Fixed Point Theory, 13(2012), No. 2, 357-382
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

TRUNCATION METHOD FOR INFINITE COUNTABLE SYSTEMS OF PARABOLIC DIFFERENTIAL-FUNCTIONAL EQUATIONS

STANISŁAW BRZYCHCZY*,*** AND MIROSŁAW LUŚTYK**

*Department of Differential Equations, Faculty of Applied Mathematics,
AGH University of Science and Technology,
Al. Mickiewicza 30, 30-059 Cracow, Poland
E-mail: brzych@agh.edu.pl

**Department of Differential Equations, Faculty of Applied Mathematics,
AGH University of Science and Technology,
Al. Mickiewicza 30, 30-059 Cracow, Poland
E-mail: lustyk@agh.edu.pl

Abstract. In the paper we study the truncation method for solvability of the Fourier first boundary problem for infinite countable systems of nonlinear parabolic-reaction-diffusion equations with Volterra functionals in Banach sequences spaces. These systems arise as discrete models of processes considered. In the truncation method a solution of the infinite countable system is defined as the limit when $N \to \infty$ of the sequence of approximations $\{z_N\}_{N=1,2,\dots}$, where $z_N = (z_N^1, z_N^2, \dots, z_N^N)$ are defined as solutions of the finite systems of the first N equations in N unknown functions with corresponding initial and boundary conditions. The truncation method plays an important role among approximation methods; it is very useful and commonly used in numerical computation of approximate solutions. The main results of the paper are an existence and uniqueness theorem for infinite countable systems of nonlinear parabolic-reaction-diffusion equations with Volterra functionals and a new method for the construction of truncated systems when a lower or an upper solution of the problem considered is known. This method may be used also to research of positive solutions of discrete models in infinite-dimensional Banach spaces.
Key Words and Phrases: Truncation method, truncated system, infinite countable system, infinite uncountable system, differential-functional equation, reaction-diffusion equation, uniformly parabolic equation, Banach sequence space, partially ordered Banach space, Volterra functional, positive solution.
2010 Mathematics Subject Classification: 35K57, 35R10, 35R15.

1. Introduction

Infinite systems of nonlinear ordinary differential equations and parabolic partial differential equations have been emerging in mathematics in response to problems arising in mathematics itself, e.g., numerical method of lines[1] (see Leszczyński [33],

***Corresponding author.

[1]The numerical method of lines consists of replacing spatial derivatives with difference expressions. Solutions of the problem considered are approximated by means of solutions of the infinite countable system of ordinary differential-functional equations.

Kamont and Zacharek [24]), as well as in physics, chemistry, biology, neurology and other sciences. These infinite systems of differential equations arise in the theory of branching processes, the birth process, the theory of neural nets and neuronal models, the theory of dissociation in polymers, the degenerations of polymers, and in other problems (see Tadeusiewicz [55, 56]). An infinite countable system of ordinary differential equations was originally introduced by Marian Smoluchowski [2] in 1917 [50, 49], as a model for coagulation of colloids moving according to a Brownian motion.

The application of infinite systems of differential equations to describing difficult and important processes and phenomena observed in reality, purely mathematical aspects and large computational capabilities of contemporary computers have all encouraged numerous mathematicians to focus their interest and research on such systems.

Certain important processes and phenomena (e.g., particles coagulation and polymers fragmentation in continuous mechanics) lead to physical models where it is from the very beginning assumed that the number of particles involved in the process discussed is unbounded. This assumption, in turn, leads to mathematical models which involve infinite systems of equations.

There are two main approaches to the construction of mathematical models of physical processes: a discrete version or a continuous version of description. If a variable taking countable infinite number of values is used to describe a process, a discrete model of the process is obtained. Discrete models are expressed in terms of infinite countable systems of equations.

On the other hand, if a variable taking arbitrary real values ranging over some interval (such as volume) is used, then a continuous model is obtained and expressed in terms of infinite uncountable systems of equations.

We remark that it is not possible to directly solve infinite countable and uncountable systems of differential equations. In the case of infinite systems of such equations, no existence theorems are known in the literature other than those quoted in the author's publications [9, 10, 11, 12, 13, 14, 15, 16, 17, 19]. Therefore, in practice the infinite systems of differential equations are replaced by finite systems of suitably defined differential equations. However, examples have been provided (see Szafirski[3] [52]) proving, that not always is this the case.

Under appropriate assumptions, the studying and solving of infinite countable systems an be replaced with studying and solving of finite systems, which may be solved with well-known numerical methods. The subject is comprehensively covered in recently published papers (see e.g. Pao [42, 43]. Moreover, assuming that the right-hand sides of system equations are monotone with respect to the function and functional

[2]Marian Smoluchowski (28th May 1872—5th September 1917) born in the town of Vonderbrühl near Vienna. Professor of physics, Head of the Department of Theoretical Physics at the John Casimir University of Lviv (1903–1913) and Head of the Experimental Physics Department at the Jagiellonian University (1913–1917). Elected Rector of the Jagiellonian University (15th July 1917). Smoluchowski died in Cracow. He was a prominent Polish scientist, theoretical physicist, pioneer of statistical physics and a mountaineer. His works on the Brownian motion theory play a fundamental role in theoretical physics. Extensive information about M. Smoluchowski see Fuliński [23].

[3]This information has never been published yet. It is available in [52].

arguments, and that Condition A holds, the method yields monotone sequences of approximations: one increasing and the other decreasing, both converging to the solution sought for. And what is perhaps most important here, the N-th approximation of the solution sought for may be found without the previous approximations.

The most difficult task is to give conditions under which that is possible, as such conditions depend on not only the equation, but also on the space and adopted definition of a solution.

We will be interested in finite systems of the first N equations in N unknown functions of an infinite system. Such transition from infinite systems of equations to finite ones, known as truncation, may be effected in various ways. One of the ways to describe the truncation process is assuming that we have a projection from an infinite dimensional space onto its finite dimensional subspace.

In this method a solution of the infinite system of differential equations is defined as the limit, when $N \to \infty$, of the sequence of approximations which are solutions of the truncated systems of the first N equations in N unknown functions with appropriate initial and boundary conditions [18, 20]. In other words, solution of the original problem is approximated by means of solutions of the infinite systems of differential equations.

In the second step of the method, one has to prove that each truncated system (that is for each positive integer N) has a solution in some Banach space. In the case of an infinite countable system of parabolic equations, this is the classical Banach space of convergent sequences of real-valued functions. The purpose of our next considerations is to give some conditions which will guarantee that this is the case. It is easy to see that we do not need to know the previous approximations to determine the next approximations.

It should be stressed here that there are existence theorems for finite systems of semilinear parabolic differential-functional equation of the reaction-diffusion type in the extensive literature on this subject. In this scope, existence theorems for finite systems of equations Ugowski [58, 59, 60, 61] have been proved with monotone iterative methods (see Ladde *et al.* [29]), based on the topological fixed point method (giving "exact" solutions), on the theory of continuous semigroups of linear operators with evolution system techniques, as well as finite difference methods, to mention just a few most commonly used.

The finite difference method, including its monotone variants, i.e., the three basic monotone iteration schemes of Picard, Jacobi and Gauss-Seidel (see Pao [42, 43]) is not only one of the simplest methods used in numerical analysis, but also an important theoretical method of proving existence theorems for partial differential equations. It is highly advantageous to select a finite difference method to prove the existence and uniqueness for any truncated system for the problem considered, because we thus arrive at numerically proved constructive theorems on existence. Each of these solutions is an approximation of a solution of this problem and may be numerically calculated and plotted, or tabulated.

In [2, p. 344] Amman notices that in the case of solving countable systems of equations as the discrete coagulation-fragmentation models with diffusion, the technique used in practically all papers is the natural one: it starts with a study of finite systems obtained by truncation to the first N equations followed by passing to the limit as $N \to \infty$.

We remark that to solve infinite countable systems of ordinary differential equations, parabolic partial differential equations and integro-differential equations of parabolic type, numerous authors have applied also the truncation method; see for example: Ball and Carr [4], Deimling [21], Lachowicz and Wrzosek [28], Moszyński and Pokrzywa [39] Persidskiĭ [44], Rzepecki [48], Tychonov [57], Wrzosek [63, 64, 65], Valeev [62] and Zautykov [62].

It should be emphasized here that with his research in numerous paper, Dariusz Wrzosek has made a important and lasting contributions to the development of the theory of infinite countable systems of semilinear parabolic differential equations, and in particular to the theory of infinite systems of discrete coagulation-fragmentation equations with diffusion.

The truncation method is also the fundamental approximation method of studying solvability of infinite uncountable systems of ordinary differential equations, as well as integro-differential and differential-functional equations of parabolic type (Lamb [31], McLaughlin *et al.* [34, 35, 36, 37] and Laurençot [32]). A finite truncated system may be obtained from an uncountably infinite system with use of a projection. The truncation method for infinite uncountable systems of parabolic-reaction-diffusion equations has been investigated in a separates paper. Therefore, the truncation method described above is extremely useful and widely used in practice.

2. Preliminaries

2.1. **Notation and definitions.** [4]

Let D be a domain in the time-space $(t,x) = (t, x_1, x_2, \ldots, x_m)$ and S be an arbitrary set of indices (finite or infinite).

Let $\mathcal{B}(S)$ be the real Banach space of mappings

$$w \colon S \to \mathbb{R}, \quad i \mapsto w(i) := w^i,$$

with the finite norm

$$\|w\|_{\mathcal{B}(S)} := \sup\left\{\left|w^i\right| : i \in S\right\}.$$

We use the symbol $|\cdot|$ to denote the absolute value of a real number, and we use index notation $w = \{w^i\}_{i\in S}$, where S is a non-empty index set.

The space ℓ^∞ is the Banach sequence space of all real-valued bounded sequences $w = \{w^j\}_{j\in\mathbb{N}} = (w^1, w^2, \ldots)$, with the finite norm

$$\|w\|_{\ell^\infty} := \sup\left\{\left|w^j\right| : j \in \mathbb{N}\right\}.$$

The partial order "$\leq$" in the space ℓ^∞ is defined by the positive cone

$$\ell^\infty_+ := \left\{w \colon w = \{w^j\}_{j\in\mathbb{N}} \in \ell^\infty,\ w^j \geq 0 \quad \text{for} \quad j \in \mathbb{N}\right\}$$

[4]The notation w or $w(\cdot,\cdot)$ (where $w \equiv w(\cdot,\cdot)$) denotes that w is regarded as an element of the set of admissible functions, while $w(t,x)$ stands for the value of this function at the point (t,x). However, sometimes, to stress the dependence of a function w on the variables t and x, we will write $w = w(t,x)$ and hope that this will not confuse the reader.

in the following way

$$u \le v \Longleftrightarrow v - u \in \ell^\infty_+.$$

If S is a finite set of indices with r-elements, i.e., $S = \{1, 2, \dots, r\}$ then $\mathcal{B}(S) = \mathbb{R}^r$. For an infinite countable set S, there is $\mathcal{B}(S) = \mathcal{B}(\mathbb{N}) = \ell^\infty$. For an infinite uncountable set S, there is $\mathcal{B}(S) = \mathcal{B}(\mathbb{R}^\infty)$.

We introduce three spaces of sequences of real-valued functions (see Kantorovič *et al.* [26, pp. 147–149]) equipped with the norms induced by the norm of the space ℓ^∞.

Denote by $\mathscr{C}_{\mathbb{N}}(\overline{D}) := \mathscr{C}^0_{\mathbb{N}}(\overline{D})$ the space of infinite sequences $z = (z^1, z^2, \dots)$ of real-valued functions $z^j = z^j(t, x)$, $j \in \mathbb{N}$, defined and continuous in a domain $\overline{D}$, with the finite supremum norm

$$\|z\|_{\mathscr{C}_{\mathbb{N}}(\overline{D})} := \sup\{\left|z^j\right|_0 : j \in \mathbb{N}\},$$

where $z^j \in C(\overline{D}) := C^0(\overline{D})$, $j \in \mathbb{N}$, and

$$\left|z^j\right|_0 := \sup\{\left|z^j(t, x)\right| : (t, x) \in \overline{D}\}$$

is the norm in the space $C(\overline{D})$.

The partial order "$\le$" in the space $\mathscr{C}_{\mathbb{N}}(\overline{D})$ is defined by means of the positive cone

$$\mathscr{C}^+_{\mathbb{N}}(\overline{D}) := \left\{w \colon w = \{w^j\}_{j \in \mathbb{N}} \in \mathscr{C}_{\mathbb{N}}(\overline{D}),\ w^j(t, x) \ge 0 \quad \text{for} \quad (t, x) \in \overline{D} \text{ and } j \in \mathbb{N}\right\}$$

in the following way

$$u \le v \Longleftrightarrow v - u \in \mathscr{C}^+_{\mathbb{N}}(\overline{D}).$$

From this it follows that the inequality $u \le v$ is to be understood componentwise (natural ordering), i.e., $u^j \le v^j$ for all $j \in \mathbb{N}$.

Inequality $u \le v$ is to be understood both componentwise and pointwise, i.e., $u^j(t, x) \le v^j(t, x)$ for arbitrary $(t, x) \in \overline{D}$ and all $j \in \mathbb{N}$.

We also introduce the space $\mathscr{C}_{N,0}(\overline{D})$ consisting of those infinite sequences in $\mathscr{C}_{\mathbb{N}}(\overline{D})$ which have a finite number of non-zero terms (i.e., almost all terms of each such sequences are equal to 0), namely $z_{N,0} = (z^1_N, z^2_N, \dots, z^N_N, 0, 0, \dots)$ with the finite norm

$$\|z_{N,0}\|_{\mathscr{C}_{N,0}(\overline{D})} = \max\{\left|z^j_N\right|_0 : j = 1, 2, \dots, N\}.$$

The space $C_N(\overline{D})$ is the space of finite sequences $z_N = (z^1_N, z^2_N, \dots, z^N_N)$ of real-valued functions $z^j_N = z^j_N(t, x)$ for $j = 1, 2, \dots, N$, defined and continuous in a domain $\overline{D}$, with the finite norm

$$\|z_N\|_{C_N(\overline{D})} := \max\{\left|z^j\right|_0 : j = 1, 2, \dots, N\}.$$

Convention. We adhere to the convention that every finite sequence

$$z_N = (z^1_N, \dots, z^j_N, \dots, z^N_N) \in C_N(\overline{D})$$

is treated as infinite one

$$z_{N,0} = (z^1_N, \dots, z^j_N, \dots, z^N_N, 0, 0, \dots) \in \mathscr{C}_{N,0}(\overline{D})$$

and we note

$$z_N \cong z_{N,0} \quad \text{for all} \quad N \in \mathbb{N}.$$

In this sense, the space $C_N(\overline{D})$ is identified with the space $\mathscr{C}_{N,0}(\overline{D})$ and we note

$$C_N(\overline{D}) \cong \mathscr{C}_{N,0}(\overline{D}).$$

Finally in this sense, the space $C_N(\overline{D})$ may be treated as the subspace of the space $\mathscr{C}_{\mathbb{N}}(\overline{D})$.

We remark that the partial ordering in the space $\mathscr{C}_{\mathbb{N}}(\overline{D})$ induces a corresponding partial ordering in the subspaces $\mathscr{C}_{N,0}(\overline{D})$ and $C_N(\overline{D})$.

2.2. **Formulation of problems.** Let us consider weakly coupled[5] *infinite countable systems* of semilinear parabolic-reaction-diffusion equations with functionals of the form

$$\mathcal{F}^j[z^j](t,x) = f^j(t,x,z(t,x),z) \quad \text{for} \quad j \in \mathbb{N}, \tag{1}$$

where

$$\mathcal{F}^j := \mathcal{D}_t - \mathcal{L}^j, \qquad \mathcal{L}^j := \sum_{i,k=1}^{m} a^j_{ik}(t,x)\mathcal{D}^2_{x_i x_k} - \sum_{i=1}^{m} b^j_i(t,x)\mathcal{D}_{x_i}$$

$x = (x_1, \ldots, x_m)$, $(t,x) \in (0,T] \times G := D$, $0 < T < \infty$, where T can be arbitrarily large, $G \subset \mathbb{R}^m$ and G is an open, bounded domain, whose boundary ∂G is an $(m-1)$-dimensional surface of a class $C^{2+\alpha}$ $(0 < \alpha < 1)$, $S_0 := \{(t,x) : t = 0, x \in \overline{G}\}$, $\sigma := [0,T] \times \partial G$ is a lateral surface of the cylindrical domain D, $\Gamma := S_0 \cup \sigma$ is the parabolic boundary of domain D and $\overline{D} := D \cup \Gamma$, $\mathbb{N}$ is the set of natural numbers and N is an arbitrary fixed natural number. Diagonal operators[6] $\mathcal{F}^j$, $j \in \mathbb{N}$ are uniformly parabolic in $\overline{D}$, z stands for the mapping

$$z\colon \mathbb{N} \times \overline{D} \to \ell^\infty, \quad (j,t,x) \mapsto z(j,t,x) := z^j(t,x),$$

composed of unknown functions $z = \{z^j\}_{j\in\mathbb{N}}$, and $f = \{f^j\}_{j\in\mathbb{N}}$ are given nonlinear functions

$$f\colon \mathbb{N} \times \overline{D} \times \ell^\infty \times \mathscr{C}_{\mathbb{N}}(\overline{D}) \to \ell^\infty, \quad (j,t,x,y,s) \mapsto f(j,t,x,y,s) := f^j(t,x,y,s).$$

The reaction functions (reaction terms) describing kinetic behavior of the problem are functionals with respect to the last variable and we assume that they are Volterra functionals (i.e. satisfy the Volterra condition). This means that the values of these functions depend on the past history of the modelled processes.

If we introduce the function $\tilde{f} = \{\tilde{f}^j\}_{j\in\mathbb{N}}$ setting

$$\tilde{f}^j(t,x,z) := f^j(t,x,z(t,x),z), \quad j \in \mathbb{N},$$

where

$$\tilde{f}\colon \mathbb{N} \times \overline{D} \times \mathscr{C}_{\mathbb{N}}(\overline{D}) \to \ell^\infty, \quad (j,t,x,s) \mapsto \tilde{f}(j,t,x,s) := \tilde{f}^j(t,x,s),$$

then we will write the equations of system (1) in another form

$$\mathcal{F}^j[z^j](t,x) = \tilde{f}^j(t,x,z), \quad j \in \mathbb{N} \tag{2}$$

which may be useful in our further considerations.

[5]This means that every equation contains derivatives of one unknown function only.

[6]Operators $\mathcal{F}^j$, $j \in \mathbb{N}$, are diagonal if $\mathcal{F}^j$ depends on z^j only, for all $j \in \mathbb{N}$.

Both forms (1) and (2) of the system considered have some disadvantages (Kamont [25]). Namely, if the reaction functions containing operators are of the form

$$f(t,x,z(t,x),z) \quad \text{or} \quad \tilde{f}(t,x,z)$$

then it is difficult to formulate conditions concerning the existence of solutions. For instance, we have not been able to do so in the case of equations with deviated argument.

The situation is different if the right-hand sides are of the form

$$\hat{f}^j(t,x,z(t,x),V[z](t,x)), \quad j \in \mathbb{N},$$

i.e., are superpositions of functions defined on certain subsets of the space $\mathbb{R}^{m+1} \times \ell^\infty \times \ell^\infty$ and some operators $V = \{V^j\}_{j\in\mathbb{N}}$.

Thus

$$\hat{f}^j : \overline{D} \times \ell^\infty \times \ell^\infty \to \mathbb{R}, \quad (t,x,y,s) \mapsto \hat{f}^j(t,x,y,s), \quad j \in \mathbb{N},$$

and

$$V^j : C_{\mathbb{N}}(\overline{D},\mathbb{R}) \to \mathbb{R}, \quad z \mapsto V^j[z], \quad j \in \mathbb{N},$$

that is

$$V : C_{\mathbb{N}}(\overline{D},\mathbb{R}) \to \ell^\infty, \quad z \mapsto V[z].$$

Therefore, differential-functional system (1) will take the following form

$$\mathcal{F}^j[z^j](t,x) = \hat{f}^j(t,x,z(t,x),V[z](t,x)), \quad j \in \mathbb{N}, \tag{3}$$

where $V[z] = \{V^j[z]\}_{j\in\mathbb{N}}$.

Now we may adopt separate assumptions on the functions $\hat{f}^j$ and operator V. We may assume different conditions concerning the type of dependence of $\hat{f}^j$ on the variables t and x.

The objective of using different expressions is to facilitate the understanding of the problem, as well as to facilitate, if not make possible, a precise formulation and wording of appropriate assumptions.

Finally, it should be noted that, formally, all three expressions (1), (2) and (3) are equivalent and may be used alternately, as the context requires.

For system (1) (or (2)) we will consider the *so-called* Fourier first boundary value problem:

Find the regular (classical) solution of infinite countable system of equation (1) *(*(2) *or* (3)*) in* $\overline{D}$ *fulfilling the initial-boundary condition*

$$z(t,x) = \phi(t,x) \quad \textit{for} \quad (t,x) \in \Gamma. \tag{4}$$

Remark 2.1. *Initial-boundary condition* (4) *is equivalent to the initial condition*

$$z(0,x) = \phi_0(x) \quad \textit{for} \quad x \in G \tag{5}$$

and the Dirichlet boundary condition

$$z(t,x) = h(t,x) \quad \textit{for} \quad (t,x) \in \sigma \tag{6}$$

with the compatibility condition[7]

$$h(0,x) = \phi_0(x) \quad \textit{for} \quad x \in \partial G. \tag{7}$$

[7]For the definition of a compatibility condition see, e.g., Ladyzhenskaya *et al.* [30, p. 319].

The truncation method is one of the basic approximation methods of solving initial and initial-boundary problems for countable infinite systems of equations and it is very useful and commonly used in practical computations of applied solutions. In this method, a solution z of infinite countable system (1) is defined as the limit, when $N \to \infty$, of the approximation sequence $\{z_N\}_{N=1,2,\dots}$, where $z_N = (z_N^1, z_N^2, \cdots, z_N^N)$, are defined as solutions of finite systems of the first N equations of the system (1) in N unknown functions for an arbitrary N, $N \in \mathbb{N}$, (which are called truncated systems) of the form

$$\begin{aligned} &\mathcal{F}^j[z^j](t,x) = \\ &= \tilde{f}^j_{N,\psi}\big(t,x,z_{N,\psi}(t,x),z_{N,\psi}\big) := \\ &= \tilde{f}^j_{N,\psi}\big(t,x,z_N^1(t,x),\dots,z_N^j(t,x),\dots,z_N^N(t,x),\psi^{N+1}(t,x),\psi^{N+2}(t,x),\dots, \\ &\qquad z_N^1,\dots,z_N^j,\dots,z_N^N,\psi^{N+1},\psi^{N+2},\dots\big) \text{ for } (t,x) \in D \text{ and } j = 1,2,\dots,N \end{aligned} \tag{8}$$

with the corresponding initial condition

$$z_N^j(0,x) = \phi_0^j(x) \quad \text{for} \quad x \in G \quad \text{and} \quad j = 1,2,\dots,N, \tag{9}$$

and the boundary condition

$$z_N^j(t,x) = h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j = 1,2,\dots,N, \tag{10}$$

with the compatibility condition (7) under appropriate assumptions which would guarantee the existence and uniqueness of a solution of problem (8), (9), (10).

The remaining terms $z_N^{N+1}, z_N^{N+2}, \dots$ of the approximation sequence $\{z_N\}$ are defined as follows:

$$z_N^j(t,x) = \psi^j(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = N+1, N+2, \dots, \tag{11}$$

where the function $\psi = \{\psi^j\}_{j\in\mathbb{N}}$, $\psi^j = \psi^j(t,x)$ for $(t,x) \in \overline{D}$, satisfies the initial condition

$$\psi(0,x) = \phi_0(x) \quad \text{for} \quad x \in G, \tag{12}$$

and the boundary condition

$$\psi(t,x) = h(t,x) \quad \text{for} \quad (t,x) \in \sigma \tag{13}$$

with the compatibility condition (7) and will be defined some way later on.

2.3. **Examples of Volterra functionals.** The following examples of Volterra functionals have been considered in papers by Bellout [8], Nickel [40, 41], Redlinger [46, 47] and Rzepecki [48]:

$$\tilde{f}_1(t,x,z) = \int_0^t m(t-\tau)K(z(\tau,x))\,d\tau, \tag{14}$$

$$\tilde{f}_2(t,x,z) = \int_0^t K(t,\tau,x,z(\tau,x))\,d\tau, \tag{15}$$

$$\tilde{f}_3(t,x,z) = \int_0^t\int_G K(t,\tau,x,\xi,z(\tau,\xi))\,d\tau\,d\xi, \tag{16}$$

$$\tilde{f}_4(t,x,z) = z(\theta t,x) \quad \text{with} \quad 0 \le \theta \le 1, \tag{17}$$

$$\tilde{f}_5(t,x,z) = z(t-\tau,x) \quad \text{with} \quad \tau > 0. \tag{18}$$

Equations including such functionals need to be considered in appropriately chosen domains. For instance, the example including $z(t-\tau,x)$ leads to an equation with retarded argument and such equations require a modified domain of the initial condition.

Other examples include (see Wrzosek [63, 64, 65]):

$$\begin{aligned}
\tilde{f}_6^1(t,x,z) &= -z^1(t,x)\sum_{k=1}^{\infty} a_k^1 z^k(t,x) + \sum_{k=1}^{\infty} b_k^1 z^{1+k}(t,x),\\
\tilde{f}_6^j(t,x,z) &= \frac{1}{2}\sum_{k=1}^{j-1} a_k^{j-k} z^{j-k}(t,x)z^k(t,x) - z^i(t,x)\sum_{k=1}^{\infty} a_k^j z^k(t,x) + \\
&\quad + \sum_{k=1}^{\infty} b_k^i z^{j+k}(t,x) - \frac{1}{2}z^j(t,x)\sum_{k=1}^{j-1} b_k^{j-k} \quad \text{for} \quad j = 2,3\ldots,
\end{aligned} \tag{19}$$

and (see Lachowicz and Wrzosek [28]):

$$\begin{aligned}
\tilde{f}_7^1(t,x,z) &= -z^1(t,x)\sum_{k=1}^{\infty}\int_G a_k^1(x,\xi)z^k(t,\xi)d\xi + \sum_{k=1}^{\infty}\int_G B_k^1(x,\xi)z^{1+k}(t,\xi)d\xi,\\
\tilde{f}_7^j(t,x,z) &= \frac{1}{2}\sum_{k=1}^{j-1}\int_{G\times G} A_k^{j-k}(x,\xi,\eta)z^{j-k}(t,\xi)z^k(t,\eta)d\xi d\eta - \\
&\quad - z^j(t,x)\sum_{k=1}^{\infty}\int_G a_k^j(x,\xi)z^k(t,\xi)d\xi + \sum_{k=1}^{\infty}\int_G B_k^j(x,\xi)z^{j+k}(t,\xi)d\xi - \\
&\quad - \frac{1}{2}z^j(t,x)\sum_{k=1}^{j-1} b_k^{j-k}(x) \qquad \text{for} \quad j = 2,3,\ldots,
\end{aligned} \tag{20}$$

where $\int_G A_k^j(x,\xi,\eta)dx = a_k^j(\xi,\eta)$ and $\int_G B_k^j(x,\xi)dx = b_k^j(\xi)$ are the nonnegative coefficients of coagulation a_k^j and fragmentation b_k^j rates, while f_6^j, f_7^j are Volterra functionals.

The theory of monotone iterative methods covers some of these examples only, namely the examples $\tilde{f}_1$, $\tilde{f}_2$ and $\tilde{f}_3$. The theory presented does not cover the examples f_4 and $\tilde{f}_5$; the same applies to the examples $\tilde{f}_6^j$, $\tilde{f}_7^j$. Systems with right-hand sides of this type are, however, considered in the literature, but other methods are required to solve them. In the case of examples $\tilde{f}_4^j$, $\tilde{f}_6^j$, $\tilde{f}_7^j$, the truncation method applied by the authors may be used. Thus they obtain a solution which is global in time, uniquely defined and mass-preserving.

2.4. **Methods for the construction of truncated systems.** Now we will give a few examples of the construction of truncated systems of N equations in N unknown functions corresponding to the infinite countable system (1) or (2), as applying by various authors.

1. Let us consider the infinite countable system of parabolic-reaction-diffusion equations of the form (2) with initial and boundary conditions (5)–(7), where nonlinear reaction functions are given as in paper [28] by Lachowicz and Wrzosek. Therefore, we consider the system

$$\mathcal{F}^j[z^j](t,x) = \tilde{f}^j(t,x,z) := \tilde{f}^j(t,x,z^1,z^2,\ldots) \tag{21}$$

for $(t,x) \in D$ and $j \in \mathbb{N}$, where $z = (z^1,z^2,\ldots) \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ and the functions $\tilde{f}^j = \tilde{f}^j(t,x,z)$, $j \in \mathbb{N}$, have the special form

$$\begin{aligned} \tilde{f}^1(t,x,z) &= -z^1(t,x)\sum_{k=1}^{\infty} a_k^1 z^k(t,x) + \sum_{k=1}^{\infty} b_k^1 z^{1+k}(t,x), \\ \tilde{f}^j(t,x,z) &= \frac{1}{2}\sum_{k=1}^{j-1} a_k^{j-k} z^{j-k}(t,x) z^k(t,x) - z^j(t,x)\sum_{k=1}^{\infty} a_k^j z^k(t,x) + \\ &\quad + \sum_{k=1}^{\infty} b_k^j z^{j+k}(t,x) - \frac{1}{2} z^j(t,x)\sum_{k=1}^{j-1} b_k^{j-k} \quad \text{for} \quad j = 2,3\ldots. \end{aligned} \tag{22}$$

where the coagulation rates a_k^j and fragmentation rates b_k^j are nonnegative constants such that $a_k^j = a_j^k$ and $b_k^j = b_j^k$.

We truncate system (21), (22), assuming

$$a_k^j \equiv 0 \quad \text{and} \quad b_k^j \equiv 0 \quad \text{for} \quad j > N \quad \text{or} \quad k > N. \tag{23}$$

Then we obtain the following associated truncated system of N equations in N unknown functions of the form

$$\mathcal{F}^j[z_N^j](t,x) = \tilde{f}_N^j(t,x,z_N) := \tilde{f}_N^j(t,x,z_N^1,\ldots,z_N^j,\ldots,z_N^N) \tag{24}$$

for $(t,x)\in D$ and $j=1,2,\ldots,N$, where $z_N=(z_N^1,\ldots,z_N^j,\ldots,z_N^N)\in C_N(\overline{D})$ and

$$\begin{aligned}\tilde f_N^j(t,x,z_N)&=\frac{1}{2}\sum_{k=1}^{j-1}a_k^{j-k}z_N^{j-k}(t,x)z_N^k(t,x)-z_N^j\sum_{k=1}^{N-j}a_k^j z_N^k(t,x)+\\&\quad+\sum_{k=1}^{N-j}b_k^j z_N^{j+k}(t,x)-\frac{1}{2}z_N^j(t,x)\sum_{k=1}^{j-1}b_k^{j-k}\quad\text{for}\quad j=2,3\ldots,\end{aligned}\tag{25}$$

with the corresponding initial and boundary conditions.

Ball and Carr [4] used such truncating for systems of ordinary differential equations, while Wrzosek [63] applied it to coagulation-fragmentation systems with diffusion.
2. Wrzosek in [63] considered the truncated systems of $2N$ equations in $2N$ unknown functions defined for any integer $N\geq 2$, by setting in (21)–(22):

$$a_k^j\equiv 0\quad\text{for}\quad j>N\quad\text{or}\quad k>N\quad\text{and}\quad b_k^j\equiv 0\quad\text{for}\quad j+k>N.\tag{26}$$

The truncated systems corresponding to the first $2N$ equations of system (21), (22) have the following form

$$\mathcal{F}^j[z_N^j](t,x)=\tilde f_N^j\big(t,x,z_N^1,\ldots,z_N^j,\ldots,z_N^N\big)\tag{27}$$

where

$$\tilde f_N^1(t,x,z_N)=-z_N^1(t,x)\sum_{k=1}^{N}a_k^1 z_N^k(t,x)+\sum_{k=1}^{N-1}b_k^1 z_N^{i+k}(t,x),$$

$$\begin{aligned}\tilde f_N^j(t,x,z_N)&=\frac{1}{2}\sum_{k=1}^{j-1}a_k^{j-k}z_N^{j-k}(t,x)z_N^k(t,x)-z_N^j(t,x)\sum_{k=1}^{N}a_k^j z_N^k(t,x)+\\&\quad+\sum_{k=1}^{N}b_k^j z_N^{j+k}(t,x)-\frac{1}{2}\sum_{k=1}^{j-1}b_k^{j-k}z_N^j(t,x)\quad\text{for}\quad j=2,3\ldots,N,\end{aligned}\tag{28}$$

$$\tilde f_N^j(t,x,z_N)=\frac{1}{2}\sum_{k=1}^{N}a_k^{j-k}z_N^{j-k}(t,x)z_N^k(t,x)\quad\text{for}\quad N+1\leq j\leq 2N.$$

3. The discrete model of nonlocal coagulation-fragmentation process is expressed in terms of an infinite countable system of integro-differential semilinear equations of the form (see Lachowicz and Wrzosek [28]):

$$\mathcal{F}^j[z^j](t,x)=\mathbf{F}^j[z^j](t,x)\quad\text{for}\quad(t,x)\in D\quad\text{and}\quad j\in\mathbb{N}\tag{29}$$

where

$$\mathbf{F}^j[z](t,x):=f^j(t,x,z(t,x),z),\quad j\in\mathbb{N}$$

are the nonlinear nonlocal, coagulation-fragmentation operators. They are the nonlinear Nemytskiĭ operators[8] generated by the functions f^j, $j\in\mathbb{N}$.

[8]The nonlinear Nemytskiĭ operator is sometimes also called the superposition operator, composition operator, or substitution operator. This type of operators plays an important role in the theory of nonlinear equations. The nonlinear Nemytskiĭ operator$\mathbf{F}=\{\mathbf{F}^j\}_{j\in\mathbb{N}}$, is generated by the function $f^j=f^j(t,x,y,s)$ or $\tilde f^j=\tilde f^j(t,x,s)$, $j\in\mathbb{N}$, and defined for sufficiently regular functions

$$\beta:\overline{D}\ni(t,x)\mapsto\beta(t,x)\in\ell^\infty$$

There is

$$
\begin{aligned}
\mathbf{F}^1[z](t,x) = &-z^1(t,x)\sum_{k=1}^{\infty}\int_G a_k^1(x,y)z^k(t,y)dy+\\
&+\sum_{k=1}^{\infty}\int_G \mathfrak{B}_k^1(x,y)z^k(t,y)dy,\\
\mathbf{F}^j[z](t,x) = &\frac{1}{2}\sum_{k=1}^{j-1}\int_{G\times G}\mathcal{A}_k^{j-k}(x,y,\xi)z^{j-k}(t,y)z^k(t,\xi)dyd\xi-\\
&-z^j(t,x)\sum_{k=1}^{\infty}\int_G a_k^j(x,y)z^k(t,y)dy+\\
&+\sum_{k=1}^{\infty}\int_G \mathfrak{B}_k^j(x,y)z^{j-k}(t,y)dy-\\
&-\frac{1}{2}z^j(t,x)\sum_{k=1}^{j-1}b_k^{j-k}(x) \quad \text{for} \quad j=2,3,\ldots
\end{aligned}
\tag{30}
$$

Truncated systems may be obtained from infinite countable system (29), (30) by setting

$$a_k^j \equiv 0 \quad \text{and} \quad b_k^j \equiv 0 \quad \text{for} \quad j+k>N. \tag{31}$$

There is $\mathcal{F}^j[z_N^j](t,x) = \mathbf{F}_N^j[z_N](t,x)$ for $(t,x)\in D$ and $j=1,2,\ldots,N$ where

$$
\begin{aligned}
\mathbf{F}_N^1[z_N](t,x) = &-z_N^1(t,x)\sum_{k=1}^{N-1}\int_G a_k^1(x,y)z_N^k(t,y)dy+\\
&+\sum_{k=1}^{N-1}\int_G \mathfrak{B}_k^1(x,y)z_N^{k+1}(t,y)dy,\\
\mathbf{F}_N^j[z_N](t,x) = &\frac{1}{2}\sum_{k=1}^{j-1}\int_{G\times G}\mathcal{A}_k^{j-k}(x,y,\xi)z_N^{j-k}(t,y)z_N^k(t,\xi)dyd\xi-\\
&-z_N^j(t,x)\sum_{k=1}^{N-1}\int_G a_k^j(x,y)z_N^k(t,y)dy+\\
&+\sum_{k=1}^{\infty}\int_G \mathfrak{B}_k^j(x,y)z_N^{j+k}(t,y)dy-\\
&-\frac{1}{2}z_N^j(t,x)\sum_{k=1}^{j-1}b_k^{j-k}(x) \quad \text{for} \quad j=2,3,\ldots,N.
\end{aligned}
\tag{32}
$$

---

as follows: $\mathbf{F}:\beta\mapsto\mathbf{F}[\beta]$, where $\mathbf{F}^j[\beta](t,x) := f^j(t,x,\beta(t,x),\beta)$, $j\in\mathbb{N}$. Extensive information about the nonlinear Nemytskiĭ operator can be found in the book by Krasnosel'skiĭ [27] and the monograph by Appell and Zabrejko [3].

4. Another approach is possible, whereby the truncating is performed by accepting that

$$z_{N,0}^{N+1}(t,x) = z_{N,0}^{N+2}(t,x) = \ldots = 0 \quad \text{for} \quad (t,x) \in \overline{D} \tag{33}$$

and considering the system comprising the first N equations of system (2) in N unknown functions

$$\begin{aligned} &\mathcal{F}^j[z_{N,0}^j](t,x) = \tilde{f}^j\Big(t,x,z_{N,0}^1,\ldots,z_{N,0}^j,\ldots,z_{N,0}^N,0,0,\ldots\Big) := \\ &:= \tilde{f}_{N,0}^j(t,x,z_{N,0}^1,\ldots,z_{N,0}^j,\ldots,z_{N,0}^N) \quad \text{for } (t,x)\in D \quad \text{and} \quad j=1,2,\ldots,N, \end{aligned} \tag{34}$$

with the homogeneous initial condition

$$z_{N,0}^j(0,x) = 0 \quad \text{for} \quad x \in G \quad \text{and} \quad j=1,2,\ldots,N \tag{35}$$

and boundary condition

$$z_{N,0}^j(t,x) = 0 \quad \text{for} \quad (t,x)\in\sigma \quad \text{and} \quad j=1,2,\ldots,N. \tag{36}$$

We observe that in all four examples, while constructing truncated systems, the authors made a material use of the specific form of nonlinear term (22), that is functions of the type $\tilde{f}^j(t,x,z)$, which are finite sums of expressions linear with respect to the unknown functions $z^1, z^2, \ldots$ as well as sums of series of those unknown functions. This specific form supports the construction of truncated systems by assuming that certain coefficients a_k^j and b_k^j in the first N equations of the system are identically equal to zero and omitting the equations with numbers larger than N. It means, though, that a truncated system changes with the number N of the approximation z_N, that is it changes with the number of the unknowns.

5. Finally, we present a well known idea for the construction of truncated systems by using the projection operator.

Let us consider the infinite-dimensional Banach space $\mathscr{C}_{\mathbb{N}}(\overline{D})$ and the finite N-dimensional subspace $\mathscr{C}_{N,0}(\overline{D})$.

We define the operator ρ_N as follows

$$\begin{aligned} \rho_N: \quad \mathscr{C}_{\mathbb{N}}(\overline{D}) \to \mathscr{C}_{N,0}(\overline{D}), \\ (z^1,z^2,\ldots) = z \mapsto \rho_N[z] := z_{N,0} = (z^1,\ldots,z^j,\ldots,z^N,0,0,\ldots) \end{aligned} \tag{37}$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ and $N \in \mathbb{N}$.

This means that

$$\rho_N[z] := \begin{cases} z^j & \text{if} \quad 0 \le j \le N, \\ 0 & \text{if} \quad j > N \end{cases} \tag{38}$$

for all $z \in \mathscr{C}_{\mathbb{N}}$, $N \in \mathbb{N}$.

It is easy to see that

$$\rho_N^2[z] = \rho_N\big[\rho_N[z]\big] = \rho_N[z_{N,0}] = z_{N,0} = \rho_N[z]$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ and for an arbitrary N, $N \in \mathbb{N}$. Therefore, ρ_N is the projection operator from an infinite-dimensional space onto its finite-dimensional subspaces.

Using Convention, we obtain

$$\rho_N: \quad \mathscr{C}_{\mathbb{N}}(\overline{D}) \to C_N(\overline{D}), \qquad z \mapsto \rho_N[z] := z_N \tag{39}$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ and all $N \in \mathbb{N}$.

Therefore, the truncating of the infinite system of differential equations considered in $\mathscr{C}_{\mathbb{N}}(\overline{D})$ may be treated as a projection of this system onto the finite-dimensional subspace $C_N(\overline{D})$.

2.5. **Conditions and assumptions.** We will assume that the *operators* $\mathcal{L}^j$, $j \in \mathbb{N}$, are *uniformly elliptic in* $\overline{D}$ i.e., that there exists a constant $\mu_0 > 0$ independent of t, x, ξ and j, such that the following inequalities hold:

$$\sum_{i,k=1}^{m} a_{ik}^j(t,x)\xi_i\xi_k \geq \mu_0|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^m, \quad (t,x) \in \overline{D} \quad \text{and} \quad j \in \mathbb{N}$$

where $|\xi|^2 = \sum_{j=1}^{m} \xi_j^2$.

If the operators $\mathcal{L}^j$, $j \in \mathbb{N}$, are uniformly elliptic in $\overline{D}$, then the operators $\mathcal{F}^j$, $j \in \mathbb{N}$, are *uniformly parabolic* in $\overline{D}$ and infinite *system* of equations (1) is called *uniformly parabolic* in $\overline{D}$.

We will assume that the nonlinear reaction functions $f^j = f^j(t,x,y,s)$ or functions $\tilde{f}^j = \tilde{f}^j(t,x,s)$, $j \in \mathbb{N}$, are continuous functions in their respective domains and we will introduce the following conditions:

Condition *W*. Functions $f^j = f^j(t,x,y,s)$, $j \in \mathbb{N}$, are *increasing* with respect to the functional argument s, i.e., for arbitrary $s, \tilde{s} \in \mathscr{C}_{\mathbb{N}}(\overline{D})$, there is

$$s \leq \tilde{s} \Longrightarrow f^j(t,x,y,s) \leq f^j(t,x,y,\tilde{s}) \quad \text{for} \quad (t,x) \in \overline{D}, \quad y \in \ell^\infty.$$

Condition *V*. Functions $f^j = f^j(t,x,y,s)$, $j \in \mathbb{N}$, are *Volterra functionals* with respect to the last argument s (or satisfy the *Volterra condition*), if for arbitrary $(t,x) \in \overline{D}$, $y \in \ell^\infty$ and for all $s, \tilde{s} \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ such that $s^j(\bar{t},x) = \tilde{s}^j(\bar{t},x)$ for $0 \leq \bar{t} \leq t$, $j \in \mathbb{N}$, there is $f^j(t,x,y,s) = f^j(t,x,y,\tilde{s})$.

Condition *L*. Functions $f^j = f^j(t,x,y,s)$, $j \in \mathbb{N}$, fulfil the *Lipschitz condition* with respect to y and s, if for arbitrary $y, \tilde{y} \in \ell^\infty$ and s, $\tilde{s} \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ the inequality

$$\left|f^j(t,x,y,s) - f^j(t,x,\tilde{y},\tilde{s})\right| \leq L_1 \left\|y - \tilde{y}\right\|_{\ell^\infty} + L_2 \left\|s - \tilde{s}\right\|_0 \quad \text{for} \quad (t,x) \in \overline{D}$$

holds, where L_1, L_2 are positive constants.

Condition *L.** Functions $f^j(t,x,y,s)$, $j \in \mathbb{N}$, fulfil the *Lipschitz-Volterra condition* with respect to the functional argument s if for arbitrary $s, \tilde{s} \in C_{\mathbb{N}}(\overline{D})$ the inequality

$$\left|f^i(t,x,y,s) - f^i(t,x,y,\tilde{s})\right| \leq L \left\|s - \tilde{s}\right\|_{0,t} \quad \text{for} \quad (t,x) \in \overline{D},\ y \in \ell^\infty$$

holds, where L is a positive constant.[9]

[9]It is easy to see that (V) and $(L) \Longleftrightarrow (L^*)$. The fact that in Condition (L^*) there is $\|s - \tilde{s}\|_{0,t}$ means that for functions $s, \tilde{s} \in C_{\mathbb{N}}(\overline{D})$, such that, $s(\bar{t},x) = \tilde{s}(\bar{t},x)$ for $0 \leq \bar{t} \leq t$, there is $f^j(t,x,y,s) = f^j(t,x,y,\tilde{s})$, $j \in \mathbb{N}$, i.e., the functions f^j are functionals in s taking the same values. Therefore, the functions f^j satisfy Volterra conditions (V), i.e., the functions f^i are functionals in s of the Volterra type. Moreover, if f^j, $j \in \mathbb{N}$, satisfy Condition (L^*), then Lipschitz Condition (L) holds, because $\|s - \tilde{s}\|_{0,t} \leq \|s - \tilde{s}\|_0$. The reverse implication is obvious.

Assumption H_a. We will assume that all the coefficients $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$ and $b^i_j = b^i_j(t,x)$ $(j,k = 1,\ldots,m,\ i \in S)$ of the operators $\mathcal{L}^i$, $i \in S$, are uniformly Hölder continuous with respect to t and x in $\overline{D}$ with exponent α $(0 < \alpha < 1)$ and their Hölder norms are uniformly bounded, i.e.,

$$\left|a^i_{jk}\right|_{0+\alpha} \leq k, \quad \left|b^i_j\right|_{0+\alpha} \leq k, \quad j,k = 1,\ldots,m, \quad i \in S,$$

where k is a positive constant.

Assumption H_f. Functions $f^i(t,x,y,s), i \in S$, are *uniformly Hölder continuous* with exponent α $(0 < \alpha < 1)$ with respect to t and x in $\overline{D}$, and their Hölder norms $|f^i|_{0+\alpha}$ are uniformly bounded, i.e., $f(\cdot,\cdot,s) \in C^{0+\alpha}_S(\overline{D})$.

Assumption $H_{\phi_0,h}$. If initial-boundary conditions are of form (5), (6) then we assume that $\phi_0 \in C^{2+\alpha}_S(\overline{G})$, $h \in C^{2+\alpha}_S(\sigma)$, where $0 < \alpha < 1$, and the compatibility condition

$$h(0,x) = \phi_0(x) \quad \text{for} \quad x \in \partial G$$

holds.

Remark 2.2. *If the initial boundary-condition is of the form* (4), $\phi \in C^{2+\alpha}_N(\Gamma)$ *and a boundary* $\partial G \in C^{2+\alpha}$, *then without loss of generality we can consider the homogeneous initial-boundary condition*

$$z(t,x) = 0 \quad \textit{for} \quad (t,x) \in \Gamma \tag{40}$$

(*cp.* [17, p. 35]).

A *function* $w \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ will be called *regular* in $\overline{D}$ if w^j, $j \in \mathbb{N}$, have continuous derivatives $\mathcal{D}_t w^j$, $\mathcal{D}_x w^j$, $\mathcal{D}^2_{xx} w^j$ in D, i.e., $w \in \mathscr{C}^{\mathrm{reg}}_{\mathbb{N}}(\overline{D}) := \mathscr{C}_{\mathbb{N}}(\overline{D}) \cap \mathscr{C}^{1,2}_{\mathbb{N}}(D)$.

Functions $u, v \in \mathscr{C}^{\mathrm{reg}}_{\mathbb{N}}(\overline{D})$ satisfying the infinite countable systems of inequalities

$$\begin{cases} \mathcal{F}^j[u^j](t,x) \leq f^j(t,x,u(t,x),u) \quad \text{for} \quad (t,x) \in D, \\ u^j(0,x) \leq \phi^j_0(x) \quad \text{for} \quad x \in G, \\ u^j(t,x) \leq h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j \in \mathbb{N}, \end{cases} \tag{41}$$

$$\begin{cases} \mathcal{F}^j[v^j](t,x) \geq f^j(t,x,v(t,x),v) \quad \text{for} \quad (t,x) \in D, \\ v^j(0,x) \leq \phi^j_0(x) \quad \text{for} \quad x \in G, \\ v^j(t,x) \leq h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j \in \mathbb{N}, \end{cases} \tag{42}$$

are called, respectively, a *lower* and an *upper solution* of parabolic problem (1), (5)–(7) in $\overline{D}$.

We will adopt the following fundamental assumptions:

Assumption A. We assume that there exists at least one pair u_0 and v_0, respectively of a lower and an upper solution of problem (1), (5)–(7) in $\overline{D}$.

A pair u and v of a lower and an upper solution of problem (1), (5)–(7) in $\overline{D}$, is called an *ordered pair*, if $u \leq v$ in $\overline{D}$.

We notice that the inequality $u_0 \leq v_0$ does not follow directly from inequalities (41) and (42). Therefore, we will adopt the following assumption:

Assumption A_0. We assume that there exists at least one ordered pair u_0 and v_0 of a lower and an upper solution of problem (1), (5)–(7) in $\overline{D}$, respectively.

For a given ordered pair of a lower and an upper solution, respectively, of problem (1), (5)–(7) in $\overline{D}$ we define[10] the *sector* $\langle u_0, v_0\rangle$ in the space $\mathscr{C}_{\mathbb{N}}(\overline{D})$ as the *functional interval* formed by u_0 and v_0:

$$\langle u_0, v_0\rangle := \left\{ s \in \mathscr{C}_{\mathbb{N}}(\overline{D})\colon u_0(t,x) \le s(t,x) \le v_0(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \right\}. \tag{43}$$

This sector is a closed set in the space $\mathscr{C}_{\mathbb{N}}(\overline{D})$.

Analogously, we define the sector $\left\langle \underline{m}, \overline{M}\right\rangle$ in the space ℓ^∞ as follows

$$\left\langle \underline{m}, \overline{M}\right\rangle := \left\{ y \in \ell^\infty \colon \underline{m} \le y \le \overline{M} \right\},$$

where

$$\underline{m}^j = \inf_{\overline{D}} u_0^j(t,x), \ \underline{m} = \{\underline{m}^j\}_{j\in\mathbb{N}},$$
$$\overline{M}^j = \sup_{\overline{D}} v_0^j(t,x), \ \overline{M} = \{\overline{M}^j\}_{j\in\mathbb{N}}.$$

Finally we define the set

$$\mathcal{K} := \left\{ (t,x,y,s)\colon (t,x) \in \overline{D}, \quad y \in \left\langle \underline{m}, \overline{M}\right\rangle, \quad s \in \langle u_0, v_0\rangle \right\}. \tag{44}$$

Remark 2.3. *If Assumption A_0 holds and we define the sector $\langle u_0, v_0\rangle$ generated by the lower u_0 and upper v_0 solution, and the set $\mathcal{K}$ then the remaining assumptions on the functions f^j may be weakened to hold locally only in the set $\mathcal{K}$. Therefore, all our considerations will be true within the sector $\langle u_0, v_0\rangle$, only.*

Analogously, we define an ordered pair of a lower u_0 and an upper v_0 solution of problem (2), (5)–(7) in $\overline{D}$. Next, for a given ordered pair of a lower u_0 and an upper v_0 solution of problem (2), (5)–(7) in $\overline{D}$, we define the sector $\langle u_0, v_0\rangle$ and the set $\tilde{\mathcal{K}}$ as

$$\tilde{\mathcal{K}} := \left\{ (t,x,s)\colon (t,x) \in \overline{D}, \quad s \in \langle u_0, v_0\rangle \right\}. \tag{45}$$

3. A new method for the construction of truncated systems

3.1. **A new truncation method.** While using the method, theorems are exploited concerning differential inequalities of the parabolic type, as Mlak and Olech's ([38]) idea of proving an existence theorem for infinite countable systems of ordinary differential equations.

A. If $\alpha = \alpha(t,x)$ is a lower solution of problem (2), (5)–(7) in $\overline{D}$ such that:

$$\begin{cases} \mathcal{F}^j[\alpha^j](t,x) \le \tilde{f}^j(t,x,\alpha) \quad \text{for} \quad (t,x) \in D, \\ \alpha^j(0,x) = \phi_0^j(x) \quad \text{for} \quad x \in G, \\ \alpha^j(t,x) = h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \qquad j \in \mathbb{N}, \end{cases} \tag{46}$$

[10] The partial ordering in a set $\mathcal{X}$ induces also a corresponding partial ordering in a subset W of $\mathcal{X}$ and if $u, v \in W$ with $u \le v$, then

$$\langle u, v\rangle := \{s \in W, \quad u \le s \le v\}$$

denotes the *sector*, formed by the ordered pair u and v.

then we construct the finite system of N equations, where N is an arbitrary natural number ($N \in \mathbb{N}$) by truncating this system to the first N equations and substituting

$$z^j_{N,\alpha}(t,x) = \alpha^j(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = N+1,\, N+2, \ldots .$$

We obtain the finite truncated system of equations

$$\begin{aligned} \mathcal{F}^j[z^j_{N,\alpha}](t,x) = \tilde{f}^j\big(t,x,z^1_{N,\alpha},\ldots,z^j_{N,\alpha},\ldots,z^N_{N,\alpha},\alpha^{N+1},\alpha^{N+2},\ldots\big) := \\ := \tilde{f}^j_{N,\alpha}(t,x,z^1_{N,\alpha},\ldots,z^j_{N,\alpha},\ldots,z^N_{N,\alpha}) \text{ for } (t,x) \in D \text{ and } j = 1,2,\ldots,N, \end{aligned} \tag{47}$$

with the corresponding initial condition

$$z^j_{N,\alpha}(0,x) = \phi^j_0(x) \quad \text{for} \quad x \in G \quad \text{and} \quad j = 1,2,\ldots,N \tag{48}$$

and boundary condition

$$z^j_{N,\alpha}(t,x) = h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j = 1,2,\ldots,N \tag{49}$$

for an arbitrary N, $N \in \mathbb{N}$.

Moreover, if we define the remaining terms of the approximation sequence $\{z_{N,\alpha}\}_{N=1,2,\ldots}$ received

$$z^j_{N,\alpha}(t,x) := \alpha^j(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = N+1,\, N+2, \ldots . \tag{50}$$

then this approximation sequence is defined as follows:

$$z_{N,\alpha} = \Big(z^1_{N,\alpha},.\, z^j_{N,\alpha},\ldots,z^N_{N,\alpha},\alpha^{N+1},\alpha^{N+2},\ldots\Big) \quad \text{for} \quad N = 1,2,\ldots$$

B. Analogously, if $\beta = \beta(t,x)$ is an upper solution of problem (2), (5)–(7) in $\overline{D}$ such that:

$$\begin{cases} \mathcal{F}^j[\beta^j](t,x) \geq \tilde{f}^j(t,x,\beta) \quad \text{for} \quad (t,x) \in D, \\ \beta^j(0,x) = \phi_0(x) \quad \text{for} \quad x \in G, \\ \beta^j(t,x) = h(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j \in \mathbb{N}, \end{cases} \tag{51}$$

then we construct the finite system of N equations by truncating this system to the first N equations, for an arbitrary N, $N \in \mathbb{N}$, and substituting

$$z^j_{N,\beta}(t,x) = \beta^j(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = N+1,\, N+2, \ldots .$$

We obtain the finite truncated system of equations

$$\begin{aligned} \mathcal{F}^j[z^j_{N,\beta}](t,x) = \tilde{f}^j\big(t,x,z^1_{N,\beta},\ldots,z^j_{N,\beta},\ldots,z^N_{N,\beta},\beta^{N+1},\beta^{N+2},\ldots\big) := \\ := \tilde{f}^j_{N,\beta}(t,x,z^1_{N,\beta},\ldots,z^j_{N,\beta},\ldots,z^N_{N,\beta}) \text{ for } (t,x) \in D, \text{ and } j = 1,2,\ldots,N, \end{aligned} \tag{52}$$

with the corresponding initial

$$z^j_{N,\beta}(0,x) = \phi^j_0(x) \quad \text{for} \quad x \in G \quad \text{and} \quad j = 1,2,\ldots,N \tag{53}$$

and boundary condition

$$z^j_{N,\beta}(t,x) = h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j = 1,2,\ldots,N. \tag{54}$$

We define the remaining terms of the approximation sequence $\{z_{N,\beta}\}_{N=1,2\ldots}$ received as follows:

$$z^j_{N,\beta}(t,x) := \beta^j(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = N+1,\, N+2, \ldots \tag{55}$$

and the suitable approximation sequence $\{z_{N,\beta}\}$ is defined as follows:

$$z_{N,\beta} = \left(z^1_{N,\beta}, \dots z^j_{N,\beta}, \dots, z^N_{N,\beta}, \beta^{N+1}, \beta^{N+2}, \dots\right).$$

C. Finally, if $u_0 = u_0(t,x) \equiv 0$ in $\overline{D}$ is a lower solution of system (2) with the homogeneous initial and boundary conditions in $\overline{D}$ then

$$0 \le \tilde{f}^j(t,x,0) \quad \text{for} \quad (t,x) \in D \quad \text{and} \quad j \in \mathbb{N}, \tag{56}$$

and we construct the finite truncated system of N equations in the following form:

$$\begin{aligned} &\mathcal{F}^j[z^j_{N,0}](t,x) = \tilde{f}^j\left(t,x,z^1_{N,0},\dots,z^j_{N,0},\dots,z^N_{N,0},0,0,\dots\right) := \\ &:= \tilde{f}^j_{N,0}(t,x,z^1_{N,0},\dots,z^2_{N,0},\dots,z^N_{N,0}) \text{ for } (t,x)\in D \text{ and } j=1,2,\dots,N, \end{aligned} \tag{57}$$

with the homogeneous initial

$$z^j_{N,0}(0,x) = 0 \quad \text{for} \quad x \in G \quad \text{and} \quad j = 1,2,\dots,N \tag{58}$$

and boundary conditions

$$z^j_{N,0}(t,x) = 0 \quad \text{for} \quad (t,x) \in \Gamma \quad \text{and} \quad j = 1,2,\dots,N, \tag{59}$$

where N is an arbitrary natural number.

We define the remaining functions received

$$z^j_{N,0}(t,x) = 0 \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = N+1,\ N+2,\dots, \tag{60}$$

and the approximation sequence $\{z_{N,0}\}$ are defined as follows

$$z_{N,0} = \left(z^1_{N,0},\dots,z^j_{N,0},\dots,z^N_{N,0},0,0,\dots\right).$$

3.2. **Existence of monotone approximation sequences.** Now we will study the solvability of infinite countable problem (2), (6), (7) in the partially ordered Banach sequence space $\mathscr{C}^{2+\alpha}_{\mathbb{N}}(\overline{D})$ by using the truncation method. First, we will define the successive terms of approximation sequences as regular solutions of the finite truncated systems with initial and boundary conditions in $\overline{D}$. Under appropriate assumptions, we will prove the existence and uniqueness of a global-in-time regular solution of finite truncated system in the sector $\langle u_0, v_0\rangle$, which lies in the Banach space $\mathscr{C}^{2+\alpha}_{\mathbb{N}}(\overline{D})$.

We will give a theorem on the approximation of the solution of initial-boundary value problem (2), (5)–(7) by solutions of corresponding truncated approximation problems. In other words, we shall give some conditions for the reaction functions of equations of infinite system (2) enabling the consideration of (2) to be reduced to the consideration of finite truncated systems.

Let us consider the initial-boundary value problem for infinite countable system of parabolic-reaction-diffusion equations

$$\begin{cases} \mathcal{F}^j[z^j](t,x) = \tilde{f}^j(t,x,z) := \tilde{f}^j(t,x,z^1,z^2,\dots) \quad \text{for} \quad (t,x)\in D, \\ z^j(0,x) = \phi^j_0(x) \quad \text{for} \quad x \in G, \\ z^j(t,x) = h^j(t,x) \quad \text{for} \quad (t,x)\in\sigma \quad \text{and} \quad j\in\mathbb{N}, \end{cases} \tag{61}$$

and the corresponding problems for finite truncated systems

$$\begin{cases} \mathcal{F}^j[w_N^j](t,x) = \tilde{f}^j\Big(t,x,w_N^1,\dots,w_N^j,\dots,w_N^N,0,0,\dots\Big) := \\ := \tilde{f}^j_{N,0}\Big(t,x,w_N^1,\dots,w_N^j,\dots,w_N^N\Big) \quad \text{for } (t,x)\in D, \\ w_N^j(0,x) = \phi_0^j(x) \quad \text{for} \quad x\in G, \\ w_N^j(t,x) = h^j(t,x) \quad \text{for} \quad (t,x)\in\sigma \text{ and } j=1,2,\dots,N, \end{cases} \tag{62}$$

where N is an arbitrary, $N\in\mathbb{N}$.

If the remaining terms $w_N^{N+1}, w_N^{N+2},\dots$ of the approximation sequences $\{w_{N,0}\}$ are defined as follows:

$$w_N^j(t,x) = 0 \quad \text{for} \quad (t,x)\in\overline{D} \quad \text{and} \quad j=N+1,N+2,\dots, \tag{63}$$

then these sequences have the following forms

$$w_{N,0} = (w_N^1,\dots,w_N^j,\dots,w_N^N,0,0,\dots) \quad \text{for} \quad N=1,2,\dots.$$

Theorem 1. *Let us consider infinite countable system* (2) *with initial and boundary conditions* (5), (6) *and compatibility conditions* (7). *Let assumptions* A_0, (H_a), (H_f), $(H_{\phi_0,h})$ *hold, and conditions* (L), (V), (W) *hold in the set* $\tilde{\mathcal{K}}$.

If we define the successive terms of the approximation sequences $\{u_{N,u_0}\}$ *and* $\{v_{N,v_0}\}$ *as regular solutions in* $\overline{D}$ *of the following truncated systems of semilinear parabolic differential-functional equations*

$$\begin{aligned} \mathcal{F}^j[u^j_{N,u_0}](t,x) &= \tilde{f}^j\big(t,x,u^1_{N,u_0},\dots,u^j_{N,u_0},\dots,u^N_{N,u_0},u_0^{N+1},u_0^{N+2},\dots\big) := \\ &= \tilde{f}^j_{N,u_0}(t,x,z^1_{N,u_0},\dots,z^j_{N,u_0},\dots,z^N_{N,u_0}) \end{aligned} \tag{64}$$

for $(t,x)\in D$ *and* $j=1,2,\dots,N$, *with the initial and boundary conditions*

$$u^j_{N,u_0}(0,x) = \phi_0^j(x) \qquad \textit{for} \quad x\in G \quad \textit{and} \quad j=1,2,\dots,N, \tag{65}$$

$$u^j_{N,u_0}(t,x) = h^j(t,x) \qquad \textit{for} \quad (t,x)\in\sigma \quad \textit{and} \quad j=1,2,\dots,N, \tag{66}$$

and

$$u^j_{N,u_0}(t,x) = u_0^j(t,x) \qquad \textit{for} \quad (t,x)\in\overline{D} \quad \textit{and} \quad j=N+1,\ N+2,\dots, \tag{67}$$

as well as

$$\begin{aligned} \mathcal{F}^j[v^j_{N,v_0}](t,x) &= \tilde{f}^j\big(t,x,v^1_{N,v_0},\dots,v^j_{N,v_0},\dots,v^N_{N,v_0},v_0^{N+1},v_0^{N+2},\dots\big) := \\ &= \tilde{f}^j_{N,v_0}(t,x,u^1_{N,u_0},\dots,u^2_{N,u_0},\dots,u^N_{N,u_0}) \end{aligned} \tag{68}$$

for $(t,x)\in D$ *and* $j=1,2,\dots,N$, *with the initial and boundary conditions*

$$v^j_{N,v_0}(0,x) = \phi_0^j(x) \quad \textit{for} \quad x\in G \quad \textit{and} \quad j=1,2,\dots,N, \tag{69}$$

$$v^j_{N,v_0}(t,x) = h^j(t,x) \quad \textit{for} \quad (t,x)\in\sigma \quad \textit{and} \quad j=1,2,\dots,N, \tag{70}$$

and

$$v^j_{N,v_0}(t,x) = v_0^j(t,x) \quad \textit{for} \quad (t,x)\in\overline{D} \quad \textit{and} \quad j=N+1,\ N+2,\dots, \tag{71}$$

where u_0 and v_0 are an ordered pair of a lower and an upper solution, respectively, of parabolic problem (2), (5)–(7) *in $\overline{D}$ given by Assumption A_0, which is an initial pair in the iterating process, then*

1° *these sequences are well defined as a global-in-time regular unique solutions of suitable problems, and u_{N,u_0}, $v_{N,v_0} \in \mathscr{C}_{\mathbb{N}}^{0+\alpha}(\overline{D})$, for $N = 1, 2, \ldots$;*

2° *the sequence $\{u_{N,u_0}\}$ is monotone non-decreasing, the sequence $\{v_{N,v_0}\}$ is monotone non-increasing and the following inequalities hold:*

$$\begin{aligned} u_0(t,x) \le \ldots \le u_{N,u_0}(t,x) \le u_{N+1,u_0}(t,x) \le \ldots \\ \ldots \le v_{N+1,v_0}(t,x) \le v_{N,v_0}(t,x) \le \ldots \le v_0(t,x) \end{aligned} \tag{72}$$

for $(t,x) \in \overline{D}$ and $N = 1, 2, \ldots$.

Proof 1°. There is u_0, $v_0 \in \mathscr{C}_{\mathbb{N}}^{0+\alpha}(\overline{D})$. By assumptions (H_f) and (L), we conclude that the right-hand sides of equations of systems (64) and (68) are of class $\mathscr{C}_{\mathbb{N}}^{0+\alpha}(\overline{D})$ (see [17, Lemma 2.1, p. 42]). If assumptions (H_a), $(H_{\phi,h})$, (W), and (L) hold, then we may prove (see [9, Th. 2.1, p. 40] and [17]) that there exist the regular unique solutions u_1 and v_1 of finite systems of equations (64) and (68), with suitable initial and boundary conditions, u_1, $v_1 \in C_N^{2+\alpha}(\overline{D})$, and u_{1,u_0}, $v_{1,v_0} \in \mathscr{C}_{\mathbb{N}}^{0+\alpha}(\overline{D})$.

Analogously, by induction, we prove that the approximation sequences $\{u_{N,u_0}\}$, $\{v_{N,v_0}\}$ are well defined, $u_{N,u_0}, v_{N,v_0} \in C_N^{2+\alpha}(\overline{D})$ and u_{N,u_0}, $v_{N,v_0} \in \mathscr{C}_{\mathbb{N}}^{0+\alpha}(\overline{D})$. □

Proof 2°. Since u_0 is a lower solution of problem (2), (5)–(7) in $\overline{D}$, then it satisfies

$$\begin{cases} \mathcal{F}^j[u_0^j](t,x) \le \tilde{f}^j\big(t,x,u_0^1,\ldots,u_0^j,\ldots,u_0^N,u_0^{N+1},u_0^{N+2},\ldots\big) \quad \text{for} \quad (t,x) \in D \\ u_0^j(0,x) \le \phi_0^j(x) \quad \text{for} \quad x \in G, \\ u_0^j(t,x) \le h^j(t,x) \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j = 1,2,\ldots,N. \end{cases} \tag{73}$$

Applying the theorem on weak partial differential-functional inequalities of parabolic type for finite systems (see Szarski [53]) to problems (59)–(62) and (68)–(70), we get

$$u_0^j(t,x) \le u_{N,u_0}^j(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = 1,2,\ldots,N. \tag{74}$$

Hence, by (67) there is

$$u_0(t,x) \le u_{N,u_0}(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad N = 1,2,\ldots \tag{75}$$

The function u_{N,u_0} is the solution of problem (59)–(69), hence there is

$$\mathcal{F}^j[u_{N,u_0}^j](t,x) = \tilde{f}^j\big(t,x,u_{N,u_0}^1,\ldots,u_{N,u_0}^j,\ldots,u_{N,u_0}^N,u_0^{N+1},u_0^{N+2},\ldots\big) \tag{76}$$

for $j = 1,2,\ldots,N$, $(t,x) \in D$.

Since u_0 is a lower solution then by (75) and (W) for $j = N+1$ there is

$$\begin{aligned} \mathcal{F}^{N+1}[u_0^{N+1}](t,x) \le \tilde{f}^{N+1}\big(t,x,u_0^1,\ldots,u_0^j,\ldots,u_0^N,u_0^{N+1},u_0^{N+2},\ldots\big) \le \\ \le \tilde{f}^{N+1}\big(t,x,u_{N,u_0}^1,\ldots,u_{N,u_0}^j,\ldots,u_{N,u_0}^N,u_0^{N+1},u_0^{N+2},\ldots\big). \end{aligned} \tag{77}$$

The function u_{N+1,u_0} is the solution of problem (59)–(69), i.e.,

$$\begin{aligned} &\mathcal{F}^j[u_{N+1,u_0}^j](t,x) = \\ &= \tilde{f}^j\big(t,x,u_{N+1,u_0}^1,\ldots,u_{N+1,u_0}^j,\ldots,u_{N+1,u_0}^N,u_{N+1,u_0}^{N+1},u_0^{N+2},\ldots\big) \end{aligned} \tag{78}$$

for $(t,x) \in D$ and $j = 1,2,\ldots,N+1$.

Applying the above theorem on weak inequalities to systems (76), (77) and (78) with suitable initial and boundary conditions, we obtain

$$u^j_{N,u_0}(t,x) \le u^j_{N+1,u_0}(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad j = 1,2,\ldots,N,$$

and

$$u_0^{N+1}(t,x) \le u^{N+1}_{N+1,u_0}(t,x) \quad \text{for} \quad (t,x) \in \overline{D}.$$

This means that

$$u_{N,u_0}(t,x) \le u_{N+1,u_0}(t,x) \quad \text{for} \quad (t,x) \in \overline{D}. \tag{79}$$

Analogously, we obtain

$$v_{N,u_0}(t,x) \le v_0(t,x) \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad N = 1,2,\ldots$$

and

$$v_{N+1,v_0}(t,x) \le v_{N,v_0}(t,x) \quad \text{for} \quad (t,x) \in \overline{D}. \tag{80}$$

By (68), A_0 and condition (W), there is

$$\begin{aligned}\mathcal{F}^j[v^j_{N,v_0}](t,x) = \tilde{f}^j\big(t,x,v^1_{N,v_0},\ldots,v^j_{N,v_0},\ldots,v^N_{N,v_0},v_0^{N+1},v_0^{N+2},\ldots\big) \ge \\ \ge \tilde{f}^j\big(t,x,v^1_{N,v_0},\ldots,v^j_{N,v_0},\ldots,v^N_{N,v_0},u_0^{N+1},u_0^{N+2},\ldots\big)\end{aligned} \tag{81}$$

for $(t,x) \in D$ and $j = 1,2,\ldots,N$, with conditions (69), (70).

From (64), (81), (67), (71), by the theorem on weak inequalities we get

$$u_{N,u_0}(t,x) \le v_{N,v_0}(t,x) \quad \text{for} \quad (t,x) \in \overline{D}. \tag{82}$$

By induction, from (79), (80) and (82), we derive inequalities (72).

Therefore, if we assume Condition (W) then the truncation method is the monotone approximation method.

□

Remark 3.1. *Theorem* 1 *may be replaced by another theorem on the existence and uniqueness of solutions of finite (truncated) systems which under appropriate assumptions would guarantee the existence and uniqueness of solutions of the problems* (64)–(66) *and* (68)–(70).

4. Application of the truncation method

4.1. **Theorem on existence and uniqueness of global in time regular solutions of infinite countable systems of semilinear parabolic equations.** As an application of the truncation method, we give a theorem on existence and uniqueness of global-in-time regular solutions of infinite countable systems of parabolic-reaction-diffusion equations with homogenous initial and boundary conditions of the form

$$\begin{cases}\mathcal{F}^j[z^j](t,x) = \tilde{f}^j(t,x,z) := \tilde{f}^j\big(t,x,z^1,z^2,\ldots\big) \quad \text{for} \quad (t,x) \in D, \\ z^j(0,x) = 0 \quad \text{for} \quad x \in G, \\ z^j(t,x) = 0 \quad \text{for} \quad (t,x) \in \sigma \quad \text{and} \quad j \in \mathbb{N}.\end{cases} \tag{83}$$

The proof will start with rewriting the differential problem for an infinite countable system of equations equivalently as the infinite countable system of integral equations

$$z^j(t,x) = \int_0^t \int_G \mathcal{G}^j(t,x;\tau,\xi)\, \tilde{f}^j(\tau,\xi,z^1,z^2,\ldots)d\tau d\xi \tag{84}$$

for $(t,x) \in \overline{D}$ and $j \in \mathbb{N}$, where $\mathcal{G}^j(t,x;\tau,\xi)$, $j \in \mathbb{N}$, are Green functions for suitable linear problems in domain D (see Ladyzhenskaya *et al.* [30, pp. 406–413]).

Analogously, the problem for the following finite system of differential equations, truncated with respect to initial system (83), with a homogenous initial-boundary condition

$$\begin{cases} \mathcal{F}^j[w_N^j](t,x) = \tilde{f}^j(t,x,w_N^1,\ldots,w_N^j,\ldots,w_N^N,0,0,\ldots) := \\ := \tilde{f}^j_{N,0}(t,x,w_N^1,\ldots,w_N^j,\ldots,w_N^N) \quad \text{for } (t,x) \in D, \\ w_N^j(0,x) = 0 \quad \text{for} \quad x \in G, \\ w_N^j(t,x) = 0 \quad \text{for} \quad (t,x) \in \sigma \text{ and } j = 1,2,\ldots,N \end{cases} \tag{85}$$

is equivalent to the finite system of integral equations

$$w_N^j(t,x) = \int_0^t \int_G \mathcal{G}^j(t,x;\tau,\xi)\, \tilde{f}^j(\tau,\xi,w_N^1,\ldots,w_N^j,\ldots,w_N^N,0,0,\ldots)d\tau d\xi \tag{86}$$

for $(t,x) \in \overline{D}$ and $j = 1,2,\ldots,N$.

A function $z \in \mathscr{C}_{\mathbb{N}}(\overline{D})$ is called a weak C-solution of the initial-boundary value problem for infinite countable system (83) if it satisfies the following infinite countable system of integral equations:

$$z^j(t,x) = \int_0^t \int_G \mathcal{G}^j(t,x;\tau,\xi)\tilde{f}^j(\tau,\xi,z^1,z^2,\ldots)d\tau d\xi \tag{87}$$

for $(t,x) \in \overline{D}$ and $j \in \mathbb{N}$, where $\mathcal{G}^j(t,x;\tau,\xi)$, $j \in \mathbb{N}$, are the Green functions for suitable linear problems $\mathcal{F}^j[z^j](t,x) = 0$ in domain D.

If Assumption (H_a) holds and $\partial G \subset C^{2+\alpha}$, then Greens functions do exist (see Ladyzhenskaya *et al.* [30, pp. 406–413]).

Theorem 2. *Let $D = (0,T] \times G$, $0 < T < \infty$, where $G \subset \mathbb{R}^m$ is an open, bounded and convex domain with the boundary ∂G of class $C^{2+\alpha}$. Let Conditions (L), (V), (B) hold. Assume that every truncated problem* (85) *has a global regular unique solution $w_N = (w_N^1, w_N^2, \ldots, w_N^N) \in C_N^{2+\alpha}(\overline{D})$.*

Then there exists a global regular unique solution $z = (z^1, z^2, \ldots)$ of problem (83) *in whole $\overline{D}$ such that*

$$\lim_{N\to\infty} w_{N,0}(t,x) = z(t,x) \quad \textit{uniformly in} \quad \overline{D},$$

where $w_{N,0} = (w_N^1, w_N^2, \ldots, w_N^N, 0, 0, \ldots)$ and $z \in C_{\mathbb{N}}^{2+\alpha}(\overline{D})$.

Proof. If $w_{N,0} \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$, then $w^j_N \in C^{2+\alpha}(\overline{D})$, for each $j = 1, 2, \dots, N$ and $N = 1, 2, \dots$, with a Hölder constant independent of j and N (see [19, Theorem on the existence and uniqueness for infinite countable system of linear parabolic equations]).

Domain D is bounded and convex, therefore, by imbedding theorems (see Adams [1, Th. 1.31, p. 11]), there is

$$C^{2+\alpha}(\overline{D}) \subset C^{1+\alpha}(\overline{D}) \subset C^{1}(\overline{D}) \subset C^{0+1}(\overline{D}).$$

This means that $w^j_N \in C^{0+1}(\overline{D})$ for each j and N, i.e., these functions satisfy the Lipschitz condition with a constant independent of j and N. Therefore, the family $\{w_N\}$ consists of equicontinuous functions.

From (86) and Condition (B), we conclude that

$$\left| w^j_N(t,x) \right| \le c = \text{const} \quad \text{for } (t,x) \in \overline{D} \text{ and } j = 1, 2, \dots, N \text{ where } N = 1, 2, \dots,$$

So this family consists of uniformly bounded functions.

By virtue of the Arzela-Ascoli theorem, it follows that there exists a subsequence $\{w_{N_\nu}\}$ uniformly convergent in $\overline{D}$ to a continuous function; let

$$\lim_{N_\nu \to \infty} w^j_{N_\nu}(t,x) = \overline{z}^j(t,x) \quad \text{uniformly in } \overline{D}. \tag{88}$$

This is the weak C-solution of problem (83).

Now, we prove that $\overline{z}$ is a regular solution of problem (83) and $\overline{z} \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$. From Theorem on the unique solvability of the Fourier firs initial-boundary problem (see Friedman [22, Th. 6 and 7, p. 65]) it follows that $w^j_{N_\nu} \in C^{0+\alpha}(\overline{D})$ with the Hölder constant independent of j and N_ν. Therefore, $\overline{z}^j \in C^{0+\alpha}(\overline{D})$ for all $j \in \mathbb{N}$, with the same Hölder constant and $\overline{z} \in C^{0+\alpha}_{\mathbb{N}}(\overline{D})$.

Let us consider the problem

$$\begin{cases} \mathcal{F}^j[z^j](t,x) = \tilde{f}^j(t,x,\overline{z}^1,\overline{z}^2,\dots) := \tilde{\mathbf{F}}^j[\overline{z}](t,x) & \text{for} \quad (t,x) \in D, \\ z^j(0,x) = 0 \quad \text{for} \quad (t,x) \in G, \\ z^j(t,x) = 0 \quad \text{for} \quad (t,x) \in \sigma \text{ and } j \in \mathbb{N}. \end{cases} \tag{89}$$

From Assumptions (H_f) and (L) it follows that

$$\tilde{\mathbf{F}} : \mathscr{C}^{0+\alpha}_{\mathbb{N}}(\overline{D}) \to \mathscr{C}^{0+\alpha}_{\mathbb{N}}(\overline{D})$$

i.e., $\tilde{\mathbf{F}}^j[\overline{z}] \in C^{0+\alpha}(\overline{D})$, $j \in \mathbb{N}$, for $\overline{z} \in \mathscr{C}^{0+\alpha}_{\mathbb{N}}(\overline{D})$ (see [19]).

By Assumption (H_a) and the theorem on the existence and uniqueness for infinite countable system of linear parabolic equations (see [19]) we conclude that $z = \overline{z} \in \mathscr{C}^{2+\alpha}_{\mathbb{N}}(\overline{D})$.

The uniqueness of the solution of the considered problem (83) is guaranteed by the Lipschitz condition and follows from Szarski's uniqueness criterion [54]. $\square$

Remark 4.1. *Condition* (B) *plays a crucial role in proving the theorems. An analogous condition appears in Banaś and Lecko* [5, 6, 7]*, Persidskiĭ's* [44, 45]*, Rzepecki* [48]*, as well as Valeev and Zhautykov's* [66, 62] *papers, where countable systems of ordinary differential equations are studied. According to Spałek's remark* [51]*, such an assumption makes sense and is physically justified.*

Acknowledgement. The authors are very grateful to Professor Zdzisław Jackiewicz, for valuable comments and important remarks.

The research was partially supported by the AGH University of Science and Technology under local Grant No. 11.420.03.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] H. Amann, *Coagulation-fragmentation processes*, Arch. Ration. Mech., **151**(2000), 339–366.

[3] J. Appell and P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Tracts in Mathematics, Vol. 95, Cambridge University Press, Cambridge, 1990.

[4] J.M. Ball and J. Carr, *The discrete coagulation-fragmentation equations: existence, uniqueness and density conservation*, J. Statist. Phys., **61**(1990), no. 1-2, 203–234.

[5] J. Banaś and M. Lecko, *An existence theorem for a class of infinite systems of integral equations*, Math. Comput. Modelling, **34**(2001), no. 5-6, 533–539.

[6] J. Banaś and M. Lecko, *Solvability of infinite systems of differential equations in Banach sequence space*, J. Comput. Appl. Math., **137**(2001), no. 2, 363–375.

[7] J. Banaś and M. Lecko, *On solutions of an inifinite system of differential equations*, Dynam. Systems Appl., **11**(2002), no. 2, 221–230.

[8] H. Bellout, *Blow-up of solutions of parabolic equations with nonlinear memory*, J. Diff. Eq., **70**(1978), 42–68.

[9] S. Brzychczy, *Existence of solutions and monotone iterative method for infinite systems of parabolic differential-functional equations*, Ann. Polon. Math., **72**(1999), no. 1, 15–24.

[10] S. Brzychczy, *Chaplygin's method for infinite systems of parabolic differential-functional equations*, Univ. Iagel. Acta Math., **38**(2000), 153–162.

[11] S. Brzychczy, *Some variant of iteration method for infinite systems of parabolic differential-functional equations*, Opuscula Math., **20**(2000), 41–50.

[12] S. Brzychczy, *Existence and uniqueness of solutions of nonlinear infinite systems of parabolic differential-functional equations*, Ann. Polon. Math., **77**(2001), no. 1, 1–9.

[13] S. Brzychczy, *On the existence of solutions of nonlinear infinite systems of parabolic differential-functional equations*, Univ. Iagel. Acta Math., **40**(2002), 31–38.

[14] S. Brzychczy, *Existence of solutions of nonlinear infinite systems of parabolic differential-functional equations*, Math. Comput. Modelling, **36**(2002), no. 4-5, 435–443.

[15] S. Brzychczy, *Existence and uniqueness of solutions of infinite systems of semilinear parabolic differential-functional equations in arbitrary domains in ordered Banach spaces*, Math. Comput. Modelling, **36**(2002), 1183–1192.

[16] S. Brzychczy, *Monotone iterative methods for infinite systems of reaction-diffusion-convection equations with functional dependence*, Opuscula Math., **25**(2005), 29–99.

[17] S. Brzychczy, *Infinite Systems of Parabolic Differential-Functional Equations*, Monograph, AGH University of Science and Technology Press, Kracow, 2006.

[18] S. Brzychczy and R. Poznanski, *Neuronal models in infinite-dimensional spaces and their finite-dimensional projections*. Part I, J. Integr. Nurosci., **9**(2010), 11–30.

[19] S. Brzychczy and L. Górniewicz, *Continuous and discrete models of neural systems in infinite-dimensional abstract spaces* Neurocomputing, (in press).

[20] S. Brzychczy and R. Poznanski, *Neuronal models in infinite-dimensional spaces and their finite-dimensional projections*. Part II, (in preparation).

[21] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Math., Vol. 596, Springer-Verlag, Berlin, 1977.

[22] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1964.

[23] A. Fuliński, *On Marian Smoluchowski's life and contribution to physics*, Acta Phys. Polon., B 29, **6**(1998), 1523–1537.

[24] Z. Kamont and S. Zacharek, *The line method for parabolic differential-functional equations with initial boundary conditions of the Dirichlet type*, Atti Sem. Mat. Fis. Univ. Modena, **35**(1987), 249–262.

[25] Z. Kamont, *On adaptation of notations to the case of differential-functional equations*, 2005 (private communication).

[26] L. Kantorovič and G. Akilov, *Funktionalanalysis in normieren Räumen*, Akademic Verlag, Berlin, 1964 [German]. English ed.: *Functional Analysis*, Pergamon Press, Oxford, 1964.

[27] M.A. Krasnosel'skiĭ, *Topological Methods in the Theory of Nonlinear Integral Equations*, Gostehizdat, Moscow, 1956 [in Russian]. English translation: Macmillan, New York, 1964.

[28] M. Lachowicz and D. Wrzosek, *A nonlocal coagulation-fragmentation model*, Appl. Math., **27**(2000), no. 1, 45–66.

[29] G.S. Ladde, V. Lakshmikantham and A.S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman Advanced Publishing Program, Boston, 1985.

[30] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka Press, Moscow, 1967 [Russian]. Translation of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, RI, 1968.

[31] W. Lamb, *Existence and uniqueness results for the continuous coagulation and fragmentation equation*, Math. Meth. Appl. Sci., **27**(2004), 703–721.

[32] Ph. Laurençot, *On a class of conditions coagulation-fragmentation equation*, J. Diff. Eq., **167**(2000), 245–174.

[33] H. Leszczyński, *On the method of lines for a heat nonlinear equation with functional dependence*, Ann. Polon. Math., **69**(1998), no. 1, 61–74.

[34] D.J. McLaughlin, W. Lamb and A.C. Bride, *Existence results for non-autonomous multiple-fragmentation models*, Math. Meth. Appl. Sci., **20**(1997), 1313–1323.

[35] D.J. McLaughlin, W. Lamb and A.C. Bride, *A semigroup approach to fragmentation models.*, SIAM J. Math. Anal., **28**(1997), 1158–1172.

[36] D.J. McLaughlin, W. Lamb and A.C. Bride, *An existence and uniqueness theorem for a coagulation and multiple-fragmentation equation*, SIAM J. Math. Anal., **28**(1997), 1173–1190.

[37] McLaughlin, D.J., Lamb, W. and A.C. McBride, *Existence and uniqueness results for the non-autonomous coagulation and multiple-fragmentation equation*, Math. Meth. Appl. Sci., **21**(1998), 1067–1084.

[38] W. Mlak and C. Olech, *Integration of infinite systems of differential inequalities*, Ann. Polon. Math., **13**(1963), 105–112.

[39] K. Moszyński and A. Pokrzywa, *Sur les systémes infinis d'équations différentielles ordinaires dans certains espaces de Fréchet*, Dissertationes Math., 115, PWN, Polish Sci. Publ., Warszawa, 1974.

[40] K. Nickel, *Das Lemma von Max Müller-Nagumo-Westphal für stark gekoppelte Systeme parabolischer Functional-Differentialgleichungen*, Math. Z., **161**(1978), 221–234.

[41] K. Nickel, *Bounds for the set of solutions of functional-differential equations*, Ann. Polon. Math., **42**(1983), 241–257.

[42] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[43] C.V. Pao, *Numerical analysis of coupled systems of nonlinear parabolic equations*, SIAM J. Numer. Anal., **36**(1999), no. 2, 393–416.

[44] K.P. Persidskiĭ, *Infinite countable systems of differential equations and stability of their solutions*, Part I, Izv. Akad. Nauk. Kaz. SSR, 7 (11), (1958), pp. 52–71 Part II, *ibid.*, **8(12)**(1959), 45–64 Part III, *Fundamental theorems on solvability of solutions of countable many differential equations*, ibid., **9 (13)**(1960), 11-34 [Russian]. [Russian].

[45] K.P. Persidskiĭ, *Selected Works*, Vol. 2, Izdat. Nauka, Kaz. SSR, Alma-Ata, 1976 [Russian].

[46] R. Redlinger, *Existence theorems for semilinear parabolic systems with functionals*, Nonlinear Anal., **6**(1984), 667–682.

[47] R. Redlinger, *On Volterra's population equation with diffusion*, SIAM J. Math. Anal., **16(1)**(1985), 135-142.

[48] B. Rzepecki, *On infinite systems of differential equations with deviated argument*, Part I, Ann. Polon. Math., **31**(1975), 159–169, Part II, *ibid.*, **34**(1977), 251–264.

[49] M. Smoluchowski, *Drei Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen*, Physik. Z., **17**(1916), 557–585.
[50] M. Smoluchowski, *Versuch einer mathematischen Theorie der Koagulationskinetik kolloiden Lösungen*, Z. Phys. Chem., **92**(1917), 129–168.
[51] J. Spałek, (non-publication, private comunication) 2005.
[52] B. Szafirski, *Information* (non-publication, private comunication) 2010.
[53] J. Szarski, *Strong maximum principle for non-linear parabolic differential-functional inequalities*, Ann. Polon. Math., **49**(1974), 207–214.
[54] J. Szarski, *Comparison theorem for infinite systems of parabolic differential-functional equations and strongly coupled infinite systems of parabolic equations*, Bull. Acad. Polon. Sci., Sér. Sci. Math., **27(11-12)**(1979), 739–846.
[55] R. Tadeusiewicz (ed.), *Theoretical Neurocybernetics* [Polish] Warsaw University Press, Warszawa 2009.
[56] R. Tadeusiewicz, *New trends in Neurocybernetics*, Computer Methods in Materials, **10**(2010), no. 1, 1–7.
[57] A.N. Tychonov, *On an infinite system of differential equations*, Mat. Sb., **41**(1934), no. 4, 551–555 [German and Russian].
[58] H. Ugowski, *On integro – differential equations of parabolic and elliptic type*, Ann. Polon. Math., **22**(1970), 255–275.
[59] H. Ugowski, *On integro-differential equations of parabolic type*, Ann. Polon. Math., **25**(1971), 9–22.
[60] H. Ugowski, *Some theorems on the estimate and existence of solutions of integro-differential equations of parabolic type*, Ann. Polon. Math., **25**(1972), 311–323.
[61] H. Ugowski, *On a certain non-linear initial-boundary value problem for integro-differential equations of parabolic type*, Ann. Polon. Math., **28**(1973), 249–259.
[62] K. G. Valeev and O. A. Zhautykov, *Infinite Systems of Differential Equations*, Izdat. Nauka, Kaz. SSR, Alma-Ata, 1974 [Russian].
[63] D. Wrzosek, *Existence of solutions for the discrete coagulation-fragmentation model with diffusion*, Topol. Methods Nonlinear Anal., **9**(1997), 279–296.
[64] D. Wrzosek, *Mass-conserving solutions to the discrete coagulation-fragmentation model with diffusion*, Nonlinear Anal., **49**(2002), 297–314.
[65] D. Wrzosek, *Weak solutions to the Cauchy problem for the diffusive discrete coagulation-fragmentation system*, J. Math. Anal. Appl., **289**(2004), 405–418.
[66] O. A. Zhautykov, *Infinite systems of differential equations and their applications*, Differ. Uravn., **1**(1965), 162–170 [Russian].

Received: March 9, 2011; Accepted: April 10, 2011.