Fixed Point Theory, 13(2012), No. 2, 383-402
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

STRONG CONVERGENCE THEOREMS FOR GENERAL VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS IN q-UNIFORMLY SMOOTH BANACH SPACES

GANG CAI AND SHANGQUAN BU

Department of Mathematical Sciences, Tsinghua University, 100084 Beijing, China
E-mail: caigang-aaaa@163.com (G. Cai), sbu@math.tsinghua.edu.cn (S. Bu)

Abstract. In this paper, we introduce a new iterative algorithm for finding a common element of the set of solutions of a general variational inequality and the set of common fixed points of an infinite family of nonexpansive mappings in q-uniformly smooth Banach space. We obtain some strong convergence theorems under suitable conditions. Furthermore we give an appropriate example such that all conditions of this result are satisfied. Our results extend the recent results announced by many others.

Key Words and Phrases: Strong convergence, general variational inequality, fixed point, Banach space.
2010 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

Throughout this paper, we denote by X and X^* a real Banach space and the dual space of X, respectively. Let C be a subset of X and T be a self-mapping of C. We use $F(T)$ to denote the fixed points of T. Let $q > 1$ be a real number. The(generalized)duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^q\,,\ \|x^*\| = \|x\|^{q-1}\right\},\ \forall x \in X.$$

In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If X is a Hilbert space, then $J = I$, where I is the identity mapping. It is well-known that if X is smooth, then J_q is single-valued, which is denoted by j_q.

Recall that a mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\|\,,\ \forall\, x, y \in C. \tag{1.1}$$

A mapping $T : C \to C$ is said to be L-Lipschitzian, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\,\|x - y\|\,,\ \forall\, x, y \in C. \tag{1.2}$$

This work was supported by the NSF of China (No. 10731020, No. 11171172) and the Specialized Research Fund for the Doctoral Program of Higher Education (No. 200800030059).

A mapping $A : C \to X$ is said to be α-strongly accretive if there exists $j_q(x-y) \in J_q(x-y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \alpha \|x-y\|^q, \ \forall\, x, y \in C. \tag{1.3}$$

A mapping $A : C \to X$ is said to be α-inverse-strongly accretive if there exists $j_q(x-y) \in J_q(x-y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq \alpha \|Ax-Ay\|^q, \ \forall\, x, y \in C. \tag{1.4}$$

A mapping $A : C \to X$ is said to be relaxed (c, d)-cocoercive if there exists $j_q(x-y) \in J_q(x-y)$ and two constants $c, d \geq 0$ such that

$$\langle Ax - Ay, j_q(x-y)\rangle \geq (-c)\|Ax-Ay\|^q + d\|x-y\|^q, \ \forall\, x, y \in C. \tag{1.5}$$

A mapping $f : C \to C$ is said to be a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x-y\|, \ \forall\, x, y \in C.$$

We use the notation Π_C to denote the collection of all contractions on C, i.e., $\Pi_C = \{f : C \to C \text{ a contraction}\}$.

Example 1.1. Let C be a subset of Hilbert space H. Define $Ax = \frac{1}{2}x, \forall\, x \in C$, then A is $\frac{1}{3}$-strongly accretive.

Example 1.2. Let C be a subset of Hilbert space H. Define $Ax = \frac{2}{3}x, \forall\, x \in C$, then A is $\frac{3}{4}$-inverse-strongly accretive.

Example 1.3. Let C be a subset of Hilbert space H. Define $Ax = \frac{3}{4}x, \forall\, x \in C$, then A is relaxed $(\frac{4}{9}, \frac{1}{2})$-cocoercive.

Let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. A retraction Q is said to be orthogonal if for each $x, x - Q(x)$ is normal to D in the sense of R.C. James [9].

It is well known (see [4]) that if X is a Banach space, a projection mapping is a sunny nonexpansive retraction Q of X onto C. If X is uniformly smooth and there exists a nonexpansive retraction of X onto C, then there exists a nonexpansive projection of X onto C. If X is a real smooth Banach space, then Q is an orthogonal projection of X onto C if and only if

$$Q(x) \in C \text{ and } \langle Q(x) - x, j_q(Q(x) - y)\rangle \leq 0, \ \forall\, y \in C. \tag{1.6}$$

Example 1.4 ([10])**.** If X is strictly convex and uniformly smooth and $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $F(T)$, then the set $F(T)$ is a sunny nonexpansive retraction of C.

let C be a nonempty closed convex subset of a real Hilbert space H. Recall that the classical variational inequality, denoted by $VI(A,C)$, is to find an $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \ \forall\, x \in C.$$

Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [3-12] and the references therein.

Let $A, B : C \to H$ be two mappings. Recently, Ceng et al. [6] considered the following general variational inequality problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \ \forall\, x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \ \forall\, x \in C, \end{cases} \tag{1.7}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$ and $x^* = y^*$, then problem (1.7) reduces to the classical variational inequality $VI(A,C)$.

Let C be a nonempty closed convex subset of a real Banach space X. Very recently, Yao et al. [20] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \ \forall\, x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \ \forall\, x \in C, \end{cases} \tag{1.8}$$

which is called the system of general variational inequalities in a real Banach spaces, where $A, B : C \to X$ are two operators.

In order to find a solution of problem (1.8), Yao et al. [20] proved the following strong convergence theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X which admits a weakly sequentially continuous duality mapping. Let Q_C be the sunny nonexpansive retraction from X onto C. Let the mappings $A, B : C \to X$ be α-inverse-strongly accretive with $\alpha \geq K^2$ and β-inverse-strongly accretive with $\beta \geq K^2$, respectively, where K is defined by Lemma 2.3. Suppose the set of fixed points Ω of the mapping $G : C \to C$ defined by $G(x) = Q_C[Q_C(x - Bx) - AQ_C(x - Bx)], \forall\, x \in C$ is nonempty. For a given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = Q_C(x_n - Bx_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), n \geq 0. \end{cases} \tag{1.9}$$

Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:
(i) $\alpha_n + \beta_n + \gamma_n = 1, \ \forall\, n \geq 0$;
(ii) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$.
Then $\{x_n\}$ converges strongly to $Q'u$, where $Q'u$ is the sunny nonexpansive retraction of C onto $F(G)$.

Some questions arise naturally:

(1) Could we extend a system of variational inequality problem (1.8) to more general variational inequality problem which includes (1.8) as a special case?

(2) Could we extend Theorem 1.1 from 2-uniformly smooth Banach space to q-uniformly smooth Banach space,where $1 < q \leq 2$? At the same time, could we remove the space condition that X is uniformly convex Banach Space which admits a uniformly sequentially continuous duality mapping?

(3) Could we modify the iterative algorithm (1.9) such that we can find the common element of the set of solutions of the general variational inequality problem (1.10) and the set of common fixed points of an infinite family of nonexpansive mappings?

(4) Could we replace u with $f(x_n)$, where $f \in \Pi_C$?

(5) Could we extend Theorem 1.1 from inverse-strongly accretive mappings to Lipchitzian and relaxed cocoercive mappings?

(6) Could we weaken the condition $\lim_{n\to\infty} \alpha_n = 0$ such that Theorem 1.1 also holds when $\lim_{n\to\infty} \alpha_n \neq 0$?

The purpose of this paper is to give affirmative answers to the questions raised above. Let C be a nonempty closed convex subset of a real Banach space X. For given two operators $A, B : C \to X$, we consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j_q(x - x^*)\rangle \geq 0, \ \forall\, x \in C, \\ \langle \mu Bx^* + y^* - x^*, j_q(x - y^*)\rangle \geq 0, \ \forall\, x \in C, \end{cases} \tag{1.10}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. If $\lambda = \mu = 1$ and $q = 2$, the problem (1.10) reduces to problem (1.8). If X is a Hilbert space, then (1.10) becomes the problem (1.7). Consequently, our variational inequality problem (1.10) contains (1.7) or (1.8) as a special case.

In this paper, we introduce a new iterative algorithm for finding a common element of the set of solutions of a general variational inequality (1.10) and the set of common fixed points of an infinite family of nonexpansive mappings in q-uniformly smooth Banach space. Furthermore we prove some strong convergence theorems under suitable conditions. Then we give an appropriate example such that all conditions of this result are satisfied and the condition $\alpha_n \to 0$[Theorem 1.1] is not satisfied. The results presented in this paper extend and improve the results of Yao et al. [20], Ceng et al. [6] and many others.

2. Preliminaries

Let $S(X) = \{x \in X : \|x\| = 1\}$.Then the norm of X is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{Δ}$$

exists for each $x, y \in S(X)$. In this case, X is said to be smooth. The norm of X is said to be uniformly Gâteaux differentiable if for each $y \in S(X)$, the limit(Δ)is attained uniformly for $x \in S(X)$. The norm of the X is said to be Frêchet differentiable, if for each $x \in S(X)$, the limit(Δ)is attained uniformly for $y \in S(X)$. The norm of X is called uniformly Fréchet differentiable, if the limit(Δ)is attained uniformly for $x, y \in S(X)$. It is well-known that(uniform)Fréchet differentiability of the norm X implies(uniform)Gâteaux differentiability of norm X.

Let $\rho_X : [0,\infty) \longrightarrow [0,\infty)$ be the modulus of smoothness of X defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(X),\ \|y\| \leq t\right\}.$$

A Banach space X is said to be uniformly smooth if $\dfrac{\rho_X(t)}{t} \to 0$ as $t \to 0$. A Banach space X is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$. It is well-known that X is uniformly smooth if and only if the norm of X is uniformly Fréchet differentiable. If X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth, and hence the norm of X is uniformly Fréchet differentiable, in particular, the norm of X is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p,2\}$-uniformly smooth for every $p > 1$.

In order to obtain our main results, we collect the following Lemmas.

Lemma 2.1 ([19])**.** *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n,\ n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([17])**.** *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the following condition: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n,\ n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3 ([18])**.** *Let X be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q x\rangle + C_q \|y\|^q,$$

for all $x, y \in X$. In particular, if X is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2,$$

for all $x, y \in X$.

Lemma 2.4 ([12], p. 63)**.** *Let $q > 1$. Then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q} b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.5. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space X. Let P_C be the sunny nonexpansive retraction from X onto C. Let $A, B : C \to X$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.10) if and only if $x^* = P_C(y^* - \lambda Ay^*)$ where $y^* = P_C(x^* - \mu Bx^*)$.*

Proof. We can rewrite (1.10) as

$$\begin{cases} \langle (y^* - \lambda A y^*) - x^*, j_q(x - x^*)\rangle \leq 0, \ \forall\, x \in C, \\ \langle (x^* - \mu B x^*) - y^*, j_q(x - y^*)\rangle \leq 0, \ \forall\, x \in C. \end{cases} \tag{2.1}$$

From (1.6), we can deduce that (2.1) is equivalent to

$$\begin{cases} x^* = P_C(y^* - \lambda A y^*), \\ y^* = P_C(x^* - \mu B x^*). \end{cases}$$

This completes the proof. □

Lemma 2.6. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space X. Let the mapping $A : C \to X$ be relaxed (c, d)-cocoercive and L_A-Lipschitzian. Then, we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q + (q\lambda c L_A^q - q\lambda d + C_q \lambda^q L_A^q)\, \|x - y\|^q\,,$$

where $\lambda > 0$. In particular, if $\lambda \leq (\frac{qd - qcL_A^q}{C_q L_A^q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Proof. From Lemma 2.3, we have for all $x, y \in C$

$$\begin{aligned} &\|(I - \lambda A)x - (I - \lambda A)y\|^q \\ &= \|x - y - \lambda(Ax - Ay)\|^q \\ &\leq \|x - y\|^q - q\lambda \langle Ax - Ay, j_q(x - y)\rangle + C_q \lambda^q \|Ax - Ay\|^q \\ &\leq \|x - y\|^q - q\lambda(-c\,\|Ax - Ay\|^q + d\,\|x - y\|^q) + C_q \lambda^q L_A^q \|x - y\|^q \\ &\leq \|x - y\|^q + (q\lambda c L_A^q - q\lambda d + C_q \lambda^q L_A^q)\, \|x - y\|^q\,. \end{aligned}$$

It is easy to see that $I - \lambda A$ is nonexpansive if $\lambda \leq (\frac{qd - qcL_A^q}{C_q L_A^q})^{\frac{1}{q-1}}$. This completes the proof. □

Lemma 2.7. *Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space X. Let P_C be the sunny nonexpansive retraction from X onto C. Let the mapping $A : C \to X$ be (c, d)-cocoercive and L_A-Lipschitzian and let $B : C \to X$ be (c', d')-cocoercive and L_B-Lipschitzian. Let $G : C \to C$ be a mapping defined by*

$$G(x) = P_C\left[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)\right],\ \forall\, x \in C.$$

If $0 < \lambda \leq (\frac{qd - qcL_A^q}{C_q L_A^q})^{\frac{1}{q-1}}$ and $0 < \mu \leq (\frac{qd' - qc' L_B^q}{C_q L_B^q})^{\frac{1}{q-1}}$, then $G : C \to C$ is nonexpansive.

Proof. For all $x, y \in C$, by Lemma 2.6, we have

$$\begin{aligned} &\|G(x) - G(y)\| \\ &= ||P_C\left[P_C(x - \mu Bx) - \lambda A P_C(y - \mu By)\right] \\ &\quad - P_C\left[P_C(y - \mu By) - \lambda A P_C(y - \mu By)\right]|| \\ &\leq \|(I - \lambda A)P_C(I - \mu B)x - (I - \lambda A)P_C(I - \mu B)y\| \\ &\leq \|P_C(I - \mu B)x - P_C(I - \mu B)y\| \\ &\leq \|(I - \mu B)x - (I - \mu B)y\| \\ &\leq \|x - y\|\,, \end{aligned}$$

which implies that G is nonexpansive. This completes the proof. □

Motivated and inspired by Theorem 4.1 of Xu [19], we obtain the following Lemma.

Lemma 2.8. *Let X be a q-uniformly smooth Banach space, C be a closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$ with contractive constant $\alpha \in (0, 1)$. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ for $t \in (0, 1)$ converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by*

$$Q(f) := \lim_{t\to 0} x_t,\ f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (I - f)Q(f), j_q(Q(f) - p)\rangle \leq 0,\ f \in \Pi_C, p \in F(T).$$

Proof. First we show that $\{x_t\}$ is bounded. Indeed take a $p \in F(T)$, we have

$$\begin{aligned} \|x_t - p\| &= \|(1-t)(T(x_t) - p) + t(f(x_t) - f(p)) + t(f(p) - p)\| \\ &\leq (1-t)\|T(x_t) - p\| + t\|f(x_t) - f(p)\| + t\|f(p) - p\| \\ &\leq (1-t)\|x_t - p\| + t\alpha\|x_t - p\| + t\|f(p) - p\|, \end{aligned}$$

which implies that

$$\|x_t - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\|,$$

and hence $\{x_t\}$ is bounded. Assume $t_n \to 0$. Set $x_n := x_{t_n}$ and define $\mu : C \to R$ by

$$\mu(x) = LIM\|x_n - x\|^q,\ x \in C,$$

where LIM is a Banach limit on l^∞. Let

$$K = \left\{x \in C : \mu(x) = \min_{x\in C} LIM\|x_n - x\|^q\right\}.$$

We see easily that K is a nonempty closed convex bounded subset of X. Since

$$\|x_n - Tx_n\| = t_n\|f(x_n) - Tx_n\| \to 0 \text{ as } n \to \infty,$$

and hence

$$\begin{aligned} \mu(Tx) &= LIM\|x_n - Tx\|^q \\ &\leq LIM(\|x_n - Tx_n\| + \|Tx_n - Tx\|)^q \\ &\leq LIM\|Tx_n - Tx\|^q \\ &\leq LIM\|x_n - x\|^q \\ &= \mu(x). \end{aligned}$$

It follows that $T(K) \subset K$; that is, K is invariant under T. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, T has a fixed point, say z in K. Since z is also a minimizer of μ over C, it follows that, for $t \in (0,1)$ and $x \in C$,

$$\begin{aligned}0 &\leq \frac{\mu(z+t(x-z))-\mu(z)}{t}\\ &= LIM\frac{\|(x_n-z)+t(z-x)\|^q-\|x_n-z\|^q}{t}\\ &= LIM\frac{\langle (x_n-z)+t(z-x), j_q((x_n-z)+t(z-x))\rangle - \|x_n-z\|^q}{t}.\end{aligned}$$

The uniform smoothness of X implies that the duality map j_q is norm-to-norm uniformly continuous on bounded sets of X. Letting $t \to 0$, we find that two limits above can be interchanged and obtain

$$0 \leq LIM\,\langle z-x, j_q(x_n-z)\rangle\,,$$

which implies

$$LIM\,\langle x-z, j_q(x_n-z)\rangle \leq 0,\ \ x \in C. \tag{2.2}$$

Since $x_t - z = t(f(x_t)-z)+(1-t)(Tx_t-z)$,

$$\begin{aligned}\|x_t-z\|^q &= t\,\langle f(x_t)-z, j_q(x_t-z)\rangle + (1-t)\,\langle Tx_t-z, j_q(x_t-z)\rangle\\ &\leq t\,\langle f(x_t)-z, j_q(x_t-z)\rangle + (1-t)\,\|x_t-z\|^q\,.\end{aligned}$$

Hence

$$\begin{aligned}\|x_t-z\|^q &\leq \langle f(x_t)-z, j_q(x_t-z)\rangle\\ &\leq \langle f(x_t)-x, j_q(x_t-z)\rangle + \langle x-z, j_q(x_t-z)\rangle\,.\end{aligned} \tag{2.3}$$

Therefore by (2.2), we have for $x \in C$

$$\begin{aligned}LIM\,\|x_n-z\|^q &\leq LIM\,\langle f(x_n)-x, j_q(x_n-z)\rangle + LIM\,\langle x-z, j_q(x_n-z)\rangle\\ &\leq LIM\,\langle f(x_n)-x, j_q(x_n-z)\rangle\\ &\leq LIM\,\|f(x_n)-x\|\,\|x_n-z\|^{q-1}\,.\end{aligned}$$

In particular,

$$\begin{aligned}LIM\,\|x_n-z\|^q &\leq LIM\,\|f(x_n)-f(z)\|\,\|x_n-z\|^{q-1}\\ &\leq \alpha LIM\,\|x_n-z\|^q\,.\end{aligned}$$

Hence $LIM\,\|x_n-z\|^q = 0$ and there exists a subsequence which is still denoted $\{x_n\}$ such that $x_n \to z$.

Now assume there exists another subsequence $\{x_m\}$ of $\{x_t\}$ such that $x_m \to z' \in F(T)$. It follows from (2.3) that

$$\|z'-z\|^q \leq \langle f(z')-z, j_q(z'-z)\rangle\,. \tag{2.4}$$

Interchange z' and z to obtain

$$\|z-z'\|^q \leq \langle f(z)-z', j_q(z-z')\rangle\,. \tag{2.5}$$

Adding up (2.4) and (2.5) yields

$$\begin{aligned} 2\left\|z'-z\right\|^q &\le \langle f(z')-f(z)+z'-z, j_q(z'-z)\rangle \\ &\le (1+\alpha)\left\|z'-z\right\|^q . \end{aligned}$$

Since $\alpha \in (0,1)$, this implies $z' = z$. Hence $x_t \to z$ as $t \to 0$.

Define $Q : \Pi_C \to F(T)$ by $Q(f) := \lim_{t\to 0} x_t$. Since $x_t = tf(x_t) + (1-t)Tx_t$, we have

$$(I-f)x_t = -\frac{1-t}{t}(I-T)x_t.$$

Hence for $p \in F(T)$,

$$\begin{aligned} \langle (I-f)x_t, j_q(x_t-p)\rangle &= -\frac{1-t}{t}\langle (I-T)x_t-(I-T)p, j_q(x_t-p)\rangle \\ &\le 0. \end{aligned}$$

Letting $t \to 0$ yields

$$\langle (I-f)Q(f), j_q(Q(f)-p)\rangle \le 0.$$

This completes the proof. □

Lemma 2.9. *Let C be a closed convex subset of a real q-uniformly smooth Banach space X, and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Assume $\{x_n\}$ is a bounded sequence such that $x_n - Tx_n \to 0$ as $n \to \infty$. Let $x_t = tf(x_t)+(1-t)Tx_t$, $\forall\, t \in (0,1)$, where $f \in \Pi_C$ with contractive constant $\alpha \in (0,1)$. Assume that $Q(f) := \lim_{t\to 0} x_t$ exists. Then*

$$\limsup_{n\to\infty} \langle (f-I)Q(f), j_q(x_n-Q(f))\rangle \le 0.$$

Proof. Set $M = \sup\left\{\|x_n-x_t\|^{q-1} : t\in(0,1), n \ge 0\right\}$. Then we have

$$\begin{aligned} &\|x_t-x_n\|^q \\ &= t\left\|f(x_t)-x_n, j_q(x_t-x_n)\right\| + (1-t)\langle Tx_t-x_n, j_q(x_t-x_n)\rangle \\ &= t\langle f(x_t)-x_t, j_q(x_t-x_n)\rangle + t\|x_t-x_n\|^q \\ &\quad + (1-t)\langle Tx_t-Tx_n, j_q(x_t-x_n)\rangle + (1-t)\langle Tx_n-x_n, j_q(x_t-x_n)\rangle \\ &\le t\langle f(x_t)-x_t, j_q(x_t-x_n)\rangle + t\|x_t-x_n\|^q + (1-t)\|x_t-x_n\|^q \\ &\quad + M\|x_n-Tx_n\| \\ &= t\langle f(x_t)-x_t, j_q(x_t-x_n)\rangle + \|x_t-x_n\|^q \\ &\quad + M\|x_n-Tx_n\|, \end{aligned}$$

which implies

$$\langle f(x_t)-x_t, j_q(x_n-x_t)\rangle \le \frac{M}{t}\|x_n-Tx_n\|.$$

Fixing t and letting $n\to\infty$ yields

$$\limsup_{n\to\infty}\langle f(x_t)-x_t, j_q(x_n-x_t)\rangle \le 0.$$

Since X is uniformly smooth, $j_q : X \to X^*$ is uniformly continuous on any bounded set of X, which ensures that the limits $\limsup\limits_{n\to\infty}$ and $\limsup\limits_{t\to 0}$ are interchangeable, we have

$$\limsup_{n\to\infty} \langle (f-I)Q(f), j_q(x_n - Q(f))\rangle \leq 0.$$

This completes the proof. □

Lemma 2.10 ([1])**.** *Let C be a nonempty closed convex subset of a Banach space X. Let $T_1, T_2, \cdots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} T_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.*

Lemma 2.11 ([3])**.** *Let C be a closed convex subset of a strictly convex Banach space X. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = \lambda T_1 x + (1-\lambda) T_2 x, \forall\, x \in C,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

3. Main results

Theorem 3.1. *Let C be a closed convex subset of a real q-uniformly smooth Banach space X $(q > 1)$ which is also a sunny nonexpansive retraction of X. Let the mapping $A : C \to X$ be (c,d)-cocoercive and L_A-Lipschitzian and let $B : C \to X$ be (c', d')-cocoercive and L_B-Lipschitzian. $f \in \Pi_C$ with the coefficient $0 < \alpha < 1$. Let G be the mapping defined by Lemma 2.7. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself with $F := F(G) \cap \cap_{n=1}^{\infty} F(T_i) \neq \emptyset$. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ z_n = Q_C(y_n - \lambda A y_n), \\ k_n = \delta_n T_n x_n + (1-\delta_n) z_n \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n k_n, n \geq 1, \end{cases} \tag{3.1}$$

where Q_C is a sunny nonexpansive retraction of X onto C, $0 < \lambda \leq (\frac{qd - qcL_A^q}{C_q L_A^q})^{\frac{1}{q-1}}$ and $0 < \mu \leq (\frac{qd' - qc'L_B^q}{C_q L_B^q})^{\frac{1}{q-1}}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(iv) $\limsup\limits_{n\to\infty} \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| = 0$;

(v) $\liminf\limits_{n\to\infty} \gamma_n > 0$;

(vi) $\lim\limits_{n\to\infty} \delta_n = \delta \in (0,1)$.

Assume that $\sum_{n=1}^{\infty} \sup_{x \in D} \|T_{n+1}x - T_n x\| < \infty$ *for any bounded subset* D *of* C *and let* T *be a mapping of* C *into itself defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and suppose that* $F(T) = \cap_{n=1}^{\infty} F(T_n)$*. Then* $x_n \to Q(f) \Leftrightarrow \alpha_n(f(x_n) - x_n) \to 0$*, where* $Q(f) \in F$ *solves the variational inequality*

$$\langle (I-f)Q(f), j_q(Q(f)-p)\rangle \leq 0,\ \ f \in \Pi_C, p \in F.$$

Proof. Take $x^* \in F$. From Lemma 2.5, we have $x^* = Q_C[Q_C(x^* - \mu Bx^*) - \lambda A Q_C(x^* - \mu Bx^*)]$. Put $y^* = Q_C(x^* - \mu Bx^*)$, then $x^* = Q_C(y^* - \lambda Ay^*)$. It follows from Lemma 2.7 that

$$\begin{aligned}
\|z_n - x^*\| &= \|Q_C(y_n - \lambda Ay_n) - Q_C(y^* - \lambda Ay^*)\| \\
&\leq \|(I-\lambda A)y_n - (I-\lambda A)y^*\| \\
&\leq \|y_n - y^*\| \\
&= \|Q_C(x_n - \mu Bx_n) - Q_C(x^* - \mu Bx^*)\| \\
&\leq \|(I-\mu B)x_n - (I-\mu B)x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|k_n - x^*\| &= \|\delta_n(T_n x_n - x^*) + (1-\delta_n)(z_n - x^*)\| \\
&\leq \delta_n \|T_n x_n - x^*\| + (1-\delta_n)\|z_n - x^*\| \\
&\leq \delta_n \|x_n - x^*\| + (1-\delta_n)\|x_n - x^*\| \\
&= \|x_n - x^*\|. \qquad (3.2)
\end{aligned}$$

By (3.2), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(f(x_n) - x^*) + \beta_n(x_n - x^*) + \gamma_n(k_n - x^*)\| \\
&\leq \alpha_n \|f(x_n) - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|k_n - x^*\| \\
&\leq \alpha_n \|f(x_n) - f(x^*)\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|x_n - x^*\| \\
&\quad + \gamma_n \|x_n - x^*\| \\
&\leq \alpha_n \alpha \|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|f(x^*) - x^*\| \\
&= [1 - \alpha_n(1-\alpha)]\|x_n - x^*\| + \alpha_n(1-\alpha)\frac{\|f(x^*) - x^*\|}{1-\alpha} \\
&\leq \max\left\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\alpha}\right\}, \forall\, n \geq 1.
\end{aligned}$$

Therefore $\{x_n\}$ is bounded. Hence $\{y_n\}, \{k_n\}, \{z_n\}, \{Ay_n\}$ and $\{Bx_n\}$ are also bounded.

Suppose that $\alpha_n(f(x_n)-x_n)\to 0$ as $n\to\infty$. We observe that

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|Q_C(y_{n+1}-\lambda Ay_{n+1})-Q_C(y_n-\lambda Ay_n)\|\\ &\le \|(I-\lambda A)y_{n+1}-(I-\lambda A)y_n\|\\ &\le \|y_{n+1}-y_n\|\\ &= \|Q_C(x_{n+1}-\mu Bx_{n+1})-Q_C(x_n-\mu Bx_n)\|\\ &\le \|(I-\mu B)x_{n+1}-(I-\mu B)x_n\|\\ &\le \|x_{n+1}-x_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}&\|k_{n+1}-k_n\|\\ &= \|\delta_{n+1}T_{n+1}x_{n+1}+(1-\delta_{n+1})z_{n+1}-\delta_nT_nx_n-(1-\delta_n)z_n\|\\ &= ||(\delta_{n+1}-\delta_n)(T_{n+1}x_{n+1}-z_{n+1})+\delta_n(T_{n+1}x_{n+1}-T_nx_n)\\ &\quad+(1-\delta_n)(z_{n+1}-z_n)||\\ &\le |\delta_{n+1}-\delta_n|\,\|T_{n+1}x_{n+1}-z_{n+1}\|+\delta_n\|T_{n+1}x_{n+1}-T_nx_n\|\\ &\quad+(1-\delta_n)\|z_{n+1}-z_n\|\\ &\le |\delta_{n+1}-\delta_n|\,\|T_{n+1}x_{n+1}-z_{n+1}\|+\delta_n\|T_{n+1}x_{n+1}-T_nx_{n+1}\|\\ &\quad+\delta_n\|T_nx_{n+1}-T_nx_n\|+(1-\delta_n)\|z_{n+1}-z_n\|\\ &\le |\delta_{n+1}-\delta_n|\,\|T_{n+1}x_{n+1}-z_{n+1}\|+\delta_n\|T_{n+1}x_{n+1}-T_nx_{n+1}\|\\ &\quad+\delta_n\|x_{n+1}-x_n\|+(1-\delta_n)\|x_{n+1}-x_n\|\\ &= |\delta_{n+1}-\delta_n|\,\|T_{n+1}x_{n+1}-z_{n+1}\|+\delta_n\|T_{n+1}x_{n+1}-T_nx_{n+1}\|+\|x_{n+1}-x_n\|.\end{aligned} \tag{3.3}$$

Put $x_{n+1}=\beta_nx_n+(1-\beta_n)l_n$ for all $n\ge 1$. Then, we have

$$\begin{aligned}l_{n+1}-l_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}k_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+\gamma_nk_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}f(x_n)+\frac{\gamma_{n+1}}{1-\beta_{n+1}}k_{n+1}-\frac{\gamma_n}{1-\beta_n}k_n\\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_{n+1})+\frac{\alpha_n}{1-\beta_n}(f(x_{n+1})-f(x_n))\\ &\quad+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)k_{n+1}+\frac{\gamma_n}{1-\beta_n}(k_{n+1}-k_n)\\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)(f(x_{n+1})-k_{n+1})+\frac{\alpha_n}{1-\beta_n}(f(x_{n+1})-f(x_n))\\ &\quad+\frac{\gamma_n}{1-\beta_n}(k_{n+1}-k_n).\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\begin{aligned}
&\|l_{n+1} - l_n\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| (\|f(x_{n+1})\| + \|k_{n+1}\|) + \frac{\alpha_n \alpha}{1-\beta_n} \|x_{n+1} - x_n\| \\
&\quad + \frac{\gamma_n}{1-\beta_n} \|x_{n+1} - x_n\| + \frac{\gamma_n |\delta_{n+1} - \delta_n|}{1-\beta_n} \|T_{n+1}x_{n+1} - z_{n+1}\| \\
&\quad + \frac{\gamma_n \delta_n}{1-\beta_n} \|T_{n+1}x_{n+1} - T_n x_{n+1}\| \\
&= \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| (\|f(x_{n+1})\| + \|k_{n+1}\|) \\
&\quad + \frac{1-\beta_n - \alpha_n(1-\alpha)}{1-\beta_n} \|x_{n+1} - x_n\| + \frac{\gamma_n |\delta_{n+1} - \delta_n|}{1-\beta_n} \|T_{n+1}x_{n+1} - z_{n+1}\| \\
&\quad + \frac{\gamma_n \delta_n}{1-\beta_n} \|T_{n+1}x_{n+1} - T_n x_{n+1}\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| (\|f(x_{n+1})\| + \|k_{n+1}\|) + \|x_{n+1} - x_n\| \\
&\quad + |\delta_{n+1} - \delta_n| \|T_{n+1}x_{n+1} - z_{n+1}\| + \|T_{n+1}x_{n+1} - T_n x_{n+1}\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \\
&\leq \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| (\|f(x_{n+1})\| + \|k_{n+1}\|) \\
&\quad + |\delta_{n+1} - \delta_n| \|T_{n+1}x_{n+1} - z_{n+1}\| + \|T_{n+1}x_{n+1} - T_n x_{n+1}\|.
\end{aligned}$$

By conditions (iv),(vi) and the assumption on T_n,we obtain

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 2.2 that $\lim_{n\to\infty} \|l_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n) \|l_n - x_n\| = 0. \tag{3.5}$$

From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(f(x_n) - x_n) + \gamma_n(k_n - x_n)\| \\
&\geq \gamma_n \|k_n - x_n\| - \|\alpha_n(f(x_n) - x_n)\|,
\end{aligned}$$

which implies

$$\|k_n - x_n\| \leq \frac{1}{\gamma_n}(\|\alpha_n(f(x_n) - x_n)\| + \|x_{n+1} - x_n\|).$$

Noticing that condition (v), (3.5) and $\lim_{n\to\infty} \alpha_n(f(x_n) - x_n) \to 0$, we have

$$\lim_{n\to\infty} \|k_n - x_n\| = 0. \tag{3.6}$$

Define a mapping $U : C \to C$ as $Ux = \delta Tx + (1-\delta)Gx$. From Lemma 2.11, we know that U is nonexpansive and

$$F(U) = F(T) \cap F(G) = \cap_{n=1}^{\infty} F(T_i) \cap F(G) = F.$$

Since condition (vi) and the assumption on T_n, we have

$$\begin{aligned}
\|k_n - Ux_n\| &= \|\delta_n T_n x_n + (1-\delta_n) z_n - \delta T x_n - (1-\delta) G x_n\| \\
&= \|\delta_n T_n x_n + (1-\delta_n) z_n - \delta T x_n - (1-\delta) z_n\| \\
&= \|(\delta_n - \delta)(T_n x_n - z_n) + \delta (T_n x_n - T x_n)\| \\
&\leq |\delta_n - \delta| \, \|T_n x_n - z_n\| + \delta \, \|T_n x_n - T x_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (3.7)
\end{aligned}$$

Combining (3.6) and (3.7), we have

$$\|x_n - Ux_n\| \leq \|x_n - k_n\| + \|k_n - Ux_n\| \to 0 \text{ as } n \to \infty. \qquad (3.8)$$

Next we show that

$$\limsup_{n\to\infty} \langle f(z) - z, j_q(x_n - z)\rangle \leq 0, \qquad (3.9)$$

where

$$z = Q(f), \; Q(f) = \lim_{t\to 0} x_t$$

and x_t is the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x) + (1-t)Ux, t \in (0,1).$$

By Lemma 2.8, we have $Q(f) \in F(U) = F$ solves the variational inequality

$$\langle (I - f)Q(f), j_q(Q(f) - p)\rangle \leq 0, \; \forall p \in F.$$

By (3.8) and Lemma 2.9, we see that

$$\limsup_{n\to\infty} \langle f(z) - z, j_q(x_n - z)\rangle \leq 0.$$

Therefore (3.9) holds.

Finally we prove that $x_n \to z$ as $n \to \infty$. Putting

$$\sigma_n = \max\left\{\langle f(z) - z, j_q(x_{n+1} - z)\rangle, 0\right\},$$

we have $\sigma_n \to 0$ as $n \to \infty$.

By virtue of Lemma 2.4 and (3.2), we have

$$\begin{aligned}
&\|x_{n+1}-z\|^q \\
&= \langle \alpha_n(f(x_n)-z), j_q(x_{n+1}-z)\rangle + \langle \beta_n(x_n-z), j_q(x_{n+1}-z)\rangle \\
&\quad + \langle \gamma_n(k_n-z), j_q(x_{n+1}-z)\rangle \\
&= \alpha_n \langle f(x_n)-f(z), j_q(x_{n+1}-z)\rangle + \alpha_n \langle f(z)-z, j_q(x_{n+1}-z)\rangle \\
&\quad + \beta_n \langle x_n-z, j_q(x_{n+1}-z)\rangle + \gamma_n \langle k_n-z, j_q(x_{n+1}-z)\rangle \\
&\le \alpha_n\alpha \|x_n-z\| \|x_{n+1}-z\|^{q-1} + \beta_n \|x_n-z\| \|x_{n+1}-z\|^{q-1} \\
&\quad + \gamma_n \|k_n-z\| \|x_{n+1}-z\|^{q-1} + \alpha_n\sigma_n \\
&\le \alpha_n\alpha \|x_n-z\| \|x_{n+1}-z\|^{q-1} + \beta_n \|x_n-z\| \|x_{n+1}-z\|^{q-1} \\
&\quad + \gamma_n \|x_n-z\| \|x_{n+1}-z\|^{q-1} + \alpha_n\sigma_n \\
&= [1-\alpha_n(1-\alpha)] \|x_n-z\| \|x_{n+1}-z\|^{q-1} + \alpha_n\sigma_n \\
&\le [1-\alpha_n(1-\alpha)](\frac{1}{q}\|x_n-z\|^q + \frac{q-1}{q}\|x_{n+1}-z\|^q) + \alpha_n\sigma_n \\
&\le \frac{1-\alpha_n(1-\alpha)}{q}\|x_n-z\|^q + \frac{q-1}{q}\|x_{n+1}-z\|^q + \alpha_n\sigma_n,
\end{aligned}$$

which implies that

$$\|x_{n+1}-z\|^q \le [1-\alpha_n(1-\alpha)]\|x_n-z\|^q + \alpha_n(1-\alpha)\frac{q\sigma_n}{1-\alpha}.$$

By Lemma 2.1, we have $x_n \to z$ as $n\to\infty$.

Conversely, if $x_n \to Q(f)$ as $n\to\infty$. Then from (3.1) and (3.2) we obtain that

$$\begin{aligned}
&\|\alpha_n(f(x_n)-x_n)\| \\
&= \|x_{n+1}-x_n-\gamma_n(k_n-x_n)\| \\
&\le \|x_{n+1}-Q(f)\| + \|x_n-Q(f)\| + \gamma_n\|k_n-Q(f)\| + \gamma_n\|x_n-Q(f)\| \\
&\le \|x_{n+1}-Q(f)\| + (1+2\gamma_n)\|x_n-Q(f)\| \\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let C be a closed convex subset of a real q-uniformly smooth Banach space X $(q>1)$ which is also a sunny nonexpansive retraction of X. Let the mapping $A: C\to X$ be (c,d)-cocoercive and L_A-Lipschitzian and let $B: C\to X$ be (c',d')-cocoercive and L_B-Lipschitzian. $f\in\Pi_C$ with the coefficient $0<\alpha<1$. Let G be the mapping defined by Lemma 2.7. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself with $F := F(G)\cap\cap_{n=1}^{\infty}F(T_i)\neq\emptyset$. For a given $x_1\in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}
y_n = Q_C(x_n-\mu Bx_n),\\
z_n = Q_C(y_n-\lambda Ay_n),\\
k_n = \delta_n T_n x_n + (1-\delta_n)z_n\\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n k_n, n\ge 1,
\end{cases} \tag{3.10}$$

where Q_C is a sunny nonexpansive retraction of X onto C, $0<\lambda\le(\frac{qd-qcL_A^q}{C_qL_A^q})^{\frac{1}{q-1}}$ and $0<\mu\le(\frac{qd'-qc'L_B^q}{C_qL_B^q})^{\frac{1}{q-1}}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n+\beta_n+\gamma_n=1$;

(ii) $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$, $\lim\limits_{n\to\infty}\alpha_n=0$;

(iii) $0<\liminf\limits_{n\to\infty}\beta_n\le\limsup\limits_{n\to\infty}\beta_n<1$;

(iv) $\lim\limits_{n\to\infty}\delta_n=\delta\in(0,1)$.

Assume that $\sum\limits_{n=1}^{\infty}\sup\limits_{x\in D}\|T_{n+1}x-T_nx\|<\infty$ *for any bounded subset D of C and let T be a mapping of C into X defined by $Tx=\lim\limits_{n\to\infty}T_nx$ for all $x\in C$ and suppose that $F(T)=\cap_{n=1}^{\infty}F(T_n)$. Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f)\in F$ solves the variational inequality*

$$\langle (I-f)Q(f),j_q(Q(f)-p)\rangle\le 0,\ f\in\Pi_C,p\in F.$$

Proof. By condition (ii), we see that there hold the following

(1) $\alpha_n(f(x_n)-x_n)\to 0$ as $n\to\infty$;

(2) $\limsup\limits_{n\to\infty}\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|=0$;

(3) $\liminf\limits_{n\to\infty}\gamma_n=\liminf\limits_{n\to\infty}(1-\beta_n)>0$.

Therefore, all conditions of Theorem 3.1 are satisfied. So we obtain the desired result by Theorem 3.1. This completes the proof. □

Corollary 3.3. *Let C be a closed convex subset of a real q-uniformly smooth Banach space X $(q>1)$ which is also a sunny nonexpansive retraction of X. Let the mapping $A:C\to X$ be (c,d)-cocoercive and L_A-Lipschitzian and let $B:C\to X$ be (c',d')-cocoercive and L_B-Lipschitzian. $f\in\Pi_C$ with the coefficient $0<\alpha<1$. Let G be the mapping defined by Lemma 2.7. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself with $F:=F(G)\cap\cap_{n=1}^{\infty}F(T_i)\ne\emptyset$. For a given $x_1\in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n=Q_C(x_n-\mu Bx_n),\\ z_n=Q_C(y_n-\lambda Ay_n),\\ k_n=\delta_nT_nx_n+(1-\delta_n)z_n\\ x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_nk_n,n\ge 1,\end{cases}\tag{3.11}$$

where Q_C is a sunny nonexpansive retraction of X onto C, $0<\lambda\le(\frac{qd-qcL_A^q}{C_qL_A^q})^{\frac{1}{q-1}}$ and $0<\mu\le(\frac{qd'-qc'L_B^q}{C_qL_B^q})^{\frac{1}{q-1}}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n+\beta_n+\gamma_n=1$;

(ii) $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$;

(iii) $0<\liminf\limits_{n\to\infty}\beta_n\le\limsup\limits_{n\to\infty}\beta_n<1$;

(iv) $\limsup_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0, \limsup_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$;
(v) $\liminf_{n\to\infty} \gamma_n > 0$;
(vi) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ *for any bounded subset D of C and let T be a mapping of C into X defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all $x \in C$ and suppose that* $F(T) = \cap_{n=1}^{\infty} F(T_n)$. *Then* $x_n \to Q(f) \Leftrightarrow \alpha_n(f(x_n) - x_n) \to 0$, *where $Q(f) \in F$ solves the variational inequality*

$$\langle (I - f)Q(f), j_q(Q(f) - p)\rangle \leq 0,\ f \in \Pi_C, p \in F.$$

Proof. We observe that

$$\begin{aligned} \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} &= \frac{\alpha_{n+1}(1-\beta_n) - \alpha_n(1-\beta_{n+1})}{(1-\beta_{n+1})(1-\beta_n)} \\ &= \frac{\alpha_{n+1} - \alpha_n - \alpha_{n+1}\beta_n + \alpha_n\beta_n - \alpha_n\beta_n + \alpha_n\beta_{n+1}}{(1-\beta_{n+1})(1-\beta_n)} \\ &= \frac{(\alpha_{n+1} - \alpha_n)(1-\beta_n) + \alpha_n(\beta_{n+1} - \beta_n)}{(1-\beta_{n+1})(1-\beta_n)}. \end{aligned} \tag{3.12}$$

By virtue of condition (iv), we deduce from (3.12) that $\limsup_{n\to\infty}(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}) = 0$. Consequently, all conditions of Theorem 3.1 are satisfied. So, utilizing Theorem 3.1 we obtain the desired result. □

The following example shows that all conditions of Theorem 3.1 are satisfied. But the condition $\alpha_n \to 0$ in [9,Theorem 3.1] is not satisfied.

Example 3.1. Let $X = L^2$ and C be a closed convex subset of L^2. We know that L^2 is Hilbert space and 2-uniformly smooth Banach space. Then $j_q = I$. Define mappings $A, B : C \to C$ and a contraction $f : C \to C$ with contractive constant $\frac{1}{4}$ as follows:

$$Ax = Bx = \frac{1}{2}x, T_n x = x \text{ and } f(x) = \frac{1}{4}x,\ \forall\, n \geq 1, x \in C.$$

Take $\delta_n = \frac{3}{7}$, $\alpha_n = \beta_n = \gamma_n = \frac{1}{3}$, $c = c' = 1, d = d' = \frac{1}{2}, L_A = L_B = \frac{1}{2}$. Since

$$\langle Ax - Ay, j_q(x-y)\rangle = \frac{1}{2}\langle x-y, x-y\rangle = \frac{1}{2}\|x-y\|^2$$

and

$$-c\|Ax - Ay\|^2 + d\|x-y\|^2 = -\frac{1}{4}\|x-y\|^2 + \frac{1}{2}\|x-y\|^2 = \frac{1}{4}\|x-y\|^2.$$

We know that $\frac{1}{2}\|x-y\|^2 > \frac{1}{4}\|x-y\|^2$. Therefore A, B are $(1, \frac{1}{2})$-cocoercive and $\frac{1}{2}$-Lipschitzian.

We observe

$$\|x-y\|^2 = \|x\|^2 + 2\langle y, x\rangle + \|y\|^2.$$

From Lemma 2.3, we obtain $C_q = 1$. So

$$(\frac{qd - qcL_A^q}{C_q L_A^q})^{\frac{1}{q-1}} = \frac{2 \times \frac{1}{2} - 2 \times 1 \times \frac{1}{4}}{\frac{1}{4}} = 2.$$

We can take $\lambda = \mu = \frac{1}{2}$. Define a mapping $G : C \to C$ as

$$Gx = P_C(I - \frac{1}{2}A)P_C(I - \frac{1}{2}B)x = \frac{9}{16}x.$$

Then G is nonexpansive and $F(G) = \{\theta\}$, and hence $F = \{\theta\}$. For any $x_1 \in C$, let $\{x_n\}$ be defined as follows:

$$\begin{cases} y_n = Q_C(x_n - \mu B x_n), \\ z_n = Q_C(y_n - \lambda A y_n), \\ k_n = \delta_n T_n x_n + (1 - \delta_n) z_n \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n k_n, n \geq 1. \end{cases}$$

That is

$$\begin{aligned} x_{n+1} &= \frac{1}{3}(f(x_n) + x_n + \frac{3}{4}x_n) \\ &= \frac{1}{3}(\frac{1}{4}x_n + x_n + \frac{3}{4}x_n) \\ &= \frac{2}{3}x_n. \end{aligned}$$

Hence by induction we get $\|x_{n+1} - \theta\| = \|x_{n+1}\| \leq (\frac{2}{3})^n \|x_1\|$ for all $n \geq 1$. This implies that $\{x_n\}$ converges strongly to the fixed point $\theta \in F$. Thus

$$\begin{aligned} \|\alpha_n(f(x_n) - x_n)\| &\leq \alpha_n(\|f(x_n)\| + \|x_n\|) \\ &= \frac{1}{3}(\frac{1}{4}\|x_n\| + \|x_n\|) \\ &= \frac{5}{12}\|x_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Furthermore, it can be seen easily that all conditions of Theorem 3.1 are satisfied. Since $\alpha_n = \frac{1}{3} \nrightarrow 0$, the condition $\alpha_n \to 0$ in [9,Theorem 3.1] is not satisfied.

References

[1] K. Aoyama et al., *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal., **67**(2007), 2350-2360.

[2] A. Bnouhachem, M. Aslam Noor, Z. Hao, *Some new extragradient iterative methods for variational inequalities*, Nonlinear Anal., **70**(2009), 1321-1329.

[3] R.E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc., **179**(1973), 251-262.

[4] R.E. Bruck, *Nonexpansive projections on subsets of Banach space*, Pacific J. Math., **47**(1973), 341-355.

[5] L.C. Ceng, J.C. Yao, *Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems*, Taiwanese J. Math., **10**(2006), 1293-1303.

[6] L.C. Ceng, C. Wang, J.C. Yao, *Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities*, Math. Methods Oper. Res., **67**(2008), 375-390.

[7] L.C. Ceng, J.C. Yao, *Convergence and certain control conditions for hybrid viscosity approximation methods*, Nonlinear Anal., **73**(2010), 2078-2087.

[8] Y.J. Cho, X. Qin, *Systems of generalized nonlinear variational inequalities and its projection methods*, Nonlinear Anal., **69**(2008), 4443-4451.
[9] R.C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., **61**(1947), 265-292.
[10] S. Kitahara, W. Takahashi, *Image recovery by convex combinations of sunny nonexpansive retractions*, Topological Meth. in Nonlinear Anal., **2**(1993), 333-342.
[11] H. Liduka, W. Takahashi, M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, Pan. Math. J., **14**(2004), 49-61.
[12] D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York, 1970.
[13] M.A. Noor, *Projection-splitting algorithms for general mixed variational inequalities*, J. Comput. Anal. Appl., **4**(2002), 47-61.
[14] M.A. Noor, *Some development in general variational inequalities*, Appl. Math. Comput., **152**(2004), 199-277.
[15] X. Qin et al., *Approximation of solutions to a system of variational inclusions in Banach spaces*, J. Ineq. Appl., Volume 2010, Article ID 916806, 16 pages,doi:10.1155/2010/916806.
[16] X. Qin, M. Shang, H. Zhou, *Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces*, Appl. Math. Comput., **200**(2008), 242-253.
[17] T. Suzuki, *Strong convergence of Krasnoselskii and Manns type sequences for one parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl., **305**(2005), 227-239.
[18] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(1991), 1127-1138.
[19] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.
[20] Y. Yao et al., *Modified extragradient methods for a system of variational inequalities in Banach spaces*, Acta. Appl. Math., **110**(2010), 1211-1224.

Received: January 4, 2011; Accepted: October 20, 2011.