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# EXISTENCE OF POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS ON INFINITE INTERVALS

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**Abstract.** In present paper, the author investigates the existence of positive solutions of boundary value problems for second-order functional differential equations on infinite intervals as follows

$$\begin{cases} x'' - p(t)x' - q(t)x + f(t, x_t, x'_t) = 0, t \in I = [0, \infty), \\ \alpha x(t) - \beta x'(t) = \xi(t) \geq 0, t \in [-\tau, 0], \xi(0) = x(\infty) = 0, \end{cases}$$

where $\alpha \geq 0, \beta > 0, \xi(t) \in C[-\tau, 0]$. By applying fixed point index theorem on cone and operator spectra theorem, the author obtains the results on existence of positive solutions of boundary value problems.

**Key Words and Phrases**: Functional differential equation, positive solution, fixed point index on cone, operator spectra theorem.
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## 1. Introduction

In recent years, many authors have paid attention to the research of boundary value problems for functional differential equations owing to its potential significant applications, see for example [1-9]. In paper [10], Bai et al. study the existence of positive solutions for boundary value problem

$$\begin{cases} x'' - \bar{p}x' - \bar{q}x + f(t, x_t, x'_t) = 0, t \in I = [0, \infty), \\ \alpha x(t) - \beta x'(t) = \xi(t) \geq 0, t \in [-\tau, 0], \xi(0) = x(\infty) = 0, \end{cases} \tag{1.1}$$

where constants $\bar{p} \geq 0, \bar{q} > 0, \alpha \geq 0, \beta > 0$, function $f$ is continuous and nonnegative.

It is worth pointing out that the method of proof used in [10] is transforming boundary value problem (simply denoted by BVP ) into an integral equation, so that the theorem on fixed point index on cone can be applied.

In this paper, we are concerned with the more general boundary value problem

$$\begin{cases} x'' - p(t)x' - q(t)x + f(t, x_t, x'_t) = 0, t \in I = [0, \infty), \\ \alpha x(t) - \beta x'(t) = \xi(t) \geq 0, t \in [-\tau, 0], \xi(0) = x(\infty) = 0, \end{cases} \tag{1.2}$$

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where $\alpha \geq 0, \beta > 0$ and $p(t), q(t) \in C(I, R)$. Obviously, BVP (1.1) can be regarded as the special case of BVP (1.2) with $p(t) = \bar{p}$, and $q(t) = \bar{q}$. Now, owing to $p(t), q(t)$ in (1.2) being variable with respect to $t$, we can not expect to transform directly BVP (1.2) into an integral equation as in [10]. In order to overcome the present difficulty here, we introduce the method of operator spectra combined with the application of the cone fixed point theorem. We successfully established existence of positive solutions of BVP (1.2), and generalized correspond the results in [10].

Now, let us begin with formal transformation on BVP(1.2).

Put

$$u(-\tau;t) = \frac{1}{\beta} e^{\frac{\alpha}{\beta}t} \int_t^0 e^{-(\frac{\alpha}{\beta})s} \xi(s) ds, t \in [-\tau, 0]; \quad u(t) = \begin{cases} 0, t \in I, \\ u(-\tau;t), t \in [-\tau, 0], \end{cases} \tag{1.3}$$

and set $y = x - u$, then by [10], BVP(1.2) can be reduced to the form

$$\begin{cases} y'' - p(t)y' - q(t)y + f(t, y_t + u_t, y'_t + u'_t) = 0, t \in I \\ \alpha y(t) - \beta y'(t) = 0, t \in [-\tau, 0], y(\infty) = 0. \end{cases} \tag{1.4}$$

Let $r_1, r_2$ be two real roots of $x^2 - \bar{p}x - \bar{q} = 0$ i.e. $r_1 = \frac{\bar{p}+\sqrt{\bar{p}^2+4\bar{q}}}{2}$, $r_2 = \frac{\bar{p}-\sqrt{\bar{p}^2+4\bar{q}}}{2}$.The constants $\bar{p}, \bar{q}$ satisfies $\bar{p} \geq 0, \bar{q} > 0$. It is easy to see that $\bar{p} \geq 0, \bar{q} > 0$ imply $r_2 < 0 < r_1$. Let $h = \frac{\alpha - \beta r_1}{\alpha - \beta r_2}$, then $\beta > 0$ yields $h < 1$. As in [10], throughout the paper, we always assume that $h > 0$.

Let $\sigma(t) \in C(I, R)$ satisfying $\int_0^\infty \sigma(t) e^{-r_1 t} dt < \infty$. Consider the following linear differential equation

$$\begin{cases} y'' - \bar{p}y' - \bar{q}y + \sigma = 0, t \in I, \\ \alpha y(t) - \beta y'(t) = 0, t \in [-\tau, 0], x(\infty) = 0. \end{cases} \tag{1.5}$$

From [10], BVP(1.5) has an unique solution $y = T\sigma(t), t \in J = [-\tau, \infty)$ as bellow

$$(T\sigma)(t) = \begin{cases} g_1(t), t \in I, \\ g_2(t), t \in [-\tau, 0], \end{cases} \tag{1.6}$$

where

$$g_1(t) = \int_0^\infty G(t,s)\sigma(s)ds, t \in I; g_2(t) = e^{\frac{\alpha}{\beta}t} g_1(0), t \in [-\tau, 0], \tag{1.7}$$

and

$$G(t,s) = \frac{1}{r_1 - r_2} \begin{cases} e^{r_2 t}(e^{-r_2 s} - h e^{-r_1 s}), 0 \leq s \leq t, \\ e^{-r_1 s}(e^{r_1 t} - h e^{r_2 t}), 0 \leq t \leq s. \end{cases} \tag{1.8}$$

By $h > 0$, we have $\alpha - \beta r_1 > 0$.

Throughout the paper, we keep the following notations:

For fixed $\mu \in (r_1, \frac{\alpha}{\beta})$, set $E = \left\{ y \in C^1[J, R] : \sup_{t \in J} |\, y(t)| e^{-\mu t} < \infty, \sup_{t \in J} |\, y'(t)| e^{-\mu t} \right\}$.

From [10], it follows that $(E, ||\cdot||_1)$ is a Banach space equipped with the norm

$$||y||_1 = \sup_{t \in J} \left\{ (|y(t)| + |y'(t)|) e^{-\mu t} \right\}, \text{ for } y \in E.$$

Let

$$X = \left\{ x \in C(I,R) : \int_0^{\infty} |x(t)| e^{-r_1 t} dt < \infty \right\}, Z = C([-\tau, 0], R),$$

and

$$E^+ = \{x \in E : x(t) \geq 0, t \in J\},$$
$$X^+ = \{x \in X : x(t) \geq 0, t \in I\},$$
$$Z^+ = \{x \in Z : x(t) \geq 0, t \in [-\tau, 0]\}.$$

It is easy to see that $(X, ||\cdot||_X)$ is a normed linear space equipped with the norm $||x||_X = \int_0^{\infty} |x(t)| e^{-r_1 t} dt$, $x \in X$, and $(Z, ||\cdot||_0)$ is a Banach space equipped with the norm $||z||_0 = \max_{t \in [-\tau,0]} |z(t)|, z \in Z$.

As usual, $\forall y \in E, \forall s \in I, \theta \in [-\tau, 0], y_s(\theta) = y(s+\theta), y'_s(\theta) = y'(s+\theta)$. Obviously, for $y \in E, s \in I$, we have $y_s, y'_s \in Z$.

For convenience, we list the following assumptions:

$(H_1)$ $f(t, \phi, \varphi) \in C(I \times Z^+ \times Z, I)$, and exists $\nu > 1$ with $\mu\nu < \frac{\beta}{\alpha}$, such that the following inequality holds

$$f(t, \phi, \varphi) \leq a(t) + b(t)(||\phi||_0^{\nu} + ||\varphi||_0^{\nu}), \text{for any } t \in I, \phi \in Z^+, \varphi \in Z,$$

where $a(t) \in X^+, b(t)e^{\nu\mu t} \in X^+$.

$(H_2)$ Exists $0 < \gamma < \delta, M > \frac{\lambda_0}{m_0 \tau_0}$, and $R_1 > 0$ such that

$$f(t, \phi, \varphi) \geq M(||\phi||_0 + ||\varphi||_0), \text{for } t \in [\gamma, \delta], (\phi, \varphi) \in Z^+ \times Z \text{ with } ||\phi||_0 + ||\varphi||_0 \geq R_1,$$

where $\lambda_0$ described as in $(H_3)$ bellow, and $m_0, \tau_0$ will be given later on.

$(H_3)$ $p(t), q(t) \in C(I, R), 0 \leq \bar{p} \doteq \sup_{t \in I} p(t) < \infty, 0 < \bar{q} \doteq \sup_{t \in I} q(t) < \infty, q_1(t) \geq \lambda_0 p_1(t), t \in I$, $q_1(t)e^{\mu t} \in X$, where $p_1(t) \doteq \bar{p} - p(t), q_1(t) \doteq \bar{q} - q(t)$, $\lambda_0 = \max\left\{\sqrt{\bar{p}^2 + 4\bar{q}}, 1\right\}$.

**Remark 1.** In the following, we always denote $r_{1,2} = \frac{\bar{p} \pm \sqrt{\bar{p}^2 + 4\bar{q}}}{2}$, and the constants $\bar{p}, \bar{q}$ described as in $(H_3)$.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, and the proofs of the main results are given in Section 3.

## 2. Preliminaries

**Lemma 1.**[10] The Green's function $G(t,s)$ given by (1.8) has following properties

$(i)$ $G(t,s) \geq 0, \forall t, s \in I$.

$(ii)$ $G(t,s)e^{-vt} \leq G(s,s)e^{-r_1 s}, \text{for } v \geq r_1, s, t \in I$.

$(iii)$ $G_t(t,s) = \frac{1}{r_1 - r_2} \begin{cases} r_2 e^{r_2 t}(e^{-r_2 s} - h e^{-r_1 s}), 0 \leq s < t, \\ e^{-r_1 s}(r_1 e^{r_1 t} - h r_2 e^{r_2 t}), 0 \leq t < s. \end{cases}$

**Lemma 2.**[10] For $t \in [\gamma, \delta] \subset I \backslash \{0\}, t, s \in I$,

(1) if $r_2 \leq -1$, then

$$(r_1 - r_2)G(t,s) + G_t(t,s) \geq \begin{cases} m_1[G(s,s) + |G_t(s-0,s)|]e^{-r_1 s}, \text{if } t < s, \\ m_1[G(s,s) + |G_t(s+0,s)|]e^{-r_1 s}, \text{if } t > s, \end{cases}$$

where $m_1 = \frac{|r_2|}{1+r_1} e^{r_2} \delta$.

(2) if $-1 < r_2 < 0$, then

$$G(t,s) + G_t(t,s) \geq \begin{cases} m_2[G(s,s) + |G_t(s-0,s)|]e^{-r_1 s}, \text{if } t < s, \\ m_2[G(s,s) + |G_t(s+0,s)|]e^{-r_1 s}, \text{if } t > s, \end{cases}$$

where $m_2 = \min\{\frac{1+r_2}{1-r_2}e^{r_2\delta}, e^{r_1\gamma} - h\frac{1+r_2}{1+r_1}e^{r_2\gamma}\}$.

**Lemma 3.**[10] Let $h \geq 0$. For $\varsigma, s \in I, \varsigma \neq s$ and $v \geq r_1$

(i) If $r_2 \leq -1$, then

$$[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma} \leq \begin{cases} (1+h)G[(s,s) + |G_t(s-0,s)|]e^{-r_1 s}, \text{if } \varsigma < s, \\ (1+h)G[(s,s) + |G_t(s+0,s)|]e^{-r_1 s}, \text{if } \varsigma > s. \end{cases}$$

(ii) If $-1 < r_2 < 0$, then

$$[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma} \leq \begin{cases} (G(s,s) + |G_t(s-0,s)|)e^{-r_1 s}, \text{if } \varsigma < s, \\ (G(s,s) + |G_t(s+0,s)|)e^{-r_1 s}, \text{if } \varsigma > s. \end{cases}$$

**Remark 2.** In Lemma 3, the inequality is dependent on the case $\varsigma < s$ or $\varsigma > s$, however, in the proof below, it will be necessary that corresponding inequality holds independently on the case $\varsigma < s$ or $\varsigma > s$. Therefor, the following Lemma will be useful.

**Lemma 4.** Let $0 < h < 1$. For $\varsigma, s \in I, \varsigma \neq s, v \geq r_1$

(i) If $r_2 \leq -1$, then

$$[G(s,s) + |G_t(s \pm 0,s)|]e^{-r_1 s} \geq m_4[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma},$$

where $m_4 = \min\{\frac{r_1}{|r_2|}, \frac{r_2(h-1)}{r_1 - r_2 h}, \frac{1}{1+h}\}$.

(ii) If $-1 < r_2 < 0$, then

$$[G(s,s) + |G_t(s \pm 0,s)]e^{-r_1 s} \geq m_5[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma},$$

where $m_5 = \min\{\frac{r_1}{|r_2|}, \frac{r_2(h-1)}{r_1 - r_2 h}, 1\}$.

*Proof.* We notice that $r_2 < 0 < r_1, 0 < h < 1, v \geq r_1, \varsigma, s \in I, \varsigma \neq s$.

(1) If $r_2 \leq -1$, we consider the case $\varsigma > s$ or $\varsigma < s$ respectively.

(i) If $\varsigma > s$, by (1.8) and (2.3) , we have

$$G(s,s) + |G_t(s-0,s)| = 1 - he^{(r_2-r_1)s} + r_1 - hr_2 e^{(r_2-r_1)s},$$

$$[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma + r_1 s} = e^{r_2(\varsigma - s) + (r_1 s - v\varsigma)}[1 - he^{(r_2-r_1)s} + |r_2|(1 - he^{(r_2-r_1)s})].$$

Owing to $e^{r_2(\varsigma-s)+(r_1 s - v\varsigma)} < 1$, by taking $0 < m_4 \leq \min\left\{1, \frac{r_1}{|r_2|}\right\}$, we have

$$r_1 - hr_2 e^{(r_2-r_1)s} > r_1 \geq m_4|r_2| \geq m_4|r_2|(1 - he^{(r_2-r_1)s}).$$

So, it follows that

$$G(s,s) + |G_t(s-0,s)| \geq m_4[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma + r_1 s}.$$

(ii) If $\varsigma < s$, similarly to the case (i), we have

$$G(s,s) + |G_t(s+0,s)| = 1 - he^{(r_2-r_1)s} + |r_2|(1 - he^{(r_2-r_1)s}),$$

$$[G(\varsigma,s) + |G_t(\varsigma,s)|]e^{-v\varsigma + r_1 s} = e^{(r_1-v)\varsigma}[1 - he^{(r_2-r_1)\varsigma} + r_1 - hr_2 e^{(r_2-r_1)\varsigma}].$$

By $\varsigma < s$, we have $1-he^{(r_2-r_1)s} \geq 1-he^{(r_2-r_1)\varsigma}$. Taking $0 < m_4 \leq \min\left\{1, \frac{r_2(h-1)}{r_1-r_2h}\right\}$, it turns out that

$$|r_2|(1-he^{(r_2-r_1)s}) \geq |r_2|(1-h) \geq m_4(r_1-hr_2) \geq m_4(r_1-hr_2e^{(r_2-r_1)\varsigma}).$$

Noting that $e^{(r_1-v)\varsigma} \leq 1$, it follows that

$$G(s,s)+|G_t(s+0,s)| \geq m_4[G(\varsigma,s)+|G_t(\varsigma,s)|]e^{(-v\varsigma+r_1)}.$$

So, taking $m_4 = \min\{\frac{r_1}{|r_2|}, \frac{r_2(h-1)}{r_1-r_2h}, \frac{1}{1+h}\}$, by above analysis (i)-(ii) combined with (i) of Lemma 3, we have

$$[G(s,s)+|G_t(s\pm 0,s)|]e^{-r_1s} \geq m_4[G(\varsigma,s)+|G_t(\varsigma,s)|]e^{-v\varsigma}.$$

(2) If $-1 < r_2 < 0$, the proof is similar to that in the case (1), so we omit it.

From Lemma 2 and lemma 4, it is easy to see that the following lemma is true.

**Lemma 5.** $\forall t \in [\gamma,\delta] \subset (0,\infty), \forall s,\varsigma \in I, v \geq r_1, s \neq \varsigma$, we have

(*i*) if $r_2 \leq -1$, then

$$(r_1-r_2)G(t,s)+G_t(t,s) \geq m[G(\varsigma,s)+|G_t(\varsigma,s)|]e^{-v\varsigma},$$

where $m = m_1m_4$, and $m_1, m_4$ described as in Lemma 2, Lemma 4,respectively.

(*ii*) if $-1 < r_2 < 0$, then

$$G(t,s)+G_t(t,s) \geq n[G(\varsigma,s)+|G_t(\varsigma,s)|]e^{-v\varsigma},$$

where $n = m_2m_5$, and $m_2, m_5$ described as in Lemma 2, Lemma 4, respectively.

**Remark 3.** Let $e_0 = \begin{cases} m, \text{ if } \mathrm{r}_2 \leq -1 \\ n, \text{ if } -1 < \mathrm{r}_2 < 0 \end{cases}$, by Lemma 5, for any fixed number $\lambda$ with $\lambda \geq \max\{\sqrt{\bar{p}^2+4\bar{q}}, 1\} = \max\{r_1-r_2, 1\}$, $\forall t \in [r,\delta] \subset (0,\infty), \forall s,\varsigma \in I, v \geq r_1, s \neq \varsigma$, the following inequality holds

$$\lambda G(t,s)+G_t(t,s) \geq e_0[G(\varsigma,s)+|G_t(\varsigma,s)|]e^{-v\varsigma}.$$

**Lemma 6.** Assume that $(H_3)$ holds, define the operator $B$ as

$(By)(t) = p_1(t)y'(t)+q_1(t)y(t), t \in I$, for $y \in E$. Then $B: E \to X$ is linear and bounded, and $||B|| \leq b$, where $b = \int_0^\infty (p_1(t)+q_1(t))e^{(\mu-r_1)t}dt$.

*Proof.* By $(H_3)$, we have $p_1(t) \geq 0, q_1(t) \geq 0, \forall t \in I$, and $p_1(t)e^{\mu t}, q_1(t)e^{\mu t} \in X$. Hence, $\forall y \in E$, we have

$$\begin{aligned}\int_0^\infty |(By)(t)|e^{-r_1t}dt &\leq \int_0^\infty (p_1(t)|y'(t)|+q_1(t)|y(t)|)e^{-r_1t}dt \\ &= \int_0^\infty (p_1(t)|y'(t)|e^{-\mu t}+q_1(t)|y(t)|e^{-\mu t})e^{(\mu-r_1)t}dt \\ &\leq ||y||_1\int_0^\infty (p_1(t)+q_1(t))e^{(\mu-r_1)t}dt = b||y||_1.\end{aligned}$$

Thus, $By \in X$. Again, it is clear that $B$ is a linear operator with $||B|| \leq b$.

**Lemma 7.** The operator $T$ defined by (1.6) maps $X$ into $E$, and is completely continuous, moreover, $TX^+ \subset E^+, ||T|| \leq d$, where $d = \frac{1}{r_1-r_2}[2+r_1-r_2(1+h)]$.

*Proof.* For any $y \in X$, from paper [10] combined with (1.5)-(1.8), we have

$$(Ty)(t)=\int_0^{\infty}G(t,s)y(s)ds, t\in I, \tag{2.1}$$

$$(Ty)'(t)=\int_0^{\infty}G_t(t,s)y(s)ds, t\in I, \tag{2.2}$$

where

$$G_t(t,s)=\frac{1}{r_1-r_2}\left\{\begin{array}{l} r_2e^{r_2t}(e^{-r_2s}-he^{-r_1s}), 0\le s<t,\\ e^{-r_1s}(r_1e^{r_1t}-hr_2e^{r_2t}), 0\le t<s.\end{array}\right. \tag{2.3}$$

$\forall t\in I$, by (1.8)-(2.3), it follows from $\mu>r_1>0, r_2<0<h<1$ and Lemma 1 that

$$\begin{aligned}
(|(Ty)(t)| &+ |(Ty)'(t)|)e^{-\mu t}\le e^{-\mu t}\int_0^{\infty}(G(t,s)+|G_t(t,s)|)|y(s)|ds\\
&\le \frac{1}{r_1-r_2}e^{-\mu t}[(1-r_2)e^{r_2t}\int_0^t(e^{-r_2s}-he^{-r_1s})|y(s)|ds\\
&+ ((1+r_1)e^{r_1t}-h(1+r_2)e^{r_2t})\int_t^{\infty}e^{-r_1s}|y(s)|ds]\\
&\le \frac{1}{r_1-r_2}e^{-\mu t}[(1-r_2)(e^{r_1t}-he^{r_2t})\int_0^t|y(s)|e^{-r_1s}ds\\
&+ ((1+r_1)e^{r_1t}-r_2he^{r_2t})\int_t^{\infty}|y(s)|e^{-r_1s}ds]\\
&\le \frac{1}{r_1-r_2}[(1-r_2)(e^{(r_1-\mu)t}-he^{(r_2-\mu)t})\int_0^{\infty}|y(s)|e^{-r_1s}ds\\
&+ ((1+r_1)e^{(r_1-\mu)t}-r_2he^{(r_2-\mu)t})\int_0^{\infty}|y(s)|e^{-r_1s}ds].
\end{aligned} \tag{2.4}$$

Since $0\le e^{(r_1-\mu)t}-he^{(r_2-\mu)t}=e^{(r_1-\mu)t}(1-he^{(r_2-r_1)t})\le 1, t\in I$, formula(2.4) implies

$$\begin{aligned}
|(Ty)(t)|+|(Ty)'(t)|e^{-\mu t} &\le \frac{1}{r_1-r_2}[2+r_1-r_2(1+h)]||y||_X\\
&= d||y||_X, \forall t\in I.
\end{aligned}$$

Thus

$$\sup_{t\in I}[|(Ty)(t)|+|(Ty)'(t)|]e^{-\mu t}\le d||y||_X. \tag{2.5}$$

(2) $\forall t\in[-\tau,0]$, by (1.6), we have $(Ty)(t)=e^{\frac{\alpha}{\beta}t}(Ty)(0)$. From the proof in [10], it follows that $(Ty)'(0)$ exists. Thus, for any $t\in[-\tau,0]$, we have $(Ty)'(t)=\frac{\alpha}{\beta}e^{\frac{\alpha}{\beta}t}(Ty)(0)$, and

$$[|(Ty)(t)|+|(Ty)'(t)|]e^{-\mu t}=(1+\frac{\alpha}{\beta})e^{(\frac{\alpha}{\beta}-\mu)t}|(Ty)(0)|, t\in[\tau,0].$$

From the assumption $\mu < \frac{\alpha}{\beta}$, we have that $e^{(\frac{\alpha}{\beta}-\mu)t}$ is increase on $[-\tau, 0]$. Hence $\forall t \in [-\tau, 0]$, we have

$$[|(Ty)(t) + (Ty)'(t)|]e^{-\mu t} \leq [|(Ty)(0) + (Ty)'(0)|]e^{-\mu 0}$$
$$\leq \sup_{t\in I}[|(Ty)(t)| + |(Ty)'(t)|]e^{-\mu t}.$$

Thus

$$\sup_{t\in[-\tau,0]}[|(Ty)(t)| + |(Ty)'(t)|]e^{-\mu t} \leq \sup_{t\in I}[|(Ty)(t)| + |(Ty)'(t)|]e^{-\mu t} \quad . \tag{2.6}$$

By (2.5)-(2.6) above, we have

$$||(Ty||_1 = \sup_{t\in J}[|(Ty)(t)| + |(Ty)'(t)|]e^{-\mu t} \leq d||y||_X.$$

Hence $||T|| \leq d$. On the other hand, by the proof in [10], it is easy to see that the operator $T$ is compact. Thus, $T$ is completely continuous. In addition, it is follows from (i) of lemma 1 that $TX^+ \subset E^+$.

**Lemma 8.** Suppose $(H_3)$ holds, then $BTX^+ \subset X^+$.

*Proof.* For any $\sigma \in X^+$, let $\phi(t) = (T\sigma)(t)$, Lemma 7 implies $\phi \in E^+$. By (1.8), (2.3) and (i) of Lemma 1, it follows that for any $0 \leq t < s$,

$$G(t,s) \geq 0, G_t(t,s) \geq 0. \tag{2.7}$$

For $0 \leq s < t$, according to assumption $\lambda_0 \geq \sqrt{p^2+4q}(= r_1 - r_2 > |r_2|), 0 < h < 1, r_2 < 0 < r_1$, it follows that

$$G_t(t,s) + \lambda_0 G(t,s) = \frac{r_2 + \lambda_0}{r_1 - r_2} e^{r_2 t}(e^{-r_2 s} - he^{-r_1 s}) \geq 0. \tag{2.8}$$

By(2.7), (2.8), we have

$$G_t(t,s) + \lambda_0 G(t,s) \geq 0, \forall t, s \in I, t \neq s. \tag{2.9}$$

Thus,

$$\phi'(t) + \lambda_0\phi(t) = \int_0^\infty (G_t(t,s) + \lambda_0 G(t,s))\sigma(s)ds \geq 0, \forall t \in I.$$

By $(H_3)$, we have

$$\begin{aligned}(B\phi)(t) &= p_1(t)\phi'(t) + q_1(t)\phi(t)\\ &\geq p_1(t)\phi'(t) + \lambda_0 p_1(t)\phi(t)\\ &= p_1(t)(\phi'(t) + \lambda_0\phi(t)) \geq 0, t \in I.\end{aligned}$$

Thus, by Lemma 6, it follows that $B\phi \in X^+$, i.e. $BT\sigma \in X^+$.

**Lemma 9**. Suppose $(H_1)$ holds. for $y \in E^+$, define $\mathbf{f}y = f(t, y_t + u_t, y'_t + u'_t)$. Then $\mathbf{f} : E^+ \to X^+$ is continuous.

*Proof.* For any $y \in E, s \in I, \theta_1, \theta_2 \in [-\tau, 0]$, we have

$$|y_s(\theta_1) + u_s(\theta_1)| \leq |y(s+\theta_1) + u(s+\theta_1)|e^{-\mu(s+\theta_1)} \cdot e^{\mu s} \leq ||y+u||_1 e^{\mu s},$$

$$|y'_s(\theta_2) + u'_s(\theta_2)| \leq |y'(s+\theta_2) + u'(s+\theta_2)|e^{-\mu(s+\theta_2)} \cdot e^{\mu s} \leq ||y+u||_1 e^{\mu s}.$$

So

$$||y_s + u_s||_0^\nu + ||y'_s + u'_s||_0^\nu \leq 2||y+u||_1^\nu e^{\nu\mu s}, s \in I. \tag{2.10}$$

Again, by $(H_1)$, for $t \in I$, we have

$$0 \leq \int_0^\infty (\mathbf{f}y)(s)e^{-r_1 s}ds \leq \int_0^\infty a(s)e^{-r_1 s}ds + 2||y+u||_1^\nu \int_0^\infty b(s)e^{(\nu\mu - r_1)s}ds < \infty. \tag{2.11}$$

It means that $\mathbf{f}y \in X^+$.

On the other hand, for any sequence $\{y_n\}_{n=0}^\infty$ in $E^+$ with $||y_n - y_0||_1 \to 0$, then, exists $N \geq 1$ such that $||y_n||_1 \leq 1 + ||y_0||_1$ when $n \geq N$. By argument similar to (2.11), we have

$$\begin{aligned}
&|(\mathbf{f}y_n - \mathbf{f}y_0)(s)|e^{-r_1 s} \leq (\mathbf{f}y_n + \mathbf{f}y_0)(s)e^{-r_1 s} \\
&\leq 2a(s)e^{-r_1 s} + 2[(1 + ||y_0||_1 + ||u||_1)^\nu + ||y_0 + u||_1^\nu]b(s)e^{(\nu\mu - r_1)s} \\
&\triangleq F(s) \in L(0,\infty), n \geq N.
\end{aligned}$$

Applying Lebesgue convergence theorem, by the continuity of $f$, we can obtain easily $||\mathbf{f}y_n - \mathbf{f}y_0||_X \to 0 (n \to \infty)$, i.e. the operator $\mathbf{f} : E^+ \to X^+$ is continuous.

**Lemma 10**. Let $a \geq 0, b \geq 0, v > 1$, then

$$a^\nu + b^\nu \leq (a+b)^\nu \leq 2^{\nu-1}(a^\nu + b^\nu).$$

*Proof.* Without loss of generality, we can assume $ab > 0$, If $a = b$, then it is obvious that the inequality holds. So we can assume that $b > a$.

(i) Let $\varphi(t) = (t+x)^\nu, t \in [0,1], x > 1$. Then $\exists \bar{t} \in (0,1)$ such that $\varphi(1) - \varphi(0) = \varphi'(\bar{t}) = \nu(\bar{t}+x)^{\nu-1}$. Owing to $\nu > 1, x > 1$, we have that $\nu(\bar{t}+x)^{\nu-1} > 1$. Thus $(1+x)^\nu - x^\nu > 1$. i.e., $(1+x)^\nu > 1 + x^\nu$. Taking $x = \frac{b}{a}$, it follows that

$$a^\nu + b^\nu < (a+b)^\nu.$$

(ii) Let $\varphi(t) = t^\nu, t > 0$. According to $\nu > 1$, we have $\varphi''(t) = \nu(\nu-1)t^{\nu-2} > 0, t > 0$. So, by property of convex function, we have

$$\varphi(\frac{x_1 + x_2}{2}) \leq \frac{1}{2}\varphi(x_1) + \frac{1}{2}\varphi(x_2), \text{ for any } x_1, x_2 \in (0,\infty).$$

Taking $x_1 = \frac{a}{a+b}, x_2 = \frac{b}{a+b}$ in above inequality, we immediately have

$$\frac{1}{2^\nu} \leq \frac{1}{2}(\frac{a}{a+b})^\nu + \frac{1}{2}(\frac{b}{a+b})^\nu.$$

And so, $(a+b)^\nu \leq 2^{\nu-1}(a^\nu + b^\nu)$.

## 3. Main results

We introduce the notations as follows:

Let $A_0 = (1+h)\int_0^\infty (G(s,s) + H_2(s))(a(s) + 2^{\nu-1}d_0^\nu b(s))e^{-r_1 s}ds$,

$B_0 = 2^{2\nu-1}(1+h)\int_0^\infty (G(s,s) + H_2(s))b(s)e^{(\nu\mu - r_1)s}ds$,

$H_2(s) = \max\{|G_t(s-0,s)|, |G_t(s+0,s)|\}$, where

$$d_0 = (1 + \frac{\alpha}{\beta})\frac{1}{\beta}\int_{-\tau}^0 e^{-\frac{\alpha}{\beta}s}\xi(s)ds + \frac{1}{\beta}||\xi||_0.$$

Again, denote $m_0 = \int_\gamma^\delta G(0,s)ds = \frac{1-h}{(r_1-r_2)r_1}(e^{-r_1\gamma} - e^{-r_1\delta}), \tau_0 = e_0(1-L), L = bd$ ( where $b, d, e_0$ are described as in lemma 6-7 and Remark 3, respectively).

**Remark 4**. In the hypothesis $(H_2), M > \frac{\lambda_0}{m_0\tau_0}$, where $m_0, \tau_0$ described above.

We are now in a position to state and prove our main results on the existence for BVP (1.2)

**Theorem 1.** Suppose $(H_1) - (H_3)$ hold, in addition, assume that $L < 1, A_0 < (\frac{1-L}{2})^{\frac{\nu}{\nu-1}} B_0^{\frac{1}{1-\nu}}$. Then BVP (1.2) has at least one positive solution $x$ with $x(t) \geq u(t), t \in J$, where $u(t)$ described as in(1.3).

*Proof.* The proof is divided into three steps.

*Step1.*

By (1.2)-(1.8), it is clear that $x \in E$ is an solution of BVP(1.2)$\Leftrightarrow y = x-u \in E, y = T(By + \mathbf{f}y) \Leftrightarrow y \in E, (I - TB)y = T\mathbf{f}y$. By Lemma 6 and Lemma 7, we have that the linear operator $TB : E \to E$ is completely continuous, and $||TB|| \leq ||T|| \cdot ||B|| \leq L < 1$. Thus, $(I - TB)^{-1}$ is defined well and bounded. Let $H = (I - TB)^{-1}T$, then $H : X \to E$ is completely continuous. Hence,$\forall \varphi \in X$, we have

$$(I - TB)y = T\varphi, y \in E \Leftrightarrow y = H\varphi \in E.$$

By Neuman expansion formula, $H$ can be expressed by

$$\begin{aligned} (H\varphi)(t) &= (I + TB + \cdots + (TB)^n + \cdots)(T\varphi)(t) \\ &= (T\varphi)(t) + (TB)(T\varphi)(t) + (TB)^n(T\varphi)(t) + \cdots, t \in J. \end{aligned} \tag{3.1}$$

Now, we shall prove the following inequality holds by induction,

$$\forall \sigma \in X^+, \forall n \geq 1, (TB)^n(T\sigma)(t) \geq 0, t \in J. \tag{3.2}$$

In fact, for $n = 1$, owing to $\sigma \in X^+$, by Lemma 8, we have $BT\sigma \in X^+$, and so, it follows from Lemma 7 that $(TB)(T\sigma) = T(BT\sigma) \in E^+$. Thus, $(TB)(T\sigma)(t) \geq 0, t \in J$. Suppose for $n = k$, inequality (3.2) holds. Then for $n = k + 1$, letting $\sigma_1 = BT\sigma$, we have $\sigma_1 \in X^+$, and it follows that

$$\begin{aligned} &(TB)^{k+1}(T\sigma)(t) = (TB)^k(TB)(T\sigma)(t) \\ &= (TB)^k(T\sigma_1)(t) \geq 0, t \in J. \end{aligned}$$

Thus, (3.2) holds. By (3.1)-(3.2), we have

$$\forall \varphi \in X^+, (H\varphi)(t) \geq (T\varphi)(t), t \in J. \tag{3.3}$$

On the other hand, $\forall \varphi \in X$, it follows from (3.1) that

$$\begin{aligned} ||H\varphi||_1 &\leq ||T\varphi||_1 + ||TB|| \cdot ||T\varphi||_1 + \cdots + ||(TB)^n|| \cdot ||T\varphi||_1 + \cdots \\ &\leq (1 + L + \cdots + L^n + \cdots)||T\varphi||_1 = \frac{1}{1-L}||T\varphi||_1. \end{aligned} \tag{3.4}$$

*Step 2 .*

We shall show that

$$\forall \varphi \in X^+, (H\varphi)'(t) + (H\varphi)(t) \geq (T\varphi)'(t) + \lambda_0(T\varphi)(t), t \in I. \tag{3.5}$$

(1) for any fixed number $\Gamma \geq \max\left\{|r_2|, (r_1 - r_2h)(1-h)^{-1}\right\}$, we have

$$|G_t(t, s)| \leq \Gamma G(t, s), 0 \leq t, s < \infty, t \neq s. \tag{3.6}$$

In fact, by (1.8), (2.3), according to $0 < h < 1$,we have

(i) if $0 \leq s < t < \infty$, then $|G_t(t, s)| = |r_2|G(t, s) \leq \Gamma G(t, s)$.

(ii) if $0 \le t < s < \infty$, then

$$\begin{aligned} 0 &\le G_t(t,s) = \frac{1}{r_1 - r_2} e^{-r_1 s} e^{r_1 t}(r_1 - hr_2 e^{(r_2-r_1)t}) \\ &\le \frac{1}{r_1 - r_2} e^{-r_1 s} e^{r_1 t}(r_1 - hr_2) \\ &\le \frac{\Gamma}{r_1 - r_2} e^{-r_1 s} e^{r_1 t}(1 - h) \\ &\le \Gamma G(t,s). \end{aligned}$$

Hence, relation (3.6) holds.

(2) Now, we are going to show that for any fixed $d > 0$, and any $\varphi \in X^+$ , the following inequality holds

$$\forall n \ge 1, |((TB)^n (T\varphi))'(t)| \le \Gamma ||TB||^n ||T\varphi||_1 e^{\mu d}, t \in [0,d]. \tag{3.7}$$

Firstly, we shall prove the following formula holds :

$$\forall n \ge 1, \exists \varphi_n \in X^+ \text{such that}\, B(TB)^n T\varphi = BT\varphi_n. \tag{3.8}$$

Indeed, for $n = 1, B(TB)T\varphi = B(T(BT\varphi))$. Letting $\varphi_1 = BT\varphi$, by Lemma 8, we have $\varphi_1 \in X^+$. Thus $B(TB)T\varphi = BT\varphi_1$. Suppose that for $n = k$, $\exists \varphi_k \in X^+$ such that $B(TB)^k T\varphi = BT\varphi_k$. Then for $n = k+1$, we have

$$B(TB)^{k+1}T\varphi = B(TB)(TB)^k(T\varphi) = B(T(B(TB)^k)T\varphi)) = B(T(BT\varphi_k)).$$

Letting $\varphi_{k+1} = BT\varphi_k$, then, it follows from Lemma 8 that $\varphi_{k+1} \in X^+$, and so, $B(TB)^{k+1}T\varphi = BT\varphi_{k+1}$. Thus, by induction, it follows that (3.8) holds.

Secondly, we come to show the following formula holds.

$$\forall n \ge 0, \exists \varphi_n \in X^+, \text{such that } (TB)^{n+1}T\varphi = T(BT\varphi_n) \in E^+. \tag{3.9}$$

In fact,

(i) If $n \ge 1$, then from (3.8), it follows that $\exists \varphi_n \in X^+$ such that

$$(TB)^{n+1}T\varphi = (TB)((TB)^n T\varphi) = T(B(TB)^n T\varphi) = T(BT\varphi_n).$$

Again,by lemma 7 -8, we have $T(BT\varphi_n) \in E^+$.

(ii) If $n = 0$, taking $\varphi_0 = \varphi \in X^+$, by Lemma 7 -8, we have $(TB)T\varphi = T(BT\varphi_0) \in E^+$.

Hence, whether for the case (i), or case(ii) above, the formula (3.9) is always true.

Now, from (3.9),(2.2), and (3.6), for any $n \geq 1, t \in [0, d]$, we have

$$\begin{aligned} |((TB)^n T\varphi)'(t)| &= |(T(BT)\varphi_{n-1})'(t)| \\ &= |\int_0^\infty G_t(t,s)(BT\varphi_{n-1})(s)ds| \\ &\leq \int_0^\infty |G_t(t,s)|(BT\varphi_{n-1})(s)ds \\ &\leq \Gamma \int_0^\infty G(t,s)(BT\varphi_{n-1})(s)ds \\ &= \Gamma(T(BT\varphi_{n-1}))(t) \\ &= \Gamma(TB)^n(T\varphi))(t) \\ &= \Gamma((TB)^n(T\varphi))(t)e^{-\mu t})e^{\mu t} \\ &\leq \Gamma||(TB)^n(T\varphi)||_1 e^{\mu t} \\ &\leq \Gamma||TB||^n||T\varphi||_1 e^{\mu t}. \end{aligned}$$

It means that $|((TB)^n T\varphi)'(t)| \leq \Gamma||TB||^n||T\varphi||_1 e^{\mu d}$, for $t \in [0, d]$.

This show that (3.7) holds. So we have

$$\sum_{n=1}^{\infty} |((TB)^n(T\varphi))'(t)| \leq \Gamma \sum_{n=1}^{\infty} ||TB||^n \cdot ||T\varphi||_1 e^{\mu d} \leq \frac{\Gamma L}{1-L}||T\varphi||_1 e^{\mu d} < \infty, t \in [0, d].$$

Hence, we can differentiate termwise the series (3.1) on $[0, d]$, and obtain

$$(H\varphi)'(t) = (T\varphi)'(t) + \sum_{n=1}^{\infty} ((TB)^n(T\varphi))'(t), t \in [0, d].$$

Thus, $\forall t \in [0, d]$, we have

$$\begin{aligned} (H\varphi)'(t) \quad + \quad & \lambda_0(H\varphi)(t) = (T\varphi)'(t) + \lambda_0(T\varphi)(t) \\ + \quad & \sum_{n=1}^{\infty} [((TB)^n T\varphi)'(t) + \lambda_0((TB)^n T\varphi)(t)]. \end{aligned} \tag{3.10}$$

By (3.9), (2.2), (2.9) and Lemma 8, for any $n \geq 1, t \in [0, d]$, we have

$$\begin{aligned} ((TB)^n(T\varphi))'(t) + \lambda_0((TB)^n T\varphi)(t) &= (T(BT\varphi_{n-1}))'(t) + \lambda_0(T(BT\varphi_{n-1}))(t) \\ &= \int_0^1 (G_t(t,s) + \lambda_0 G(t,s))(BT\varphi_{n-1})(s) \geq 0. \end{aligned}$$

So by (3.10), the following inequality holds

$$(H\varphi)'(t) + \lambda_0(H\varphi)(t) \geq (T\varphi)'(t) + \lambda_0(T\varphi)(t), t \in [0, d].$$

It means that relations (3.5) holds according to arbitrariness of $d > 0$.

*Step 3.*

Let $P = \left\{ y \in E | y(t) \geq 0, t \in J, \min_{t \in [\gamma, \delta]} [y'(t) + \lambda_0 y(t)] \geq \tau_0 ||y||_1 \right\}$,where $\tau_0 = e_0(1-L)$. Obviously, $P \neq \emptyset$ noting that $\theta \in P(\theta \equiv 0, t \in J)$, and $P$ is a cone in $E$. Putting $Q = H\mathbf{f}$, since $H : X \to E$ is completely continuous, by Lemma 9 together with (3.2), we have $Q : E^+ \to E^+$ is completely continuous.

For any $y \in P$, we have $\mathbf{f}y \in X^+$. In view of (3.1) and Remark 3, for any $t \in [\gamma, \delta], \varsigma \in I$, we get

$$\begin{aligned}(H\mathbf{f}y)'(t) + \lambda_0(H\mathbf{f}y)(t) &\geq (T\mathbf{f}y)'(t) + \lambda_0(T\mathbf{f}y)(t) \\ &= \int_0^\infty (G_t(t,s) + \lambda_0 G(t,s))(\mathbf{f}y)(s))ds \\ &\geq e_0(\int_0^\infty (G(\varsigma,s) + |G_t(\varsigma,s)|)(\mathbf{f}y)(s)ds)e^{-\mu\varsigma} \\ &\geq e_0(T\mathbf{f}y)(\varsigma) + |(T\mathbf{f}y)'(\varsigma)|)e^{-\mu\varsigma},\end{aligned}$$

and so

$$(H\mathbf{f}y)'(t) + \lambda_0(H\mathbf{f}y)(t) \geq e_0 \sup_{\varsigma\in I}[(T\mathbf{f}y)(\varsigma) + |(T\mathbf{f}y)'(\varsigma)|]e^{-\mu\zeta}, \forall t \in [\gamma, \delta].$$

So by (2.6), we have

$$\min_{t\in[\gamma,\delta]}[(H\mathbf{f}y)'(t) + \lambda_0(H\mathbf{f}y)(t)] \geq e_0||T\mathbf{f}y||_1.$$

On the other hand, inequality (3.4) yields $||T\mathbf{f}y||_1 \geq (1-L)||H\mathbf{f}y||_1$. Therefore

$$\min_{t\in[\gamma,\delta]}[(H\mathbf{f}y)'(t) + \lambda_0(H\mathbf{f}y)(t)] \geq e_0(1-L)||H\mathbf{f}y||_1.$$

Namely ,

$$\min_{t\in[\gamma,\delta]}[(Qy)'(t) + \lambda_0(Qy)(t)] \geq \tau_0||Qy||_1(\tau_0 = e_0(1-L)).$$

Thus, we arrive at $Q : P \to P$.

*Step4.*

The hypothesis $A_0 < (\frac{1-L}{2})^{\frac{\nu}{\nu-1}} B_0^{\frac{1}{1-\nu}}$ together with $\nu > 1$ implies $\frac{2A_0}{1-L} < (\frac{A_0}{B_0})^{\frac{1}{\nu}}$. We take a number $r_0 \in (\frac{2A_0}{1-L}, (\frac{A_0}{B_0})^{\frac{1}{\nu}})$. Then $r_0^\nu < \frac{A_0}{B_0}$, and so $\frac{B_0}{1-L}r_0^\nu < \frac{A_0}{1-L}$. Thus, we have

$$\frac{A_0}{1-L} + \frac{B_0}{1-L}r_0^\nu < \frac{2A_0}{1-L} < r_0 \tag{3.11}$$

Set $\Omega_{r_0} = \{y \in P : ||y||_1 < r_0\}$. For any $y \in \partial\Omega_{r_0}$, we have $(y_s + u_s, y_s' + u_s') \in Z^+ \times Z$, and $||y||_1 = r_0$. Thus, by $(H_1)$, it follows that

$$f(s, y_s + u_s, y_s' + u_s') \leq a(s) + b(s)(||y_s + u_s||_0^\nu + ||y_s' + u_s'||_0^\nu), s \in I.$$

It is easy to see that $||u_s||_0 + ||u_s'||_0 \leq d_0$. By argument similar to (2.10), we obtain

$$||y_s||_0 + ||y_s'||_0 \leq 2||y||_1 e^{\mu s}, s \in I. \tag{3.12}$$

So, by Lemma 10 together with (3.12) we have

$$\begin{aligned}||y_s + u_s||_0^\nu + ||y_s' + u_s'||_0^\nu &\leq (||y_s + u_s||_0 + ||y_s' + u_s'||_0)^\nu \\ &\leq (||y_s||_0 + ||y_s'||_0 + d_0)^\nu \\ &\leq 2^{\nu-1}((||y_s||_0 + ||y_s'||_0)^\nu + d_0^\nu) \\ &\leq 2^{2\nu-1}||y||_1^\nu e^{\nu\mu s} + 2^{\nu-1}d_0^\nu, s \in I.\end{aligned}$$

Thus,

$$f(s, y_s + u_s, y_s' + u_s') \leq (a(s) + 2^{\nu-1}d_0^\nu b(s)) + 2^{2\nu-1}b(s)e^{\nu\mu s}||y||_1^\nu, s \in I.$$

Then, by proving as paper [10], we have

$$\begin{aligned}
&[(T\mathbf{f}y)(t)+|(T\mathbf{f}y)')(t)|]e^{-\mu t}\\
&\le e^{-\mu t}\int_0^\infty (G(t,s)+|G_t(t,s)|)f(s,y_s+u_s,y_s'+u_s')ds\\
&\le (1+h)\int_0^\infty (G(s,s)+H_2(s))f(s,y_s+u_s,y_s'+u_s')e^{-r_1 s}ds\\
&\le (1+h)\int_0^\infty (G(s,s)+H_2(s))(a(s)+2^{\nu-1}d_0^\nu b(s))e^{-r_1 s}ds\\
&\quad +2^{2\nu-1}(1+h)\int_0^\infty (G(s,s)+H_2(s))b(s)e^{(\nu\mu-r_1)s}ds||y||_1^\nu\\
&= A_0+B_0||y||_1^\nu, \forall t\in I.
\end{aligned}$$

Thus, by (2.6), it follows that $||T\mathbf{f}y||_1 \le A_0 + B_0||y||_1^\nu$. So by (3.4) and (3.12), we have

$$\begin{aligned}
||Qy||_1 &= ||H\mathbf{f}y||_1 \le \frac{1}{1-L}||T\mathbf{f}y||_1 \le \frac{A_0}{1-L}+\frac{B_0}{1-L}||y||_1^\nu\\
&= \frac{A_0}{1-L}+\frac{B_0}{1-L}r_0^\nu < r_0 = ||y||_1.
\end{aligned}$$

So, the fixed point index theorem implies

$$i(Q,\Omega_{r_0},P)=1. \tag{3.13}$$

(2) Take $R_0 > \max\{\frac{\lambda_0 d_0 m_0 M}{\tau_0 m_0 M-\lambda_0}, \frac{\lambda_0(d_0+R_1)}{\tau_0}, r_0,\}$, and set $\Omega_{R_0} = \{y \in P : ||y||_1 < R_0\}$. Now, we shall prove the following inequality is true

$$||Qy||_1 \ge ||y||_1, \forall y\in \partial\Omega_{R_0}. \tag{3.14}$$

In fact, for any $y \in \partial\Omega_{R_0}$, we have $y_s+u_s \in Z^+, y_s'+u_s' \in Z, \forall s\in I$, and $||y||_1 = R_0$. Owing to $||u_s||_0 + ||u_s'||_0 \le d_0, \forall s \in I$, we have

$$\begin{aligned}
||y_s+u_s||_0+||y_s'+u_s'||_0 &\ge ||y_s||_0+||y_s'||_0-(||u_s||_0+||u_s'||_0)\\
&\ge ||y_s||_0+||y_s'||_0-d_0, s\in I. \qquad (3.15)
\end{aligned}$$

According to $y \in P, \lambda_0 \ge 1$, it follows that

$$\begin{aligned}
y(s)+|y'(s)| &= \frac{1}{\lambda_0}(\lambda_0 y(s)+\lambda_0|y'(s)|) \ge \frac{1}{\lambda_0}(\lambda_0 y(s)+|y'(s)|)\\
&\ge \frac{\tau_0}{\lambda_0}||y||_1, \forall s\in[\gamma,\delta]. \qquad (3.16)
\end{aligned}$$

Since $||y_s||_0 + ||y_s'||_0 \ge y(s) + |y'(s)||, \forall s \in [\gamma,\delta]$, by (3.15)-(3.16), we have

$$\begin{aligned}
||y_s+u_s||_0+||y_s'+u_s'||_0 &\ge \frac{\tau_0}{\lambda_0}||y||_1-d_0\\
&= \frac{\tau_0}{\lambda_0}R_0-d_0 > R_1, s\in[\gamma,\delta]. \qquad (3.17)
\end{aligned}$$

Consequently, by $(H_2)$ together with (3.17), we have

$$\begin{aligned}
f(s,y_s+u_s,y_s'+u_s') &> M(||y_s+u_s||_0+||y_s'+u'||_0)\\
&\ge M\frac{\tau_0}{\lambda_0}R_0-Md_0, s\in[\gamma,\delta].
\end{aligned}$$

Thus,

$$\begin{aligned}(T\mathbf{f}y)(0) &= \int_0^\infty G(0,s)f(s,y_s+u_s,y_s'+u_s')ds\\ &\geq \int_\gamma^\delta G(0,s)f(s,y_s+u_s,y_s'+u_s')ds\\ &\geq (\int_\gamma^\delta G(0,s)ds)(M\frac{\tau_0}{\lambda_0}R_0-Md_0)\\ &= M\frac{m_0\tau_0}{\lambda_0}R_0-Mm_0d_0>R_0.\end{aligned} \tag{3.18}$$

From (3.18) and (3.3), it follows that

$$||Qy||_1 \geq (Qy)(0)=(H\mathbf{f}y)(0)\geq (T\mathbf{f}y)(0)>R_0=||y||_1.$$

Hence, (3.14) holds, and so

$$i(Q,\Omega_{R_0},P)=0.$$

Thus $i(Q,\Omega_{R_0}\backslash\overline{\Omega}_{r_0},P)=-1$, and so $\exists y^*\in\Omega_{R_0}\backslash\overline{\Omega}_{r_0}$ with $Qy^*=y^*$. It means that $y^*\in P\backslash\{\theta\}$ is a positive solution BVP(4), and so $x=y^*+u$ is a positive solution BVP (1.2), satisfying $x(t)\geq u(t), t\in J$, $x\neq u$.This completes the proof of theorem 1.

**Example 1**. Consider the following BVP

$$\begin{cases} x''-(1-e^{-65.5t})x'-6(1-e^{-60.5t})+f(t,x_t,x_t')=0, t\in I=[0,\infty),\\ 4x-x'=1-e^{4t}, t\in[-1,0],\\ x(\infty)=0.\end{cases} \tag{3.19}$$

Set $\alpha=4,\beta=1,\xi(t)=1-e^{4t},p(t)=1-e^{-65.5t},q(t)=6(1-e^{-60.5t})$. Then, $\bar{p}=1,\bar{q}=6,p_1(t)=e^{-65.5t},q_1(t)=6e^{-60.5t},r_1=3,r_2=-2,\frac{\alpha}{\beta}=4,\lambda_0=5,h=\frac{1}{6},d_0=\frac{5}{4}e^4-e^{-4}-\frac{21}{4}$. Take $\mu=3.5$, then $b=\int_0^\infty(p_1(t)+q_1(t))e^{(\mu-r_1)t}dt=\frac{3}{26},d=\frac{1}{r_1-r_2}[(2+r_1-r_2(1+h)]=\frac{22}{3},L=bd=\frac{11}{13}<1$. Take $v=1.1$, set

$$f(t,\varphi,\phi)=f_1(t)+f_2(t)e^{-0.85t}(\int_0^\infty k(t)\varphi^{1.1}(t)dt+||\varphi||_0^{1.1}+||\phi||_0^{1.1}),$$

$\forall t\in I,(\varphi,\phi)\in Z^+\times Z$, where function $f_1,f_2,k\in C(I,R^+)$ satisfy $\int_0^\infty k(t)dt\leq 1$, and $\int_0^\infty f_1(s)e^{-3s}ds<+\infty$.

In addition, $f_2$ satisfies the following conditions :

$(D_1)$ $\int_0^\infty f_2(s)ds<\min\{1,\rho\}$, where $\rho=(\frac{9}{455})^{\frac{11}{10}}(\int_0^\infty f_1(s)e^{-3s}ds+(2d_0)^{1.1})^{-\frac{1}{10}}$.

$(D_2)$ Exists $t_0\in(0,+\infty)$ such that $f_2(t_0)>0$.

Obviously, after $f_1$ has been given, above function $f_2$ can be found easily.

It is to see that function $f$ satisfies the following relation;

$$f(t,\varphi,\phi)\leq a(t)+b(t)(||\varphi||_0^{1.1}+||\phi||_0^{1.1}),\forall t\in I,(\varphi,\phi)\in Z^+\times Z. \tag{3.20}$$

$$f(t,\varphi,\phi)\geq e^{-0.85t}f_2(t)(||\varphi||_0^{1.1}+||\phi||_0^{1.1}),\quad \forall t\in I,(\varphi,\phi)\in Z^+\times Z. \tag{3.21}$$

where $a(t)=f_1(t),b(t)=2e^{-0.85t}f_2(t)$.

Again, we easily know that

$$G(s,s)+H_2(s)=\frac{4}{5}+\frac{1}{30}e^{-5s}\leq\frac{5}{6}, s\in I. \tag{3.22}$$

By carefully calculating, taking account of (3.22), we obtain

$$A_0\leq\frac{35}{36}(\int_0^{\infty}f_1(s)e^{-3s}ds+(2d_0)^{1.1}),\quad B_0\leq 2^{0.2}\times\frac{35}{9}\int_0^{\infty}f_2(s)ds. \tag{3.23}$$

Thus, by the choice of $f_2$ together with (3.23), we have

$$(\frac{1-L}{2})^{\frac{v}{v-1}}B_0^{\frac{1}{1-v}}=(\frac{1}{13})^{11}\frac{1}{B_0^{10}}>A_0.$$

On the other hand, from $(D_2)$ , it follows that exists $0<\gamma<\delta$, and $b>0$ such that $e^{-0.85t}f_2(t)\geq b, t\in[\gamma,\delta]$. Consequently, from (3.21), and Lemma 10, we have

$$f(t,\varphi,\phi)\geq b(||\varphi||_0^{1.1}+||\phi||_0^{1.1})\geq\frac{1}{2^{0.1}}b(||\varphi||_0+||\phi||_0)^{1.1}, t\in[\gamma,\delta], (\varphi,\phi)\in Z^+\times Z.$$

Thus

$$\inf_{t\in[\gamma,\delta]}\lim_{||\varphi||_0+||\phi||_0\to+\infty}\frac{f(t,\varphi,\phi)}{||\varphi||_0+||\phi||_0}\geq\frac{b}{2^{0.1}}\lim_{||\varphi||_0+||\phi||_0\to\infty}(||\varphi||_0+||\phi||_0)^{0.1}=+\infty.$$

Hence, for $M>\frac{\lambda_0}{m_0\tau_0}$, exists $R_1>0$ such that the follows inequality holds.

$f(t,\varphi,\phi)\geq M(||\varphi||_0+||\phi||_0), t\in[\gamma,\delta], (\varphi,\phi)\in Z^+\times Z$ with $||\varphi||_0+||\phi||_0\geq R_1$.

So by Theorem 1, BVP (3.19) has a positive solution $x=y+u$.

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