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J^*-HOMOMORPHISMS AND J^*-DERIVATIONS ON J^*-ALGEBRAS FOR A GENERALIZED JENSEN TYPE FUNCTIONAL EQUATION

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Abstract. We will apply the fixed point method for proving the stability and superstability of J^*-homomorphisms and J^*-derivations associated to a generalized Jensen type functional equation between J^*-algebras.

Key Words and Phrases: Approximate J^*-homomorphism; approximate J^*-derivation; J^*-algebra; alternative fixed point; generalized Jensen functional equation.

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1. Introduction

Our knowledge concerning the continuity properties of epimorphisms on Banach algebras, Jordan-Banach algebras, and, more generally, nonassociative complete normed algebras, is now fairly complete and satisfactory (see [24, 44, 45]). A basic continuity problem consists in determining algebraic conditions on a Banach algebra A which ensure that derivations on A are continuous. In 1996, Villena [45] proved that derivations on semisimple Jordan-Banach algebras are continuous. In [24], the authors dealt with derivations acting on Banach-Jordan pairs. By a J^*-algebra we mean a closed subspace A of a C*–algebra such that $xx^*x \in B$ whenever $x \in B$. Several well known spaces have the structure of a J^*-algebra (cf.[17]). For example, (i) every Cartan factor of type I, i.e, the space of all bounded operators $B(H, K)$ between Hilbert spaces H and K; (ii) every Cartan factor of type IV, i.e, a closed *–subspace B of $B(H)$ in which the square of each operator in B is scalar multiple of indentity operator on H; (iii) every ternary algebra of operators [8, 18]. A J^*-homomorphism between J^*-algebras A and B is defined to be a $\mathbb{C}$–linear mapping $H : A \to B$ such that

$$H(aa^*a) = H(a)H(a)^*H(a)$$

for all $a \in A$, and a J^*-derivation on a J^*-algebras A is defined to be a $\mathbb{C}$–linear mapping $D : A \to A$ such that

$$D(aa^*a) = D(a)a^*a + aD(a)^*a + aa^*D(a)$$

for all $a \in A$. In particular, every $*$–homomorphism between C*–algebras is a J^*-homomorphism and every $*$–derivation on a C*–algebra is a J^*-derivation.

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms. Hyers [19] provided a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mappings and by Th.M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [41] has provided a lot of influence in the development of what we now call generalized Hyers–Ulam stability or as Hyers–Ulam–Rassias stability of functional equations. In 1994, a generalization of the Rassias theorem was obtained by Găvruţa [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about various results concerning such problems the reader is referred to [6, 9, 11, 14, 16, 20, 21, 22] and [37]–[42].

C. Park, J.C. Hou and Th.M. Rassias proved the stability of homomorphisms and derivations in Banach algebras, Banach ternary algebras, C*–algebras, Lie C*–algebras and C*–ternary algebras [25]–[35]. Moreover, in [29], Park established the stability of $*$–homomorphisms of a C*–algebra (see also [30]).

We note that a mapping f satisfying the following Jensen equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ is called Jensen. Stability of Jensen functional equation has been studied by using the direct method as well as the fixed point method at [3, 5, 20, 23, 42]. Recently, Eshaghi Gordji and Najati [12] proved the stability and superstability of J^*-homomorphisms between J^*-algebras for the Jensen type functional equation

$$f(\frac{x+y}{2}) + f(\frac{x-y}{2}) - f(x) = 0.$$

In addition, Eshaghi Gordji et al. [10] established the stability and superstability of J^*-derivations in J^*-algebras for the following Jensen type functional equation

$$rf(\frac{x+y}{r}) + rf(\frac{x-y}{r}) - 2f(x) = 0.$$

In this paper, we investigate the stability and superstability of J^*-homomorphisms and J^*-derivations in J^*-algebras for the generalized Jensen type functional equation

$$\mu f(\frac{\sum_{i=1}^{n} x_i}{n}) + \mu \sum_{j=2}^{n} f(\frac{\sum_{i=1, i\neq j}^{n} x_i - (n-1)x_j}{n}) - f(\mu x_1) = 0 \tag{1}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C}; |\lambda| = 1\}$, where $n \geq 2$.

Before proceeding to the main results, we recall a fundamental result in fixed point theory.

Theorem 1.1. [7]. *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive function $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^m x, T^{m+1}x) = \infty \quad \textit{for all } m \geq 0,$$

or there exists a natural number m_0 *such that*

- $d(T^m x, T^{m+1}x) < \infty$ *for all* $m \geq m_0$;
- *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T;
- y^* *is the unique fixed point of* T *in the set* $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

Radu and Cădariu [2, 3, 36] applied the fixed point method to the investigation of functional equations (see also [4, 13, 22]).
This paper is organized as follows: By using the fixed point method, in Section 2, we prove the superstability and stability of J^*-homomorphisms in J^*-algebras for the functional equation (1), and also using Gajda's example [14] to give a counterexample for a singular case. In Section 3, we prove the superstability and stability of J^*-derivations on J^*-algebras for the functional equation (1), and also we present a counterexample for a singular case.

Throughout this paper assume that A, B are two J^*-algebras.
For convenience, we use the following abbreviation for given a mapping $f : A \to B$,

$$\triangle f(x_1, x_2, \ldots, x_n, a) = \mu f(\frac{\sum_{i=1}^n x_i + aa^*a}{n})$$

$$+\mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + aa^*a}{n}) - f(\mu x_1)$$

for all $\mu \in \mathbb{T}$ and all $x_1, x_2, \ldots, x_n, a \in A$, where $n \geq 2$.

2. Approximation of J^*-homomorphisms in J^*-algebras

We will use the following lemma:

Lemma 2.1. *Let both* X *and* Y *be real vector spaces. If a mapping* $f : X \to Y$ *satisfies* (1) *with* $\mu = 1$, *then* $f : X \to Y$ *is additive.*

Proof. Letting $x_i = 0$ $(1 \leq i \leq n)$ in (1), we obtain $f(0) = 0$. Setting $x_1 = x$ and $x_i = 0$ $(2 \leq i \leq n)$ in (1), we get

$$nf\left(\frac{x}{n}\right) = f(x) \tag{2}$$

for all $x \in X$. Setting $x_i = 0$ $(3 \leq i \leq n)$ in (1) and using (2), we get

$$\frac{n-1}{n} f(x_1 + x_2) + \frac{1}{n} f(x_1 - (n-1)x_2) = f(x_1) \tag{3}$$

for all $x_1, x_2 \in X$. Putting $x_1 = x_1 + (n-1)x_2$ in (3), we get

$$\frac{n-1}{n} f(x_1 + nx_2) + \frac{1}{n} f(x_1) = f(x_1 + (n-1)x_2) \tag{4}$$

for all $x_1, x_2 \in X$. Replacing x_1 by 0 and x_2 by x in (4) and using (2), we get

$$f((n-1)x) = (n-1)f(x) \tag{5}$$

for all $x \in X$. Replacing x_1 by 0 and x_2 by x in (3) and using (5), we get $f(-x) = -f(x)$ for all $x \in X$, i.e., f is an odd function. Setting $x_2 = x_2 - x_1$ in (3), we get

$$\frac{n-1}{n} f(x_2) + \frac{1}{n} f(nx_1 - (n-1)x_2) = f(x_1) \tag{6}$$

for all $x_1, x_2 \in X$. Replacing x_1 by $\frac{x_1}{n}$ and x_2 by $-\frac{x_2}{n-1}$ in (6), by (2), (5) and the oddness of f, we obtain

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in X$. So f is additive. □

In the following we formulate and prove a theorem in superstability of J^*-homomorphisms for the functional equation (1).

Theorem 2.2. *Let $\ell \in \{-1, 1\}$ be given and let $0 \neq \ell|s| < \ell$. Assume $f : A \to B$ is a mapping for which $f(sx) = sf(x)$ for all $x \in A$. Suppose there exists a function $\phi : A^{n+1} \to [0, \infty)$ such that*

$$\|\triangle f(x_1, x_2, \dots, x_n, a) - \mu f(a)f(a)^* f(a)\| \leq \phi(x_1, x_2, \dots, x_n, a) \tag{7}$$

for all $x_1, \dots, x_n, a \in A$. If there exists an $L < 1$ such that

$$\phi(x_1, x_2, \dots, x_n, a) \leq \frac{L}{|s|^\ell} \phi(s^\ell x_1, s^\ell x_2, \dots, s^\ell x_n, s^\ell a) \tag{8}$$

for all $x_1, \dots, x_n, a \in A$, then f is a J^*-homomorphism.

Proof. It follows from (8) that

$$\lim_{m\to\infty} |s|^{m\ell} \phi(\frac{x_1}{s^{m\ell}}, \frac{x_2}{s^{m\ell}}, \dots, \frac{x_n}{s^{m\ell}}, \frac{a}{s^{m\ell}}) = 0 \tag{9}$$

for all $x_1, \dots, x_n, a \in A$. Setting $\mu = 1$ and $x_i = 0$ $(1 \leq i \leq n)$ in (7), we obtain

$$\begin{aligned}\|f(aa^*a) - f(a)f(a)^*f(a)\| &= \lim_{m\to\infty} |s|^{3m\ell} \|f((\frac{a}{s^{m\ell}})(\frac{a^*}{s^{m\ell}})(\frac{a}{s^{m\ell}})) \\ &\quad - f(\frac{a}{s^{m\ell}})f(\frac{a}{s^{m\ell}})^* f(\frac{a}{s^{m\ell}})\| \\ &\leq \lim_{m\to\infty} |s|^{3m\ell} \phi(0, 0, \dots, \frac{a}{s^{m\ell}}) \leq \lim_{m\to\infty} |s|^{m\ell} \phi(0, 0, \dots, \frac{a}{s^{m\ell}}) = 0\end{aligned}$$

for all $a \in A$. So

$$f(aa^*a) = f(a)f(a)^*f(a)$$

for all $a \in A$. Similarly put $a = 0$ in (7), then

$$\begin{aligned}&\|\mu f(\frac{\sum_{i=1}^n x_i}{n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}) - f(\mu x_1)\| \\ &= \lim_{m\to\infty} |s|^{m\ell} \|\mu f(\frac{\sum_{i=1}^n x_i}{s^{m\ell} n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{s^{m\ell} n}) - f(\mu \frac{x_1}{s^{m\ell}})\| \\ &\leq \lim_{m\to\infty} |s|^{m\ell} \phi(\frac{x_1}{s^{m\ell}}, \frac{x_2}{s^{m\ell}}, \dots, \frac{x_n}{s^{m\ell}}, 0) = 0\end{aligned}$$

for all $x_1, \ldots, x_n \in A$. So

$$\mu f(\frac{\sum_{i=1}^n x_i}{n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}) = f(\mu x_1)$$

for all $\mu \in \mathbb{T}$ and all $x_1, \ldots, x_n \in A$. Thus by Lemma 2.1, the mapping f is additive.

Letting $x_i = x$ $(1 \leq i \leq n)$ and $a = 0$ in (7), we have

$$\|f(\mu x) - \mu f(x)\| = \lim_{m\to\infty} |s|^{m\ell} \|f(\mu \frac{x}{s^{m\ell}}) - \mu f(\frac{x}{s^{m\ell}})\|$$
$$\leq \lim_{m\to\infty} |s|^{m\ell} \phi(\frac{x}{s^{m\ell}}, \frac{x}{s^{m\ell}}, \ldots, \frac{x}{s^{m\ell}}, 0) = 0$$

for all $\mu \in \mathbb{T}$ and all $x \in A$. One can show that the mapping $f : A \to B$ is $\mathbb{C}$–linear, and we conclude that f is a J^*−homomorphism. □

Corollary 2.3. *Let $\ell \in \{-1, 1\}$ be given and let $0 \neq \ell|s| < \ell$, $\ell p < \ell$ and δ, θ, p be non–negative real numbers. Suppose that $f : A \to B$ is a mapping satisfying $f(sx) = sf(x)$ for all $x \in A$, and the following inequality*

$$\|\triangle f(x_1, x_2, \ldots, x_n, a) - \mu f(a)f(a)^* f(a)\| \leq \frac{1+\ell}{2}\delta + \theta(\sum_{i=1}^n \|x_i\|^p + \|a\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x_1, x_2, \ldots, x_n, a \in A$, then f is a J^*−homomorphism.

Proof. Let $\phi(x_1, x_2, \ldots, x_n, a) := \frac{1+\ell}{2}\delta + \theta(\sum_{i=1}^n \|x_i\|^p + \|a\|^p)$ for all $x_1, x_2, \ldots, x_n$, $a \in A$ in Theorem 2.2. Then we choose $L = |s|^{\ell(1-p)}$ and we get the desired result. □

We prove the following generalized Hyers–Ulam stability problem for J^*−homomorphisms on J^*−algebras for the functional equation (1).

Theorem 2.4. *Let $f : A \to B$ be a mapping with $f(0) = 0$ for which there exists a function $\phi : A^{n+1} \to [0, \infty)$ satisfying (7). If there exists an $L < 1$ such that*

$$\phi(x_1, x_2, \ldots, x_n, a) \leq nL\phi(\frac{x_1}{n}, \frac{x_2}{n}, \ldots, \frac{x_n}{n}, \frac{a}{n}) \tag{10}$$

for all $x_1, \ldots, x_n, a \in A$, then there exists a unique J^*−homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \leq \frac{1}{n(1-L)}\phi(nx, 0, 0, \ldots, 0) \tag{11}$$

for all $x \in A$.

Proof. Letting $\mu = 1$, $x_1 = x$, $x_i = 0$ $(2 \leq i \leq n)$ and $a = 0$ in (7), we obtain

$$\|nf(\frac{x}{n}) - f(x)\| \leq \phi(x, 0, \ldots, 0) \tag{12}$$

for all $x \in A$. Replacing x by nx in (12), we get

$$\|\frac{1}{n}f(nx) - f(x)\| \leq \frac{1}{n}\phi(nx, 0, \ldots, 0) \tag{13}$$

for all $x \in A$. Consider the set $X := \{g \mid g : A \to B\}$ and introduce the generalized metric on X as follows:

$$d(g, h) := \inf\left\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(nx, 0, \ldots, 0), \forall x \in A\right\}.$$

It is easy to show that (X, d) is a generalized complete metric space [3, 4].

Now we define the linear mapping $T : X \to X$ by $T(h)(x) = \frac{1}{n}h(nx)$ for all $x \in A$. It is easy to see that

$$d(T(g), T(h)) \leq Ld(g, h)$$

for all $g, h \in X$. It follows from (13) that

$$d(f, T(f)) \leq \frac{1}{n} < \infty. \tag{14}$$

By Theorem 1.1, T has a unique fixed point in the set $X_1 := \{g \in X : d(f, g) < \infty\}$. Let H be the fixed point of T. H is the unique mapping with $H(nx) = nH(x)$ for all $x \in A$, such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - H(x)\| \leq C\phi(nx, 0, \ldots, 0)$$

for all $x \in A$. On the other hand we have $\lim_{m\to\infty} d(T^m(f), H) = 0$. It follows that

$$\lim_{m\to\infty} \frac{1}{n^m} f(n^m x) = H(x) \tag{15}$$

for all $x \in A$. Also by Theorem 1.1, we have

$$d(f, H) \leq \frac{1}{1-L} d(f, T(f)) \tag{16}$$

It follows from (14) and (16), that

$$d(f, H) \leq \frac{1}{n(1-L)}$$

This implies inequality (11). It follows from (10) that

$$\lim_{m\to\infty} \frac{1}{n^m} \phi(n^m x_1, n^m x_2, \ldots, n^m x_n, n^m a) = 0 \tag{17}$$

for all $x_1, \ldots, x_n, a \in A$. By the same reasoning as the proof of Theorem 2.2, One can show that the mapping $H : A \to B$ is $\mathbb{C}$–linear. It follows from (7), (15) and (17) that

$$\begin{aligned}
&\|H(aa^*a) - H(a)H(a)^*H(a)\| = \lim_{m\to\infty} \frac{1}{n^{3m}} \|H((n^m a)(n^m a^*)(n^m a)) \\
&- H(n^m a)H(n^m a)^*H(n^m a)\| \\
&\leq \lim_{m\to\infty} \frac{1}{n^{3m}} \phi(0, 0, \ldots, n^m a) \\
&\leq \lim_{m\to\infty} \frac{1}{n^m} \phi(0, 0, \ldots, n^m a) = 0
\end{aligned}$$

for all $a \in A$. Thus

$$H(aa^*a) = H(a)H(a)^*H(a)$$

for all $a \in A$. Hence $H : A \to B$ is a J^*-homomorphism. □

Corollary 2.5. *Let θ, p be non–negative real numbers such that $p < 1$. Suppose that a function $f : A \to B$ satisfies*

$$\|\triangle f(x_1, x_2, \ldots, x_n, a) - \mu f(a)f(a)^*f(a)\| \leq \theta \sum_{i=1}^{n} (\|x_i\|^p + \|a\|^p)$$

for all $\mu \in \mathbb{T}$ *and all* $x_1, \ldots, x_n, a \in A$. *Then there exists a unique* J^*-*homomorphism* $H : A \to B$ *such that*

$$\|f(x) - H(x)\| \leq \frac{\theta}{n^{1-p} - 1}\|x\|^p$$

for all $x \in A$.

The case in which $p = 1$ was excluded in Corollary 2.5. Indeed this result is not valid when $p = 1$. Here we use Gajda's example [14] to construct a Counterexample.

Example 2.6. *Let* $\phi : \mathbb{C} \to \mathbb{C}$ *be defined by*

$$\phi(x) := \begin{cases} x & \text{for} \quad |x| < 1; \\ 1 & \text{for} \quad |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ *to be defined by the formula*

$$f(x) := \sum_{m=0}^{\infty} n^{-m}\phi(n^m x)$$

Let
$D_\mu f(x_1, \ldots, x_n, a) := \mu f(\frac{\sum_{i=1}^n x_i + a\bar{a}a}{n}) + \mu \sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j + a\bar{a}a}{n}) - f(\mu x_1) -$

$\mu f(a)\overline{f(a)}f(a)$
for all $\mu \in \mathbb{T}$ *and all* $x_1, x_2, \ldots, x_n, a \in \mathbb{C}$. *Then* f *satisfies*

$$|D_\mu f(x_1, \ldots, x_n, a)| \leq \frac{n^4 + n^3 + 6n^2 - 7n + 2}{(n-1)^2}(\sum_{i=1}^n |x_i| + |a|) \tag{18}$$

for all $\mu \in \mathbb{T}$ *and all* $x_1, x_2, \ldots, x_n, a \in \mathbb{C}$, *and the range of* $|f(x) - A(x)|/|x|$ *for* $x \neq 0$ *is unbounded for each additive function* $A : \mathbb{C} \to \mathbb{C}$.

Proof. It is clear that f is bounded by $\frac{n}{n-1}$ on $\mathbb{C}$. If $\sum_{i=1}^n |x_i| + |a| = 0$ or $\sum_{i=1}^n |x_i| + |a| \geq 1$, then

$$|D_\mu f(x_1, \ldots, x_n, a)| \leq \frac{n^4 - n^2 + n}{(n-1)^3} \leq \frac{n^4 - n^2 + n}{(n-1)^3}(\sum_{i=1}^n |x_i| + |a|)$$

Now suppose that $0 < \sum_{i=1}^n |x_i| + |a| < 1$. Then there exists an integer $k \geq 0$ such that

$$\frac{1}{n^{k+1}} \leq \sum_{i=1}^n |x_i| + |a| < \frac{1}{n^k} \tag{19}$$

Therefore

$$n^t|\sum_{i=1}^n x_i + a\bar{a}a|, n^t|\sum_{i=1}^n x_i + a\bar{a}a - (n-1)x_j|, n^t|\mu x_1|, n^t|a| < 1$$

for all $j = 2, 3, \ldots, n$ and all $t = 0, 1, \ldots, k-1$. From the definition of f and (19), we have

$$|f(a)| \leq k|a| + \sum_{m=k}^{\infty} n^{-m}|\phi(n^m a)| \leq k|a| + \frac{n}{n^k(n-1)},$$

$$
\begin{aligned}
|D_\mu f(x_1,\dots,x_n,a)| &\le k|a|^3 + \frac{n(n+1)}{n^k(n-1)} + |f(a)|^3 \\
&\le (k+k^3)|a|^3 + \frac{n^2+2n}{n^k(n-1)} + \frac{3n(n-1)k^2+3n^2k}{n^{2k}(n-1)^2}|a| \\
&\le \frac{(n-1)^2k^3+3n(n-1)k^2+((n-1)^2+3n^2)k}{n^{2k}(n-1)^2}|a| + \frac{n^2+2n}{n^k(n-1)} \\
&\le \frac{2(n-1)^2+3n(n-1)+3n^2}{(n-1)^2}|a| + \frac{n^3+2n^2}{(n-1)}(\sum_{i=1}^n |x_i|+|a|) \\
&\le \frac{n^4+n^3+6n^2-7n+2}{(n-1)^2}(\sum_{i=1}^n |x_i|+|a|)
\end{aligned}
$$

Therefore f satisfies (18). Let $A:\mathbb{C}\to\mathbb{C}$ be an additive function such that

$$|f(x)-A(x)|\le \alpha|x|$$

for all $x\in\mathbb{C}$, where $\alpha>0$ is a constant. Then there exists a constant $c\in\mathbb{C}$ such that $A(x)=cx$ for all rational numbers x. Thus we have

$$|f(x)|\le(\alpha+|c|)|x| \tag{20}$$

for all rational numbers x. Let $t\in\mathbb{N}$ with $t>\alpha+|c|$. If x is a rational number in $(0,n^{1-t})$, then $n^m x\in(0,1)$ for all $m=0,1,...,t-1$. Therefore

$$f(x)\ge\sum_{m=0}^{t-1} n^{-m}\phi(n^m x)=tx>(\alpha+|c|)x$$

which contradicts (20). □

3. Approximation of J^*-derivations in J^*-algebras

In this section, we prove the superstablity and stability of J^*-derivations on J^*-algebras for the functional equation (1).

Theorem 3.1. *Let $\ell\in\{-1,1\}$ be given and let $0\ne|s|\ell>\ell$. Suppose $f:A\to A$ is a mapping for which $f(sx)=sf(x)$ for all $x\in A$. Suppose there exists a function $\psi:A^{n+1}\to[0,\infty)$ such that*

$$\|\triangle f(x_1,x_2,\dots,x_n,a)-\mu f(a)a^*a-\mu af(a)^*a-\mu aa^*f(a)\|\le\psi(x_1,x_2,\dots,x_n,a) \tag{21}$$

for all $x_1,\dots,x_n,a\in A$. If there exists an $L<1$ such that

$$\psi(x_1,x_2,\dots,x_n,a)\le\ell|s|^\ell\psi(\frac{x_1}{s^\ell},\frac{x_2}{s^\ell},\dots,\frac{x_n}{s^\ell},\frac{a}{s^\ell}) \tag{22}$$

for all $x_1,\dots,x_n,a\in A$, then f is a J^*-derivation.

Proof. By using equation $f(sx)=sf(x)$ and (21), we have $f(0)=0$ and

$$\|\mu f(\frac{\sum_{i=1}^n x_i}{n})+\mu\sum_{j=2}^n f(\frac{\sum_{i=1,i\ne j}^n x_i-(n-1)x_j}{n})-f(\mu x_1)\|$$

$$\leq |s|^{-m\ell}\psi(s^{m\ell}x_1, s^{m\ell}x_2, \dots, s^{m\ell}x_n, 0), \tag{23}$$

$$\|f(aa^*a) - f(a)a^*a - af(a)^*a - aa^*f(a)\| \leq |s|^{-3m\ell}\psi(0, 0, \dots, 0, s^{m\ell}a) \tag{24}$$

for all $x_1, \dots, x_n, a \in A$ and all integers m. It follows from (22), that

$$\lim_{m\to\infty} |s|^{-m\ell}\psi(s^{m\ell}x_1, s^{m\ell}x_2, \dots, s^{m\ell}x_n, s^{m\ell}a) = 0 \tag{25}$$

for all $x_1, \dots, x_n, a \in A$. Hence, we get from (23), (24) and (25) that

$$\mu f(\frac{\sum_{i=1}^n x_i}{n}) + \mu\sum_{j=2}^n f(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}) = f(\mu x_1),$$

$$f(aa^*a) = f(a)a^*a + af(a)^*a + aa^*f(a)$$

for all $x_1, \dots, x_n, a \in A$. Therefore f is additive and $f(\mu x) = \mu f(x)$ for all $\mu \in T$ and $x \in A$. By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping $f : A \to A$ is $\mathbb{C}$–linear, and we conclude that f is a J^*–derivation. □

Corollary 3.2. *Let $\ell \in \{-1, 1\}$ be given and let $0 \neq \ell|s| > \ell$, $\ell p > \ell$ and β, ε, p be non–negative real numbers. Suppose that $f : A \to A$ is a mapping satisfying $f(sx) = sf(x)$ for all $x \in A$, and the following inequality*

$$\|\triangle f(x_1, x_2, \dots, x_n, a) - \mu f(a)a^*a - \mu af(a)^*a - \mu aa^*f(a)\|$$
$$\leq \frac{1+\ell}{2}\beta + \varepsilon(\sum_{i=1}^n \|x_i\|^p + \|a\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x_1, x_2, \dots, x_n, a \in A$, then f is a J^*–derivation.

Theorem 3.3. *Let $f : A \to A$ be a mapping with $f(0) = 0$ for which there exists a function $\psi : A^{n+1} \to [0, \infty)$ satisfying (21). If there exists an $L < 1$ such that*

$$\psi(x_1, x_2, \dots, x_n, a) \leq nL\psi(\frac{x_1}{n}, \frac{x_2}{n}, \dots, \frac{x_n}{n}, \frac{a}{n}) \tag{26}$$

for all $x_1, \dots, x_n, a \in A$, then there exists a unique J^*–derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \leq \frac{L}{1-L}\psi(x, 0, 0, \dots, 0) \tag{27}$$

for all $x \in A$.

Proof. It follows from (26) that

$$\lim_{m\to\infty} \frac{1}{n^m}\psi(n^m x_1, n^m x_2, \dots, n^m x_n, n^m a) = 0 \tag{28}$$

for all $x_1, \dots, x_n, a_1, \dots, a_n \in A$. Consider the set $X' := \{g | g : A \to X\}$ and introduce the generalized metric on X' as follows:

$$d(g, h) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\psi(x, 0, \dots, 0), \forall x \in A\}$$

It is easy to show that (X', d) is a generalized complete metric space.

Now we define the linear mapping $J : X' \to X'$ by $J(h)(x) = \frac{1}{n}h(nx)$ for all $x \in A$. It is easy to see that

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all $g, h \in X'$.

Letting $\mu = 1$, $x_1 = x$, $x_i = 0$ $(2 \le i \le n)$ and $a = 0$ in (21), we obtain

$$\|nf(\frac{x}{n}) - f(x)\|_X \le \psi(x, 0, \ldots, 0) \tag{29}$$

for all $x \in A$. Thus by using (26), we obtain

$$\|\frac{1}{n}f(nx) - f(x)\|_X \le \frac{1}{n}\psi(nx, 0, \ldots, 0) \le L\psi(x, 0, \ldots, 0) \tag{30}$$

for all $x \in A$, that is,

$$d(f, J(f)) \le L < \infty. \tag{31}$$

By Theorem 1.1, J has a unique fixed point in the set $X_2 := \{h \in X' : d(f, h) < \infty\}$. Let D be the fixed point of J. We note that D is the unique mapping with $D(nx) = nD(x)$ for all $x \in A$, such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - D(x)\| \le C\psi(x, 0, \ldots, 0)$$

for all $x \in A$. On the other hand we have

$$\lim_{m\to\infty} d(J^m(f), D) = 0,$$

so

$$\lim_{m\to\infty} \frac{1}{n^{m\ell}} f(n^{m\ell}x) = D(x)$$

for all $x \in A$. Also by Theorem 1.1, we have

$$d(f, D) \le \frac{1}{1-L} d(f, J(f)) \tag{32}$$

It follows from (31) and (32), that

$$d(f, D) \le \frac{L}{1-L}$$

This implies inequality (27). By the same reasoning as in the proof of Theorem 2.2, one can show that the mapping $f : A \to A$ is $\mathbb{C}$–linear. It follows from (21) and (28) that

$$\begin{aligned}
&\|D(aa^*a) - D(a)a^*a - aD(a)^*a - aa^*D(a)\| \\
&= \lim_{m\to\infty} \|\frac{1}{n^{3m}}(D((n^m a)(n^m a^*)(n^m a)) - D(n^m a)(n^m a^*)(n^m a) - \\
&(n^m a)D(n^m a)^*(n^m a) - (n^m a)(n^m a^*)D(n^m a)\| \\
&\le \frac{1}{n^{3m}}\psi(0, 0, \ldots, 0, n^m a) \le \frac{1}{n^m}\psi(0, 0, \ldots, 0, n^m a) = 0
\end{aligned}$$

for all $a \in A$. Therefore

$$D(aa^*a) = D(a)a^*a + aD(a)^*a + aa^*D(a)$$

for all $a \in A$. Hence $D : A \to A$ is a J^*–derivation. □

Corollary 3.4. *Let ε, p be non–negative real numbers such that $p < 1$. Suppose that a function $f : A \to A$ satisfies*

$$\|\triangle f(x_1, x_2, \ldots, x_n, a) - \mu f(a)a^*a - \mu af(a)^*a - \mu aa^*f(a)\|$$

$$\leq \varepsilon(\sum_{i=1}^{n} \|x_i\|^p + \|a\|^p)$$

for all $\mu \in \mathbb{T}$ *and all* $x_1, \dots, x_n, a \in A$. *Then there exists a unique* J^*-*derivation* $D : A \to A$ *such that*

$$\|f(x) - D(x)\| \leq \frac{n^{p-1}\varepsilon}{1 - n^{p-1}} \|x\|^p$$

for all $x \in A$.

For the case $p = 1$, similar to the Example 2.6, we have the following counterexample.

Example 3.5. *Let* $\psi : \mathbb{C} \to \mathbb{C}$ *be defined by*

$$\psi(x) := \begin{cases} x & \text{for} \quad |x| < 1; \\ 1 & \text{for} \quad |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ *to be defined by the formula*

$$f(x) := \sum_{m=0}^{\infty} n^{-m} \psi(n^m x)$$

Let

$$D_\mu f(x_1, \dots, x_n, a) := \mu f(\frac{\sum_{i=1}^{n} x_i + a\overline{a}a}{n}) + \mu \sum_{j=2}^{n} f(\frac{\sum_{i=1, i\neq j}^{n} x_i - (n-1)x_j + a\overline{a}a}{n}) - f(\mu x_1)$$

$$- \mu f(a)\overline{a}a - \mu a\overline{f(a)}a - \mu a\overline{a}f(a)$$

for all $\mu \in \mathbb{T}$ *and all* $x_1, x_2, \dots, x_n, a \in \mathbb{C}$. *Then* f *satisfies*

$$|D_\mu f(x_1, \dots, x_n, a)| \leq \frac{n^3 + n^2 + 7n - 4}{n - 1}(\sum_{i=1}^{n} |x_i| + |a|) \tag{33}$$

for all $\mu \in \mathbb{T}$ *and all* $x_1, x_2, \dots, x_n, a \in \mathbb{C}$, *and the range of* $|f(x) - A(x)|/|x|$ *for* $x \neq 0$ *is unbounded for each additive function* $A : \mathbb{C} \to \mathbb{C}$.

Proof. It is clear that f is bounded by $\frac{n}{n-1}$ on $\mathbb{C}$. If $\sum_{i=1}^{n} |x_i| + |a| = 0$ or $\sum_{i=1}^{n} |x_i| + |a| \geq 1$, then

$$|D_\mu f(x_1, \dots, x_n, a)| \leq \frac{n^2 + (1 + 3|a|^2)n}{(n-1)} \leq \frac{n^2 + (1 + 3|a|^2)n}{(n-1)}(\sum_{i=1}^{n} |x_i| + |a|)$$

Now suppose that $0 < \sum_{i=1}^{n} |x_i| + |a| < 1$. Then there exists an integer $k \geq 0$ such that

$$\frac{1}{n^{k+1}} \leq \sum_{i=1}^{n} |x_i| + |a| < \frac{1}{n^k} \tag{34}$$

Therefore

$$n^t|\sum_{i=1}^{n} x_i + a\bar{a}a|, n^t|\sum_{i=1}^{n} x_i + a\bar{a}a - (n-1)x_j|, n^t|\mu x_1|, n^t|a| < 1$$

for all $j = 2, 3, \ldots, n$ and all $t = 0, 1, \ldots, k-1$. From the definition of f and (34), we have

$$|f(a)| \leq k|a| + \sum_{m=k}^{\infty} n^{-m}|\psi(n^m a)| \leq k|a| + \frac{n}{n^k(n-1)},$$

$$|D_\mu f(x_1, \ldots, x_n, a)| \leq k|a|^3 + \frac{n(n+1)}{n^k(n-1)} + 3|a|^2|f(a)|$$

$$\begin{aligned}
&\leq 4k|a|^3 + \frac{n^2+n}{n^k(n-1)} + \frac{3n}{n^k(n-1)}|a|^2 \\
&\leq \frac{4(n-1)k+3n}{n^k(n-1)}|a|^2 + \frac{n^2+n}{n^k(n-1)} \\
&\leq \frac{4(n-1)k+3n}{n^k(n-1)}|a| + \frac{n^3+n^2}{(n-1)}(\sum_{i=1}^{n}|x_i| + |a|) \\
&\leq \frac{n^3+n^2+7n-4}{(n-1)}(\sum_{i=1}^{n}|x_i| + |a|)
\end{aligned}$$

Therefore f satisfies (33). Let $A : \mathbb{C} \to \mathbb{C}$ be an additive function such that

$$|f(x) - A(x)| \leq \alpha|x|$$

for all $x \in \mathbb{C}$, where $\alpha > 0$ is a constant. Then there exists a constant $c \in \mathbb{C}$ such that $A(x) = cx$ for all rational numbers x. Thus we have

$$|f(x)| \leq (\alpha + |c|)|x| \tag{35}$$

for all rational numbers x. Let $t \in \mathbb{N}$ with $t > \alpha + |c|$. If x is a rational number in $(0, n^{1-t})$, then $n^m x \in (0, 1)$ for all $m = 0, 1, ..., t-1$. Hence

$$f(x) \geq \sum_{m=0}^{t-1} n^{-m}\phi(n^m x) = tx > (\alpha + |c|)x$$

which contradicts (35). □

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References

[1] T. Aoki, *On the stability of the linear transformationin Banach spaces,* J. Math. Soc. Japan, (1950), 64–66.

[2] L. Cadariu, V. Radu, *Fixed points and the stability of quadratic functional equations,* Analele Universitatii de Vest Timisoara, **41**(2003) 25-48.

[3] L. Cadariu, V. Radu, *Fixed points and the stability of Jensen's functional equation,* J. Inequal. Pure Appl. Math., **4**(2003), Art. ID 4.

[4] L. Cadariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach,* Grazer Mathematische Berichte, **346**(2004), 43-52.

[5] L. Cadariu, V. Radu, *The fixed point method to stability properties of a functional equation of Jensen type,* An. Ştiin. Univ. Al. I. Cuza Iaşi, Ser. Noua, Mat., **54**(2)(2008), 307–318.

[6] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequat. Math., **27**(1984), 76-86.

[7] J.B. Diaz, B. Margolis,*A fixed point theorem of the alternative for contractions on a generalized complete metric space,* Bull. Amer. Math. Soc., **74**(1968), 305-309.

[8] M. Elin, L. Harris, S. Reich, D. Shoikhet, *Evolution equations and geometric function theory in J^*-algebras*, J. Nonlinear Convex Anal., **3**(2002), 81–121.

[9] M. Eshaghi Gordji, H. Khodaei, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi–Banach spaces*, Nonlinear Anal., **71**(2009), 5629–5643.

[10] M. Eshaghi Gordji, M.B. Ghaemi, S. Kaboli Gharetapeh, S. Shams, A. Ebadian, *On the stability of J^*-derivations*, J. Geometry and Physics, **60**(3)(2010), 454–459.

[11] M. Eshaghi Gordji, T. Karimi, S. Kaboli Gharetapeh, *Approximately n–Jordan homomorphisms on Banach algebras,* J. Ineq. Appl. Volume 2009, Article ID 870843.

[12] M. Eshaghi Gordji, A. Najati, *Approximately J*-homomorphisms: A fixed point approach*, J. Geometry and Physics, **60**(5)(2010), 809–814.

[13] M. Eshaghi Gordji, H. Khodaei, J.M. Rassias, *Fixed point methods for the stability of general quadratic functional equation*, Fixed Point Theory, **12**(1)(2011), 71–82.

[14] Z. Gajda, *On stability of additive mappings,* Int. J. Math. Math. Sci., **14**(1991), 431–434.

[15] P. Gavruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184**(1994), 431-436.

[16] A. Grabiec, *The generalized Hyers–Ulam stability of a class of functional equations*, Publ. Math. Debrecen, **48**(1996), 217–235.

[17] L.A. Harris, *Bounded symmetric homogeneous domains in infinite–dimensional space,* in: Lecture Notes in Mathematics, vol. 364, Springer, Berlin, 1974.

[18] L.A. Harris, *Operator Siegel domains*, Proc. Roy. Soc. Edinburgh Sect. A, **79**(1977), 137–156.

[19] D.H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci., **27**(1941), 222–224.

[20] S.-M. Jung, *Hyers–Ulam–Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc., **126**(1998), 3137-3143.

[21] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press Inc., Palm Harbor, Florida, 2001.

[22] H. Khodaei, Th. M. Rassias, *Approximately generalized additive functions in several variables*, Int. J. Nonlinear Anal. Appl., **1**(2010), 22–41.

[23] Y.H. Lee, K.W. Jun, *A generalization of the Hyers–Ulam–Rassias stability of Jensens equation*, J. Math. Anal. Appl., **238**(1999), 305–315.

[24] A.F. Lopez, H. Marhnine, C. Zarhouti, *Derivations on Banach-Jordan pairs*, Quart. J. Math., **52**(2001), 269-283.

[25] C. Park, *On an approximate automorphism on a C*-algebra*, Proc. Amer. Math. Soc., **132**(6)(2004), 1739–1745.

[26] C. Park, *Linear $*-$derivations on JB^*-algebras,* Acta Math. Sci. Ser. B Engl. Ed., **25**(2005), 449–454.

[27] C. Park, *Lie $*-$homomorphisms between Lie C^*-algebras and Lie $*-$derivations on Lie C^*-algebras*, J. Math. Anal. Appl., **293**(2004), 419–434.

[28] C. Park, *Homomorphisms between Lie JC^*-algebras and Cauchy–Rassias stability of Lie JC^*-algebra derivations,* J. Lie Theory, **15**(2005), 393–414.
[29] C. Park, *Homomorphisms between Poisson JC*–algebras,* Bull. Braz. Math. Soc., **36**(2005), 79–97.
[30] C. Park, *Isomorphisms between C*-ternary algebras,* J. Math. Anal. Appl., **327**(2007), 101–115.
[31] C. Park, M. Eshaghi Gordji, *Comment on "Approximate ternary Jordan derivations on Banach ternary algebras" [Bavand Savadkouhi et al. J. Math. Phys. 50, 042303 (2009)],* J. Math. Phys. **51**, 044102 (2010); doi:10.1063/1.3299295 (7 pages).
[32] C. Park, J.C. Hou, *Homomorphisms between C*–algebras associated with the Trif functional equation and linear derivations on C*–algebras,* J. Korean Math. Soc., **41**(3)(2004), 461-477.
[33] C. Park, J.C. Hou, S.Q. Oh, *Homomorphisms between Lie JC^*-algebras Lie C^*-algebra,* Acta Math. Sinica, **21**(2005), 1391–1398.
[34] C. Park, Th.M. Rassias, *Homomorphisms in C*–ternary algebras and JB*–triples*, J. Math. Anal. Appl., **337**(2008) 13-20.
[35] C. Park, Th.M. Rassias, *Homomorphisms and derivations in proper JCQ*–triples*, J. Math. Anal. Appl., **337**(2008), 1404-1414.
[36] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**(2003) 91-96.
[37] Th.M. Rassias, *New characterization of inner product spaces*, Bull. Sci. Math., **108**(1984), 95–99.
[38] Th.M. Rassias,*On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl., **251**(2000), 264–284.
[39] Th.M. Rassias,*On the stability of functional equations and a problem of Ulam,* Acta Appl. Math., **62**(2000), 23–130.
[40] Th.M. Rassias, P. Šemrl,*On the Hyers-Ulam stability of linear mappings*, J. Math. Anal. Appl., **173**(1993), 325–338.
[41] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), 297–300.
[42] T. Trif, *Hyers–Ulam–Rassias stability of a Jensen type functional equation*, J. Math. Anal. Appl., **250**(2000), 579–588.
[43] S.M. Ulam, *Problems in Modern Mathematics*, Chapter VI, Science Editions, Wiley, New York, 1940.
[44] H. Upmeier, *Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics,* Regional Conf. Ser. in Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987.
[45] A.R. Villena, *Derivations on Jordan-Banach algebras*, Studia Math., **118**(1996), 205-229.

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