called diametral if*

$$\lim_n d\left(y_{n+1}, conv\left\{y_i\right\}_{i=1}^{n}\right) = diam\left\{y_n\right\}_{n=1}^{\infty}.$$

(5) Y *has weak normal structure (WNS) if there is no weakly null diametral sequence in* Y.

(6) *The coefficient* $R(Y)$ [6], *is defined by*

$$R(Y) = \sup\left\{\liminf_n \|x_n + x\| : \{x\}, \{x_n\} \subset B_Y, x_n \underset{w}{\to} 0\right\}.$$

It is known that the following implications hold

$$\begin{array}{ccccccc} & & & & R(X)<2 & & \\ & & & & & \not\nwarrow\ \searrow & \\ \text{GLD} & \begin{smallmatrix}\longrightarrow\\ \not\longleftarrow\end{smallmatrix} & \text{GGLD} & \begin{smallmatrix}\longrightarrow\\ \not\longleftarrow\end{smallmatrix} & \text{WNS} & \begin{smallmatrix}\longrightarrow\\ \not\longleftarrow\end{smallmatrix} & \text{wfpp} \\ & & & & \not\uparrow & \nearrow & \\ & & & & \text{Opial} & & \end{array}$$

We will see that Azimi Hagler's space X does not have the Opial property nor GLD and $R(X) = 2$, but X has GGLD and thus WNS and wfpp.

Lemma 3. *X does not have the Opial property.*

Proof. Let $\{\alpha_i\}$ as in definition 1, $0 < \delta < 1 - \alpha_3$, $x = \delta e_1$ and $x_n = e_{2n} - e_{2n+1}$. By lemma 1 $\{x_n\}$ is weakly null and $\|x_n\| = 1 + \alpha_2$. For $n > 1$, $\|x - x_n\| = \max\{\delta + \alpha_2 + \alpha_3, 1 - \delta + \alpha_2, 1 + \alpha_2\} = \max\{\delta + \alpha_2 + \alpha_3, 1 + \alpha_2\} = 1 + \alpha_2$. □

Lemma 4. *X does not have GLD.*

Proof. Let $\varepsilon = 1$ and for $k \in \mathbb{N}$, $b_k = \left(\sum_{i=1}^{2k} \alpha_i\right)^{-1}$, $x_k = b_k \sum_{i=1}^{k} (e_{2i} - e_{2i-1}) + e_{2k+1}$. Then $\|P_{2k} x_k\| = 1 = \|(I - P_{2k}) x_k\|$ and

$$1 + \alpha_{2k+1} \leq \|x_k\| =$$

$$\max\left\{\max_{1\leq r\leq 2k}\left[b_k\left(\sum_{i=1}^{r}\alpha_i\right) + \alpha_{r+1}\right], \max_{1\leq r\leq 2k-1}\left[b_k\left(\sum_{i=1}^{r}\alpha_i\right) + (1+b_k)\,\alpha_{r+1}\right]\right\} =$$

$$= \max_{1\leq r\leq 2k-1}\left[b_k\left(\sum_{i=1}^{r}\alpha_i\right) + (1+b_k)\,\alpha_{r+1}\right].$$

Suppose there is a subsequence $\{x_{k_s}\}_s$ and a fixed $1 \leq r \leq 2k_s - 1$ such that

$$\|x_{k_s}\| = b_{k_s}\left(\sum_{i=1}^{r}\alpha_i\right) + (1 + b_{k_s})\,\alpha_{r+1}.$$

Then, since $\lim_s b_{k_s} = 0$, $1 \leq \lim_s \|x_{k_s}\| = \alpha_{r+1}$ and this is impossible.

Otherwise, if $1 \leq \|x_k\| = b_k\left(\sum_{i=1}^{r_k}\alpha_i\right) + (1 + b_k)\,\alpha_{r_k+1}$, since $r_k \to \infty$, $\lim_k \alpha_{r_k+1} = 0$, and $b_k\left(\sum_{i=1}^{r_k}\alpha_i\right) \leq 1$, we obtain that $\lim_k \|x_k\| = 1$. This proves the lemma. □

Lemma 5. $R(X) = 2$

Proof. Let $y_n = \dfrac{e_{2n} - e_{2n-1}}{1+\alpha_2}$. Then, $\|y_n\| = 1 = \|e_1\|$, $e_1 - y_n = e_1 + \dfrac{e_{2n-1} - e_{2n}}{1+\alpha_2}$ and for $n \geq 2$

$$\|e_1 - y_n\| = \max\left(1 + \frac{\alpha_2 + \alpha_3}{1+\alpha_2}, \left(1 + \frac{1}{1+\alpha_2} + \frac{\alpha_2}{1+\alpha_2}\right)\right) = 2.$$

□

In order to prove that X has GGLD we need the following definition.

Definition 3. *A sequence* $\{y_n\} \subset Y$ *is called asymptotically diametral* [3] *if* $\lim_n \|y_n\|$ *exists and*

$$\lim_n \|y_n\| = \lim_n \left(\sup_{i,j \geq n} \|y_i - y_j\|\right).$$

It is easy to see that for every sequence $\{w_n\} \subset Y$ with $\lim_n \|w_n - y_n\| = 0$, $\{w_n\}$ is also asymptotically diametral and every subsequence of $\{y_n\}$ is also asymptotically diametral. It is obvious that if $\{w_n\} \subset Y$ is asymptotically diametral, then $\{-w_n\}$ is also asymptotically diametral.

In [3] it is shown that a space has GGLD if and only if it does not have a weakly null asymptotically diametral normalized block basic sequence. Next we will use this fact to prove that X has GGLD.

Theorem 2. *X has GGLD and thus X has the wfpp.*

Proof. Let $\{v_n\}$ be a weakly null normalized block basic sequence. We may assume by passing to a subsequence that there exists a block basic sequence $\{u_n\}$ with $\lim_n \|u_n - v_n\| = 0$ such that:

(1) $\|u_n\| = 1,\ n = 1, 2, ...$
(2) $\lim_n s(u_n) = \delta$ for some $\delta > 0$
(3) $0 = \langle u_n, \mathbb{N}\rangle,\ n = 1, 2, ...$

This is due to the following: Since $\{u_n\}$ is not equivalent to the canonical basis of l_1, applying lemma 2 we obtain (2). On the other hand since $f : X \to \mathbb{R}$ given by $f(x) = \langle x, \mathbb{N}\rangle$ is continuous, then $\lim_n \langle v_n, \mathbb{N}\rangle = 0$ and thus we get (3).

Let $\{G_n\}$ be an admissible sequence such that the support of u_n is contained in G_n, $n = 1, 2, ...$

Suppose that $\|u_n\| = \sum_{i=1}^{m_n} \alpha_i |\langle u_n, F_i^n\rangle|$, where $\{F_i^n\}_{i=1}^{m_n}$ is an admissible sequence of finite intervals in $\mathbb{N}$. Taking $\{-u_n\}$ instead of $\{u_n\}$ if necessary and passing to a subsequence we may assume that

(4) $\langle u_n, F_1^n\rangle > 0$ and $d = \lim \langle u_n, F_1^n\rangle$ exists.

Observe that by (2) $0 \leq d \leq \delta$.

By (2) there exists a sequence of intervals $\{L_n\}$ with $L_n \subset G_n$ so that

(5) $\lim_n |\langle u_n, L_n\rangle| = \delta$.

By (3) there exist intervals M_n and R_n so that $G_n = M_n \cup L_n \cup R_n$ and

(6) $\langle u_n, M_n\rangle + \langle u_n, R_n\rangle = -\langle u_n, L_n\rangle$
(7) We may assume that the limits $c = \lim_n \langle u_n, M_n\rangle$, $b = \lim_n \langle u_n, R_n\rangle$ and $a = \lim_n \langle u_n, L_n\rangle$ exist.

Observe that by (2) $\max(|b|,|c|) \leq \delta$ and $|a| = \delta$.

We will show that $\lim_k \sup_{n,r\geq k} \|u_n - u_r\| \geq 1+\varepsilon$, for some $\varepsilon > 0$. In order to do that we study several cases:

Let $k \in \mathbb{N}$ and $r \geq n \geq k$. In what follows we will use that $\{G_n \backslash F_1^n, F_1^r, ..., F_{m_r}^r\}$ and $\{L_n, R_n, F_1^r, ..., F_{m_r}^r\}$ are admissible subsequences as well as any subsequence of them.

I. $d = \delta$

Then $\langle u_n, G_n \backslash F_1^n\rangle \to -\delta$; hence

$$\|u_n - u_r\| \geq -\langle u_n - u_r, (G_n \backslash F_1^n) \cup F_1^r\rangle + \sum_{i=2}^{m_r} \alpha_i |\langle u_r, F_i^r\rangle| =$$
$$= |\langle u_n, G_n \backslash F_1^n\rangle| + 1 \underset{n\to\infty}{\to} 1+\delta.$$

II. $0 \leq d < \delta$

(a) $b < 0$. Then

$$\|u_n - u_r\| \geq -\langle u_n - u_r, R_n \cup F_1^r\rangle + \sum_{i=2}^{m_r} \alpha_i |\langle u_r, F_i^r\rangle| \underset{n\to\infty}{\to} 1 + |b|.$$

(b) $\delta > b > 0$. Since $|a| = \delta, |c| \leq \delta$ and $a+b+c = 0$ then $a = -\delta$ and $-\langle u_n, L_n \cup R_n\rangle \to \delta - b > 0$. Thus

$$\|u_n - u_r\| \geq -\langle u_n - u_r, L_n \cup R_n \cup F_1^r\rangle + \sum_{i=2}^{m_r} \alpha_i |\langle u_r, F_i^r\rangle| \underset{n\to\infty}{\to} 1+\delta-b.$$

(c) $b = 0$. Then

$$\|u_n - u_r\| \geq |\langle u_n, L_n \cup R_n\rangle| + \sum_{i=2}^{m_r} \alpha_i |\langle u_r, F_i^r\rangle| \underset{n\to\infty}{\to} \delta + 1 - d > 1.$$

(d) $b = \delta$. Then

$$\|u_n - u_r\| \geq |\langle u_n, R_n\rangle| + \sum_{i=2}^{m_r} \alpha_i |\langle u_r, F_i^r\rangle| \underset{n\to\infty}{\to} \delta + 1 - d > 1.$$

This proves that $\{u_n\}$ and thus $\{v_n\}$ is not asymptotically diametral and therefore X has GGLD. □

In [4] the authors proved that for Banach spaces Y with a bimonotone basis that are hereditarily asymptotically isometric to l_1, there is a family $\mathcal{F}$ of equivalent norms such that Y equipped with one of those norms has the fixed point property hereditarily. Further in [5] they showed that if Y has GGLD, then Y equipped with a norm of $\mathcal{F}$ also has GGLD. Thus the Azimi Hagler space with a norm of $\mathcal{F}$ has wfpp and is hereditarily FPP.

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