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FIXED POINT THEOREMS WITH PPF DEPENDENCE AND FUNCTIONAL DIFFERENTIAL EQUATIONS

BAPURAO C. DHAGE

Kasubai, Gurukul Colony, Ahmedpur-413 515
Dist: Latur Maharashtra, India
E-mail: bcdhage@yahoo.co.in, bcdhage@gmail.com

Abstract. In this paper, some fixed point theorems with PPF dependence in Banach spaces are proved and then applied to functional differential equations of delay type for proving the existence of solutions. Our results generalizes the fixed point theorems with PPF dependence of Bernfield *et al.* [1] under weaker conditions.

Key Words and Phrases: fixed point theorem, PPF dependence, Banach space, functional differential equations.
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1. Introduction

In a recent paper [1], the authors proved some fixed point theorems for nonlinear operators in Banach spaces, where the domain and range of the operators is not same. The fixed point theorems of this kind are called PPF dependent fixed point theorems and are useful for proving the solutions of nonlinear functional differential and integral equations which may depend upon the past, present and future. The properties of a special Razumikhin class of functions are employed in the development of existence theory of PPF solutions for certain nonlinear equations in abstract spaces.

Given a Banach space E and given a closed interval $I = [a, b]$ in $\mathbb{R}$, we consider the Banach space $E_0 = C(I, E)$ of continuous E-valued functions on I. We equip the space E_0 with the supremum norm $\|\cdot\|_{E_0}$ defined as

$$\|\phi\|_{E_0} = \sup_{t \in I} \|\phi(t)\|_E. \tag{1.1}$$

For a fixed point $c \in I$, the Razumikhin class or minimal class of functions is defined as

$$\mathcal{R}_c = \big\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\big\}. \tag{1.2}$$

Let $T : E_0 \to E$. A point $\phi^* \in E_0$ is called a PPF dependent fixed point of T if $T\phi^* = \phi^*(c)$ for some $c \in I$.

It is mentioned in Bernfield *et. al.* [1], that the Razumikhin class of functions plays a significant role in proving the existence of PPF-fixed points with different domain and range of the operators, and proved a fundamental fixed point theorem with PPF dependence for contraction operators in Banach spaces.

Definition 1.1. *An operator $T : E_0 \to E$ is called contraction if there is a real number $0 < \alpha < 1$ such that*

$$\|T\phi - T\xi\|_E \leq \alpha \|\phi - \xi\|_{E_0} \tag{1.3}$$

for all $\phi, \xi \in E_0$.

Theorem 1.1. *Suppose that $T : E_0 \to E$ is a contraction. Then the following statements hold.*

(a) *Given $\phi_0 \in E_0$ and given a point $c \in I$, every sequence $\{\phi_n\}$ satisfying $T\phi_n = \phi_{n+1}(c)$ and $\|\phi_n - \phi_{n+1}\|_{E_0} = \|\phi_n(c) - \phi_{n+1}(c)\|_E$ converges to a fixed point ϕ^* of T.*

(b) *Given $\phi_0, \xi_0 \in E_0$, let $\{\phi_n\}$ and $\{\xi_n\}$ be the sequences of iterates constructed as in (a). Then*

$$\|\phi_n - \xi_n\|_{E_0} \leq \frac{1}{1-\alpha}[\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0}] + \|\phi_0 - \xi_0\|_{E_0}.$$

If, in particular $\phi_0 = \xi_0$ and $\{\phi_n\} \not\equiv \{\xi_n\}$, then

$$\|\phi_n - \xi_n\|_{E_0} \leq \frac{2}{1-\alpha}\|\phi_0 - \xi_0\|_{E_0}.$$

(c) *Let $\mathcal{R}_c = \big\{\phi \in E_0 \mid \|\phi\|_{E_0} = \|\phi(c)\|_E\big\}$ and let $\{\phi_n\}$ and $\{\xi_n\}$ be defined as in (a). If $\phi_n - \xi_n \in \mathcal{R}_c$ for all n, then*

$$\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \xi_n.$$

(d) *If we define $\mathcal{R}_c = \{\phi \in E_0 \mid \|\phi - \phi^*\|_{E_0} = \|\phi(c) - \phi^*(c)\|_E\}$ where $\phi^*$is a fixed point of T, then ϕ^* is the only fixed point of T.*

Now we list some of our observations.

Observation I. The statement (a) in above Theorem presupposes that the Razumikhin class $\mathcal{R}_c$ of functions in E_0 is algebraically closed with respect to difference, that is, $\phi - \xi \in \mathcal{R}_c$ whenever $\phi, \xi \in \mathcal{R}_c$. Otherwise the construction of the sequence $\{\phi_n\}$ made there is not possible, because of the fact that

$$\|\phi - \xi\|_{E_0} = \|\phi(c) - \xi(c)\|_E = \|(\phi - \xi)(c)\|_E. \tag{1.4}$$

As a result, the statement (c) is superfluous and the conclusion holds automatically.

Observation II. The Razumikhin class $\mathcal{R}_c$ of functions in E_0 is is not assumed to be topologically closed, so the sequence of successive iterations constructed as in the statement (a) converges to a PPF fixed point of the operator T which may be outside of $\mathcal{R}_c$. If we assume $\mathcal{R}_c$ to be topologically closed, then the statement (d) of Theorem 1.1 follows immediately.

Observation III. If $T : E_0 \to E$ and $\mathcal{R}_c$ is closed with respect to difference, then for a given element $c \in [a, b]$ and given $\phi_0 \in E_0$ the sequence $\{\phi_n\}$ satisfying

$$\left.\begin{aligned} T\phi_n &= \phi_{n+1}(c) \\ \|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E \end{aligned}\right\} \tag{1.5}$$

is well defined for each n. This construction follows in view of the fact that if $x \in T(E_0)(c)$, then for a given $c \in [a, b]$ we have a $\phi \in E_0$ such that $\phi(c) = x$.

In this paper, we improve Theorem 1.1 and establish some interesting fixed point theorems with PPF dependence along with some applications to functional differential equations for proving existence results under generalized Lipschitz conditions. The rest of the paper is organized as follows. Section 2 deals with the PPF dependent fixed point theorems for the operators satisfying different contraction conditions. Section 3 deals with the hybrid fixed point theorem Krasnoselskii type with PPF dependence. Finally, some applications of the newly developed abstract fixed point theorems are given in section 4.

2. PPF Dependent Fixed Point Theory

2.1. **Linear contraction.** In this section we prove some generalizations of Theorem 1.1. We need the following definition in what follows.

Definition 2.1. *An operator $T : E_0 \to E$ is called strong Kannan type contraction if*

$$\|T\phi - T\xi\|_E \le \alpha[\|\phi(c) - T\phi\|_E + \|\xi(c) - T\xi\|_E] \tag{2.1}$$

for all $\phi, \xi \in E_0$ and some $c \in [a, b]$, where $0 < \alpha < 1/2$.

Theorem 2.1. *Suppose that $T : E_0 \to E$ is a strong Kannan type contraction. Then the following statements hold.*

(a) *If $\mathcal{R}_c$ is algebraically closed w.r.t the difference, then for a given $\phi_0 \in E_0$ and $c \in [a, b]$, every sequence $\{\phi_n\}$ of iterates of T defined by* (1.5) *converges to a PPF dependent fixed point of T.*

(b) *If $\phi_0, \xi_0 \in E_0$ and $\{\phi_n\}$, $\{\xi_n\}$ are sequences defined by* (1.5)*. Then,*

$$\|\phi_n - \xi_n\|_{E_0} \le \Big(\frac{1-\alpha}{1-2\alpha}\Big)[\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0}] + \|\phi_0 - \xi_0\|_{E_0}.$$

If, in particular $\phi_0 = \xi_0$ and $\{\phi_n\} \not\equiv \{\xi_n\}$, then

$$\|\phi_n - \xi_n\|_{E_0} \le \Big[\frac{2(1-\alpha)}{1-2\alpha}\Big]\|\phi_0 - \xi_0\|_{E_0}.$$

(c) *If $\mathcal{R}_c$ is topologically closed, then T has a unique fixed point in $\mathcal{R}_c$.*

Proof. Let $\phi_0 \in E_0$ be arbitrary and define a sequence $\{\phi_n\}$ in E_0 as follows. By hypothesis, $T\phi_0 \in E$. Suppose that $T\phi_0 = x_1$. Choose $\phi_1 \in E_0$ such that $x_1 = \phi_1(c)$ and $\|\phi_1 - \phi_0\|_{E_0} = \|\phi_1(c) - \phi_0(c)\|_E$. Proceeding in this way, by induction, we obtain

$$T\phi_n = \phi_{n+1}(c)$$

and

$$\|\phi_n - \phi_{n+1}\|_{E_0} = \|\phi_n(c) - \phi_{n+1}(c)\|_E$$

for all $n = 0, 1, \ldots.$.

We claim that $\{\phi_n\}$ is a Cauchy sequence in E_0. Now for any $n \in \mathbb{N}$ we have the following estimate,

$$\begin{aligned}
\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E \\
&= \|T\phi_{n-1} - T\phi_n\|_{E_0} \\
&\le \alpha[\|\phi_{n-1}(c) - T\phi_{n-1}\|_E + \|\phi_n(c) - T\phi_n\|_E] \\
&\le \alpha[\|\phi_{n-1}(c) - \phi_n(c)\|_E + \|\phi_n(c) - \phi_{n+1}(c)\|_E] \\
&\le \alpha\|\phi_{n-1} - \phi_n\|_{E_0} + \alpha\|\phi_n - \phi_{n+1}\|_{E_0}.
\end{aligned}$$

From the above inequality, it follows that

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le \lambda\|\phi_{n-1} - \phi_n\|_{E_0}$$

for all $n = 1, 2, \ldots$, where $\lambda = \dfrac{\alpha}{1-\alpha}$.

By induction,

$$\|\phi_n - \phi_{n+1}\|_{E_0} \le \lambda^n\|\phi_0 - \phi_1\|_{E_0}$$

for all $n = 1, 2, \ldots$.

If $m > n$, by triangle inequality, we obtain

$$\begin{aligned}
\|\phi_m - \phi_n\|_{E_0} &\le \|\phi_n - \phi_{n+1}\|_{E_0} + \cdots + \|\phi_{m-1} - \phi_m\|_{E_0} \\
&\le \lambda^n\|\phi_0 - \phi_1\|_{E_0} + \cdots + \lambda^{m-1}\|\phi_0 - \phi_1\|_{E_0} \\
&\le (\lambda^n + \cdots + \lambda^{m-1})\|\phi_0 - \phi_1\|_{E_0}.
\end{aligned}$$

Hence,

$$\lim_{m>n\to\infty} \|\phi_m - \phi_n\|_{E_0} = 0.$$

As a result, the sequence $\{\phi_n\}$ is Cauchy. Since E_0 is complete, $\{\phi_n\}$ converges to a limit ϕ^* in E_0, that is, $\lim\limits_{n\to\infty} \phi_n = \phi^*$. We prove that ϕ^* is a dependent fixed point of T. Now,

$$\begin{aligned}
\|T\phi^* - \phi^*(c)\|_E &\le \|T\phi^* - \phi_{n+1}(c)\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_E \\
&\le \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1} - \phi^*\|_{E_0} \\
&\le \alpha[\|\phi_n(c) - T\phi_n\|_E + \|\phi^*(c) - T\phi^*\|_E] + \|\phi_{n+1} - \phi^*\|_{E_0} \\
&\le \alpha\|\phi_n - \phi_{n+1}\|_{E_0} + \alpha\|\phi^*(c) - T\phi^*\|_E + \|\phi_{n+1} - \phi^*\|_{E_0}.
\end{aligned}$$

From this inequality, it follows that

$$\|T\phi^* - \phi^*(c)\|_E \le \frac{1}{1-\alpha}\|\phi^* - \phi_{n+1}\|_{E_0} + \frac{\alpha\lambda^n}{1-\alpha}\|\phi_0 - \phi_1\|_{E_0}.$$

Taking the limit superior in the above inequality yields $T\phi^* = \phi^*(c)$.

(b) Let $\phi_0, \xi_0 \in E_0$ and let $\{\phi_n\}$ and $\{\xi_n\}$ be two sequences of iterations of T defined by (1.5). Then,

$$\begin{aligned}
\|\phi_n - \xi_n\|_{E_0} &\le \|\phi_n - \phi_{n-1}\|_{E_0} + \|\phi_{n-1} - \xi_{n-1}\|_{E_0} + \|\phi_{n-1} - \xi_n\|_{E_0} \\
&\le \lambda^n \|\phi_0 - \phi_1\|_{E_0} + \|\phi_{n-1} - \xi_{n-1}\|_{E_0} + \lambda^n \|\xi_0 - \xi_1\|_{E_0} \\
&\le \lambda^n [\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0}] + \|\phi_{n-1} - \xi_{n-1}\|_{E_0} \\
&\le (\lambda^n + \cdots + 1)[\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0} + \|\phi_0 - \xi_0\|_{E_0}] \\
&= \left(\frac{1}{1-\lambda}\right)[\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0} + \|\phi_0 - \xi_0\|_{E_0}] \\
&= \left(\frac{1-\alpha}{1-2\alpha}\right)[\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0} + \|\phi_0 - \xi_0\|_{E_0}]. \qquad (2.2)
\end{aligned}$$

In particular, if $\phi_0 = \xi_0$, then $\phi_0(c) = \xi_0(c)$ and $T\phi_0 = T\xi_0$ which thereby gives $\phi_1(c) = \xi_1(c)$. Hence, from inequality (2.2) it follows that

$$\|\phi_n - \xi_n\|_{E_0} \le 2\left(\frac{1-\alpha}{1-2\alpha}\right)\|\phi_0 - \phi_1\|_{E_0}.$$

(c) Let $\{\phi_n\}$ and $\{\xi_n\}$ be two sequences of iterates of T at ϕ_0 and ξ_0 respectively. Since $\{\phi_n\}$ and $\{\xi_n\}$ are in $\mathcal{R}_c$, one has $\phi_n - \xi_n \in \mathcal{R}_c$ for $n = 1, 2, \ldots$. Then,

$$\begin{aligned}
\|\phi_n - \xi_n\|_{E_0} &= \|\phi_n(c) - \xi_n(c)\|_E \\
&\le \|T\phi_{n-1} - T\xi_{n-1}\|_E \\
&\le \alpha[\|\phi_{n-1(c)} - T\phi_{n-1}\|_E + \|\xi_{n-1}(c) - T\xi_{n-1}\|_E] \\
&= \alpha[\|\phi_{n-1}(c) - \phi_n(c)\|_E + \|\xi_{n-1}(c) - \xi_n(c)\|_E] \\
&\le \alpha[\|\phi_{n-1} - \phi_n\|_{E_0} + \|\xi_{n-1} - \xi_n\|_{E_0}] \\
&\le \alpha\lambda^n[\|\phi_0 - \phi_1\|_{E_0} + \|\xi_0 - \xi_1\|_{E_0}].
\end{aligned}$$

Since both sequences $\{\phi_n\}$ and $\{\xi_n\}$ in view of statement (a), we have shown that

$$\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \xi_n.$$

(d) To prove uniqueness of fixed point in $\mathcal{R}_c$, let ϕ^* and ξ^* be two fixed points of T, then

$$\begin{aligned}
\|\phi^* - \xi^*\|_{E_0} &= \|\phi^*(c) - \xi^*(c)\|_E \\
&\le \|T\phi^* - T\xi^*\|_E \\
&\le \alpha[\|\phi^*(c) - T\phi^*\|_E + \|\xi^*(c) - T\xi^*\|_E \\
&= 0
\end{aligned}$$

which yields $\phi^* = \xi^*$. This completes the proof. □

Next, we unify Theorem 1.1 and Theorem 2.1 and prove a fixed point theorem with PPF dependence for a wider class of generalized contraction operators on Banach spaces. The class of generalized contraction operators is supposed to be the most general one and includes several classes of contraction operators in metric spaces. We consider the following definition in what follows.

Definition 2.2. *An operator $T : E_0 \to E$ is called a strong generalized contraction if there exists a real number $0 < \alpha < 1$ satisfying*

$$\|T\phi - T\xi\|_E \leq \alpha \max\{\|\phi - \xi\|_{E_0}, \|\phi(c) - T\phi\|_E, \|\xi(c) - T\xi\|_E, \frac{1}{2}[\|\phi(c) - T\xi\|_E + \|\xi(c) - T\phi\|_E]\} \tag{2.3}$$

for all $\phi, \xi \in E_0$ and for some $c \in [a, b]$.

Remark 2.1. *It is clear that contractions and strong Kannan type contractions are strong generalized contractions, but the converse may not be true.*

Theorem 2.2. *Suppose that $T : E_0 \to E$ is a strong generalized contraction. Then the following statements hold.*

(a) *If $\mathcal{R}_c$ is closed with respect to difference, then for a given $\phi_0 \in E_0$, every sequence $\{\phi_n\}$ of iterates defined by (1.5) converges to a PPF dependent fixed point of T.*

(b) *If $\mathcal{R}_c$ is algebraically and topologically closed, then for a given $\phi_0 \in E_0$ every sequence $\{\phi_n\}$ of iterates defined by (1.5) converges to a unique PPF dependent fixed point of T in $\mathcal{R}_c$.*

Proof. The proof is similar to Theorem 2.1 and hence we omit the details. □

Remark 2.2. *We note that Theorems 2.1 and 2.2 do not require any continuity condition of the operator T on the domain of its definition.*

2.2. **Nonlinear contractions.** A nonlinear contraction is an important generalization of contraction operators in Banach spaces in which the contraction constant α is merged into a function. The results of this type are useful in the theory of nonlinear differential equations wherein the nonlinearity does satisfy Lipschitz condition in an usual way. The following definition is useful in what follows.

Definition 2.3. *A nonlinear operator $T : E_0 \to E$ is called a nonlinear contraction if there exists a upper continuous function from the right $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|T\phi - T\xi\|_E \leq \psi(\|\phi - \xi\|_{E_0}) \tag{2.4}$$

for all $\phi, \xi \in E_0$, where $\psi(r) < r, r > 0$. Similarly, T is called $\mathcal{B}$-contraction if there exists a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous from right and satisfies (2.4).

Note that every contraction is $\mathcal{B}$-contraction and every $\mathcal{B}$-contraction is nonlinear contraction, however the converse may not be true. The details appears in a monograph of Browder [2].

Theorem 2.3. *Suppose that $T : E_0 \to E$ is a nonlinear contraction. Then the following statements hold in E_0.*

(a) *If $\mathcal{R}_c$ is algebraically closed with respect to difference, then every sequence $\{\phi_n\}$ of successive iterates of T at each point $\phi_0 \in E_0$ defined as in* (1.5) *converges to a PPF dependent fixed point of T.*

(b) *If $\mathcal{R}_c$ is topologically closed, then ϕ^* is the only fixed point of T in $\mathcal{R}_c$.*

Proof. Let $\phi_0 \in E_0$ be arbitrary and define a sequence $\{\phi_n\}$ in E_0 by (1.5). Then,

$$\begin{aligned}\|\phi_n - \phi_{n+1}\|_{E_0} &= \|\phi_n(c) - \phi_{n+1}(c)\|_E \\ &= \|T\phi_{n-1} - T\phi_n\|_{E_0} \\ &\le \psi(\|\phi_{n-1} - \phi_n\|_{E_0})\end{aligned}$$

for each $n = 1, 2, \dots$.

Denote

$$d_n = \|\phi_n - \phi_{n+1}\|_{E_0}.$$

Then,

$$d_1 > d_2 > \cdots > d_n > \cdots .$$

Therefore, $\lim_{n\to\infty} d_n = d$ exists. Since $d \le \psi(d)$, one has $d = 0$.

We show that the sequence $\{\phi_n\}$ is Cauchy. Suppose not. Then for $\epsilon > 0$ there exists an integer $k \le n(k) \le m(k)$

$$\|\phi_{m(k)} - \phi_{n(k)}\|_{E_0}\| \ge \epsilon.$$

Let $m(k) > n(k) \ge k$ be the smallest integer satisfying

$$\|\phi_{m(k)} - \phi_{n(k)}\|_{E_0} \ge \epsilon.$$

Denote

$$r_k = \|\phi_{m(k)} - \phi_{n(k)}\|_{E_0}.$$

Then,

$$\begin{aligned}\epsilon \le r_k &= \|\phi_{m(k)} - \phi_{n(k)}\|_{E_0} \\ &\le \|\phi_{m(k)} - \phi_{m(k)-1}\|_{E_0} + \|\phi_{m(k)-1} - \phi_{n(k)-1}\|_{E_0} \\ &\qquad + \|\phi_{n(k)-1} - \phi_{n(k)}\|_{E_0} \\ &\le d_{m(k)-1} + \epsilon + d_{n(k)-1},\end{aligned}$$

and so, $\lim_{k\to\infty} r_k = \epsilon$.

Next,

$$\begin{aligned}\epsilon \le r_k &= \|\phi_{m(k)} - \phi_{n(k)}\|_{E_0} \\ &\le \|\phi_{m(k)} - \phi_{m(k)+1}\|_{E_0} + \|\phi_{m(k)+1} - \phi_{n(k)+1}\|_{E_0} \\ &\qquad + \|\phi_{m(k)+1} - \phi_{n(k)}\|_{E_0} \\ &\le d_{m(k)} + \psi(\|\phi_{m(k)} - \phi_{n(k)}\|_{E_0}) + d_{n(k)},\end{aligned}$$

or,

$$\epsilon \le \psi(\epsilon) < \epsilon$$

which is a contradiction. This shows that the sequence $\{\phi_n\}$ is Cauchy. Since E_0 is complete, there is a point $\phi^* \in E_0$ such that $\lim_{n\to\infty} \phi_n = \phi^*$. Now we show that ϕ^* is a PPF dependent fixed point of T. By triangle inequality,

$$\begin{aligned}\|T\phi^* - \phi^*(c)\|_E &\le \|T\phi^* - \phi_{n+1}(c)\|_E + \|\phi_{n+1}(c) - \phi^*(c)\|_E \\ &\le \|T\phi^* - T\phi_n\|_E + \|\phi_{n+1} - \phi^*\|_{E_0} \\ &\le \psi(\|\phi^* - \phi_n\|_{E_0}) + \|\phi_{n+1} - \phi^*\|_{E_0} \\ &\to 0 \qquad \text{as} \quad n \to \infty.\end{aligned}$$

As a result, $T\phi^* = \phi^*(c)$.

(b) To prove uniqueness, let $\phi^* \neq \xi^*$ be two PPF dependent fixed points of T. Then, from closedness of $\mathcal{R}_c$, it follows that $\phi^*, \xi^* \in \mathcal{R}_c$. Moreover,

$$\begin{aligned}\|\phi^* - \xi^*\|_{E_0} &= \|\phi^*(c) - \xi^*(c)\|_E \\ &\le \|T\phi^* - T\xi^*\|_E \\ &\le \psi(\|\phi^* - \xi^*\|_{E_0})\end{aligned}$$

which is a contradiction since $\psi(r) < r$ for $r > 0$. Hence, $\phi^* = \xi^*$. This completes the proof. □

Corollary 2.1. *Suppose that $T : E_0 \to E$ is a $\mathcal{B}$-contraction. Then the statements (a) and (b) of Theorem 2.3 hold in E_0.*

3. PPF Dependent Hybrid Fixed point Theory

It is known that the hybrid fixed point theory initiated by Krasnoselskii [7] is useful to prove the existence theorems for perturbed or hybrid differential and integral equations. See Dhage [3] and the references given therein. Below we prove such a hybrid fixed point theorem with PPF dependence which is an improvement upon the Krasnoselskii type fixed point theorem of Bernfield *et al* [1] under more general contraction conditions.

We need the following definition in what follows.

Definition 3.1. *An operator $T : E_0 \to E$ is called strong nonlinear $\mathcal{B}$-contraction if there exists a right continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|T\phi - T\xi\|_E \le \psi(\|\phi(c) - \xi(c)\|_{E_0}) \tag{3.1}$$

for all $\phi, \xi \in E_0$ and for some $c \in [a, b]$, where $\psi(r) < r$ for $r > 0$.

We remark that every strong $\mathcal{B}$-contraction is a $\mathcal{B}$-contraction and consequently a nonlinear contraction on E_0, but the converse may not be true.

Our main PPF dependent hybrid fixed point theorem is the following.

Theorem 3.1. *Let $A : E_0 \to E$ and $B : E \to E$ be two operators such that*

(a) *A is strong nonlinear $\mathcal{B}$-contraction, and*
(b) *B is continuous and compact.*

Further, if the Razumikhin class of functions $\mathcal{R}_c$ is algebraically closed with respect to difference and topologically closed, then for a given $c \in [a, b]$ the operator equation

$$A\phi + B\phi(c) = \phi(c) \tag{3.2}$$

has a solution in $\mathcal{R}_c$.

Proof. Let $\xi \in E_0$ be fixed and let $c \in [a, b]$ be given. Define an operator $T_{\xi(c)} : E_0 \to E$ by

$$T_{\xi(c)}(\phi) = A\phi + B\xi(c). \tag{3.3}$$

Clearly, $T_{\xi(c)}$ is a $\mathcal{B}$-contraction on E_0. To see this, let $\phi_1, \phi_2 \in E_0$. Then,

$$\begin{aligned} \|T_{\xi(c)}(\phi_1) - T_{\xi(c)}(\phi_2)\|_E &= \|A\phi_1 - A\phi_2\|_E \\ &\le \psi(\|\phi_1(c) - \phi_2(c)\|_E) \\ &\le \psi(\|\phi_1 - \phi_2\|_{E_0}). \end{aligned}$$

This shows that $T_{\xi(c)}$ is a strong $\mathcal{B}$-contraction and by Theorem 2.3 there is a unique PPF dependence fixed point $\phi^* \in E_0$ such that

$$T_{\xi(c)}(\phi^*) = \phi^*(c)$$

or

$$A\phi^* + B\xi(c) = \phi^*.$$

Next, we define a mapping $Q : E \to E$ by

$$Q\xi(c) = \phi^* = A\phi^* + B\xi(c). \tag{3.4}$$

It then follows that

$$\begin{aligned} \|Q\xi_1(c) - Q\xi_2(c)\|_E &= \|A\phi_1^* + B\xi_1(c) - A\phi_2^* - B\xi_2(c)\|_E \\ &\le \|A\phi_1^* - A\phi_2^*\|_E + \|B\xi_1(c) - B\xi_2(c)\|_E \\ &\le \psi(\|\phi_1^*(c) - \phi_2^*(c)\|_E) + \|B\xi_1(c) - B\xi_2(c)\|_E. \end{aligned} \tag{3.5}$$

Since B is compact, if $\{B\xi_n(c)\}$ is any sequence in E, then $\{B\xi_n(c)\}$ has a convergent subsequence. Without loss of generality, call it the same sequence. Hence, $\{B\xi_n(c)\}$ is a Cauchy sequence. From inequality (3.5), we obtain

$$\|Q\xi_m(c) - Q\xi_n(c)\|_E \le \psi(\|\phi_m^*(c) - \phi_n^*(c)\|_E) + \|B\xi_m(c) - B\xi_n(c)\|_E.$$

Taking the limit superior in above inequality yields

$$\begin{aligned} &\limsup_{m,n\to\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E \\ &\qquad \le \limsup_{m,n\to\infty} \psi(\|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E) + \limsup_{m,n\to\infty} \|B\xi_m(c) - B\xi_n(c)\|_E \\ &\qquad \le \psi\left(\limsup_{m,n\to\infty} \|Q\xi_m^*(c) - Q\xi_n^*(c)\|_E\right). \end{aligned}$$

Hence,

$$\lim_{m,n\to\infty} \|Q\xi_m(c) - Q\xi_n(c)\|_E = 0.$$

As a result, $\{Q\xi_n(c)\}$ is a Cauchy sequence. Since E is complete, $\{Q\xi_n(c)\}$ has a convergent subsequence. Now a direct application of Schauder fixed point yields that there is a point $\xi \in E_0$ such that $Q\xi^*(c) = \xi^*(c)$. Consequently $A\xi^* + B\xi^*(c) = \xi^*(c)$ and that $\xi^* \in \mathcal{R}_c$. This completes the proof. $\square$

Remark 3.1. In case of hybrid fixed point theory, it is necessary to assume all the mixed hypotheses from the points of one space only. Hence, Theorem 2.3 does not remain true if we replace strong $\mathcal{B}$-contraction with $\mathcal{B}$-contraction or a nonlinear contraction on E_0. This answers negatively a question raised in Bernfield *et al.* [1].

4. Functional Differential Equations

In this section we apply the abstract results proved in previous section to functional differential equations for proving the existence of solutions under a weaker Lipschitz condition than Bernfield *et al.* [1].

Given a closed interval $I_0 = [-r, 0]$ in $\mathbb{R}$ for some real number $r > 0$, let $\mathcal{C}$ denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|\phi\|_{\mathcal{C}} = \sup_{\theta \in I_0} |\phi(\theta)|. \tag{4.1}$$

It is clear that $\mathcal{C}$ is a Banach space with this norm called the history space of the problem under consideration.

Given a function $x \in C(I_0 \cup I, \mathbb{R})$, for each $t \in I = [0, T]$, define a function $t \to x_t \in \mathcal{C}$ by

$$x_t(\theta) = x(t + \theta), \ \theta \in I_0, \tag{4.2}$$

where the argument θ represents the delay in the argument of solutions.

Now we are equipped with the necessary details to study the nonlinear problems of nonlinear differential equations.

4.1. **First order functional differential equations.** Given a function $\phi \in \mathcal{C}$, consider the first order ordinary functional differential equation (in short FDE)

$$\left.\begin{aligned} x'(t) &= f(t, x_t) \\ x_0 &= \phi \end{aligned}\right\} \tag{4.3}$$

for all $t \in I$, where $f : I \times \mathcal{C} \to \mathbb{R}$ is continuous.

By a solution x of the FDE (4.3) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation in (4.3), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions on $J = I_0 \cup I$.

We consider the following hypothesis in what follows.

(H_1) There exist real numbers $L > 0$ and $K > 0$ such that

$$|f(t, x) - f(t, y)| \leq \frac{L\|x - y\|_{\mathcal{C}}}{K + \|x - y\|_{\mathcal{C}}}$$

for all $x, y \in \mathcal{C}$. Moreover, we assume that $LT \leq K$.

Remark 4.1. If $L < K$ in hypothesis (H_1), then it reduces to the usual Lipschitz condition of f, namely,

$$|f(t, x) - f(t, y)| \leq (L/K)\|x - y\|_{\mathcal{C}}.$$

Theorem 4.1. *Assume that* (H_1) *holds. Then the FDE* (4.1) *has a solution on* J.

Proof. Set $E = C(J, \mathbb{R})$. Define a set of functions

$$\widehat{E} = \{\hat{x} = (x_t)_{t\in I} : x_t \in \mathcal{C}, x \in C(I, \mathbb{R}) \text{ and } x_0 = \phi\}. \tag{4.4}$$

Clearly, $\hat{x} \in C(I_0, \mathbb{R}) = \mathcal{C}$. Next we show that $\widehat{E}$ is complete. Consider a Cauchy sequence $\{\hat{x}_n\}$ in $\hat{E}$. Then, $\{(x_t^n)_{t\in I}\}$ is a Cauchy sequence in $\mathcal{C}$ for each $t \in I$. This further implies that $\{x_t^m(s)\}$ is a Cauchy sequence in in $\mathbb{R}$ for each $s \in [-r, 0]$. Then $\{x_t^m(s)\}$ converges to $x_t(s)$ for each $t \in I_0$. Since $\{x_t^n\}$ is a sequence of uniformly continuous functions for a fixed $t \in I$, $x_t(s)$ is also continuous in $s \in [-r, 0]$. Hence the sequence $\{\hat{x}_n\}$ converges to $\hat{x} \in \hat{E}$. As a result, $\widehat{E}$ is complete.

Now, consider a operator Q on $\widehat{E}$ into $\mathbb{R}$ defined as

$$Q\hat{x} = Q(x_t)_{t\in I} = \begin{cases} \phi(0) + \displaystyle\int_0^t f(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{4.5}$$

Then,

$$\begin{aligned} \|Q\hat{x} - Q\hat{y}\|_E &\leq \int_0^t |f(s, x_s) - f(s, y_s)|\, ds \\ &\leq \int_0^t \frac{L\|x_s - y_s\|_{\mathcal{C}}}{K + \|x_s - y_s\|_{\mathcal{C}}}\, ds \\ &\leq \int_0^t \frac{L\|\hat{x} - \hat{y}\|_{\widehat{E}}}{K + \|\hat{x} - \hat{y}\|_{\widehat{E}}}\, ds \\ &\leq \frac{LT\|\hat{x} - \hat{y}\|_{\widehat{E}}}{K + \|\hat{x} - \hat{y}\|_{\widehat{E}}} \\ &= \psi(\|\hat{x} - \hat{y}\|_{\widehat{E}}) \end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi(r) = \dfrac{LTr}{K + r}$, $LT \leq K$. Hence, Q is a nonlinear contraction on $\widehat{E}$ since $\psi(r) < r$, $r > 0$. Now we apply Theorem 2.3 and get that Q has a PPF dependent fixed point $\hat{x}^*$ such that $Q\hat{x}^* = \hat{x}^*(0) = x^*(t)$ on $t \in I$. Further, the sequence $\{(x_t^n)_{t\in I}\}$ of successive approximations defined by

$$(x_t^{n+1})_{t\in I} = \begin{cases} \phi(0) + \displaystyle\int_0^t f\big(s, (x_s^n)_{s\in I}\big)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{4.6}$$

with $(x_t^0)_{t\in I} = \phi$ converges to x^*. □

4.2. **Hybrid differential equations.** Next, given $\phi \in \mathcal{C}$, consider the perturbed or a hybrid differential equation of delay type (in short HDE)

$$\left.\begin{aligned} x'(t) &= f(t, x_t) + g(t, x(t)) \\ x_0 &= \phi \end{aligned}\right\} \tag{4.7}$$

for all $t \in I$, where $f : I \times \mathcal{C} \to \mathbb{R}$ and $g : I \times \mathbb{R} \to \mathbb{R}$ are continuous.

By a solution x of the FDE (4.3) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation in (4.3), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions on $J = I_0 \cup I$.

We consider the following hypothesis in what follows.

(H_1) There exist real numbers $L > 0$ and $K > 0$ such that

$$|f(t, x) - f(t, y)| \leq \frac{L\|x(0) - y(0)\|_{\mathcal{C}}}{K + \|x(0) - y(0)\|_{\mathcal{C}}}$$

for all $x, y \in \mathcal{C}$. Moreover, we assume that $LT \leq K$.

(H_2) There exists a real number $M > 0$ such that

$$|g(t, x)| \leq M$$

for all $t \in I$ and $x \in \mathbb{R}$.

Theorem 4.2. *Assume that hypotheses (H_1) and (H_2) hold. Then the HDE* (4.7) *has a solution on J.*

Proof. Set $E = C(J, \mathbb{R})$ with the usual supremum norm $\| \cdot \|_E$. Define a set $\widehat{E}$ by (4.4). Then $\widehat{E}$ is a Banach space w.r.t. the norm

$$\|\hat{x}\|_{\widehat{E}} = \sup_{t \in I} \|x_t\|_{\mathcal{C}}.$$

Now the hybrid differential equation is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t f(s, x_s)\, ds + \int_0^t g(s, x(s))\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{4.8}$$

Consider two operators $A : \widehat{E} \to \mathbb{R}$ and $B : \ C(J, \mathbb{R}) \to \mathbb{R}$ defined by

$$A\hat{x} = A(x_t)_{t \in I} = \begin{cases} \phi(0) + \displaystyle\int_0^t f(s, x_s)\, ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases} \tag{4.9}$$

and

$$Bx(t) == \begin{cases} \displaystyle\int_0^t g(s, x(s))\, ds, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{4.10}$$

Then the HIE (4.8) is equivalent to the operator equation

$$A\hat{x} + B\hat{x}(0) = \hat{x}(0). \tag{4.11}$$

We shall show that the operators A and B satisfy all the condition of Theorem 3.1. First we show that A is a strong $\mathcal{B}$-contraction on $\widehat{E}$. Then,

$$\begin{aligned}
\|A\hat{x} - A\hat{y}\|_E &= \|A(x_t)_{t\in I} - A(y_t)_{t\in I}\| \\
&\leq \int_0^t |f(s,x_s) - f(s,y_s)|\, ds \\
&\leq \int_0^t \frac{L\|x_s(0) - y_s(0)\|_{\mathcal{C}}}{K + \|x_s(0) - y_s(0)\|_{\mathcal{C}}}\, ds \\
&\leq \int_0^t \frac{L\|\hat{x}(0) - \hat{y}(0)\|_E}{K + \|\hat{x}(0) - \hat{y}(0)\|_E}\, ds \\
&\leq \frac{LT\|\hat{x}(0) - \hat{y}(0)\|_E}{K + \|\hat{x}(0) - \hat{y}(0)\|_E} \\
&= \psi(\|\hat{x}(0) - \hat{y}(0)\|_E)
\end{aligned}$$

for all $\hat{x}, \hat{y} \in \widehat{E}$, where $\psi(r) = \dfrac{LTr}{K+r}$, $LT \leq K$. Hence, Q is a strong nonlinear contraction on $\widehat{E}$ since $\psi(r) < r,\ r > 0$.

Next, we show that B is compact and continuous operator on $C(J,\mathbb{R})$. Let $\{x_n\}$ be a sequence in $C(J,\mathbb{R})$ such that $x_n \to x$ as $n \to \infty$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \int_0^t g(s, x_n(s))\, ds \\
&= \int_0^t \lim_{n\to\infty} g(s, x_n(s))\, ds = Bx(t)
\end{aligned}$$

for all $t \in J$. This shows that $\{Bx_n(t)\}$ converges to $Bx(t)$ point-wise. Moreover, it can be shown as below that $\{Bx_n\}$ is an equicontinuous sequence of functions in E. Now, following the arguments similar to that given in Granas *et al.* [6], it is proved that B is a a continuous operator on S.

Secondly, we show that B is compact. To, finish, it is enough to show that $B(E)$ is uniformly bounded and equi-continuous set in E. Let $x \in E$ be arbitrary. Then,

$$|Bx(t)| \leq \int_0^t |g(s, x(s))|\, ds \leq MT$$

for all $t \in J$ which shows that $B(E)$ is uniformly bounded set in E. To show equi-continuity, let $t, \tau \in I$. Then,

$$|Bx(t) - Bx(\tau)| \leq \left| \int_\tau^t |g(s, x(s))|\, ds \right| \leq M\,|t - \tau|.$$

If $\tau \in I_0$ and $t \in I$, then $\tau \to 0$ and $t \to 0$ whenever, $|\tau - t| \to 0$. Whence it follows that

$$|Bx(t) - Bx(\tau)| \leq |Bx(\tau) - Bx(0)| + |Bx(t) - Bx(0)| \leq M\,|t - \tau|.$$

From the above inequalities it follows that $B(E)$ is an equi-continuous set in E. Now an application of Arzellá-Ascoli theorem yields that B is a compact operator on E into itself. Hence, by Theorem 3.1, the integral equation (4.8) has a solution on J. This further implies that the HDE (4.7) has a solution on J. This completes the proof. □

Remark 4.2. In recent papers [4, 5], the authors proved some fixed point theorems with PPF dependence for strong contraction mappings in ordered metric spaces, however, these results may also be extended to a wider class of nonlinear strong contractions having a greater flexibility for applications.

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