Fixed Point Theory, 13(2012), No. 2, 329-336
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL BOUNDARY CONDITIONS

BASHIR AHMAD* AND JUAN J. NIETO*,**

*Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: bashir_qau@yahoo.com

**Departamento de Análisis Matemático, Facultad de Matemáticas,
Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain
E-mail: juanjose.nieto.roig@usc.es

Abstract. In this paper, we study a class of Riemann-Liouville fractional differential equations with fractional boundary conditions. Some new existence results are obtained by applying standard fixed point theorems.
Key Words and Phrases: Riemann-Liouville calculus, fractional differential equations, fractional boundary conditions, fixed point theorems, anti-periodic boundary conditions.
2010 Mathematics Subject Classification: 34A12, 34A40, 47H10.

1. Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied by many researchers. Fractional differential equations appear naturally in various fields of science and engineering, and constitute an important field of research. As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [11, 12]. Some recent work on boundary value problems of fractional order can be found in [1, 2, 3, 4, 5, 6, 7, 8, 10, 16] and the references therein.

In this paper, we study the nonlinear differential equation of fractional order

$$D^{\alpha}u(t) = f(t, u(t)), \quad t \in [0, T], \quad \alpha \in (1, 2] \tag{1.1}$$

with the boundary conditions of fractional order

$$D^{\alpha-2}u(0^+) = b_0 D^{\alpha-2}u(T^-), \tag{1.2}$$

and

$$D^{\alpha-1}u(0^+) = b_1 D^{\alpha-1}u(T^-), \tag{1.3}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α and $b_0 \neq 1$ and $b_1 \neq 1$.

For $\alpha = 2$, we have the second order problem $u''(t) = f(t, u(t)), t \in [0, T]$ with the classical boundary conditions $u(0) = b_0 u(T),\ u'(0) = b_1 u'(T)$. For $b_0 = b_1 = 0$, we have the initial conditions, $u(0) = 0$ and $u'(0) = 0$. If $b_0 = b_1 = -1$, we get the anti-periodic boundary conditions. The theorems we present include and extend some previous results.

2. Preliminaries

Let us recall some basic definitions [9, 13].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined as*

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s),$$

provided the integral exists.

Definition 2.2. *For a function $u : (0, \infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$, $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of the real number α) is defined as*

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds = \Big(\frac{d}{dt}\Big)^n I^{n-\alpha} u(t),$$

provided it exists.

For $\alpha < 0$, we use the convention that $D^\alpha u = I^{-\alpha} u$. Also for $\beta \in [0, \alpha)$, it is valid that $D^\beta I^\alpha u = I^{\alpha-\beta} u$.

We note that for $\lambda > -1$, $\lambda \neq \alpha - 1, \alpha - 2, \dots, \alpha - n$, we have

$$D^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$

and

$$D^\alpha t^{\alpha-i} = 0 \quad , \quad i = 1, 2, \dots, n.$$

In particular, for the constant function $u(t) = 1$, we obtain

$$D^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad \alpha \notin \mathbb{N}.$$

For $\alpha \in \mathbb{N}$, we get, of course, $D^\alpha 1 = 0$ due to the poles of the gamma function at the points $0, -1, -2, ...,$.

For $\alpha > 0$, the general solution of the homogeneous equation

$$D^\alpha u(t) = 0$$

in $C(0, T) \cap L(0, T)$ is

$$u(t) = c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + \cdots + c_{n-2} t^{\alpha-2} + c_{n-1} t^{\alpha-1},$$

where $c_i,\ i = 1, 2, \dots, n-1$, are arbitrary real constants.

We always have $D^\alpha I^\alpha u = u$, and

$$I^\alpha D^\alpha u(t) = u(t) + c_0 t^{\alpha-n} + c_1 t^{\alpha-n-1} + \cdots + c_{n-2} t^{\alpha-2} + c_{n-1} t^{\alpha-1}.$$

3. Linear problem

For $T > 0$, and $\alpha \in (1,2]$, we consider the linear equation

$$D^{\alpha}u(t) = \sigma(t), \quad t \in [0,T], \tag{3.1}$$

with the boundary conditions (1.2) and (1.3). The general solution of (3.1) is given by

$$u(t) = c_1 t^{\alpha-1} + c_0 t^{\alpha-2} + I^{\alpha}\sigma(t) \tag{3.2}$$

with I^{α} the usual Riemann-Liouville fractional integral of order α.

Using (3.2), we have

$$D^{\alpha-1}u(t) = c_1\Gamma(\alpha) + I^1\sigma(t)$$

and imposing the boundary condition (1.3), we get

$$c_1 = \frac{b_1}{(1-b_1)\Gamma(\alpha)}\int_0^T \sigma(s)ds. \tag{3.3}$$

Since $I^{2-\alpha}(t^{\alpha-1}) = \Gamma(\alpha)t$ and $I^{2-\alpha}(t^{\alpha-2}) = \Gamma(\alpha-1)$, therefore, from (3.2), we have

$$D^{\alpha-2}u(t) = c_1\Gamma(\alpha)t + c_0\Gamma(\alpha-1) + I^2\sigma(t). \tag{3.4}$$

Using the boundary condition (1.2) in (3.4), we get

$$c_0 = \frac{b_0}{(1-b_0)\Gamma(\alpha-1)}\Big[c_1\Gamma(\alpha)T + \int_0^T (T-s)\sigma(s)ds\Big].$$

Thus,

$$u(t) = \int_0^T G(t,s)\sigma(s)ds, \tag{3.5}$$

where

$$G(t,s) = \begin{cases} \dfrac{b_1 t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)} + \dfrac{b_0 t^{\alpha-2}[T-(1-b_1)s]}{(1-b_0)(1-b_1)\Gamma(\alpha-1)} + \dfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s < t \le T, \\ \dfrac{b_1 t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)} + \dfrac{b_0 t^{\alpha-2}[T-(1-b_1)s]}{(1-b_0)(1-b_1)\Gamma(\alpha-1)} \quad , 0 \le t < s \le T. \end{cases} \tag{3.6}$$

Theorem 3.1. *Let $\alpha \in (1,2]$ and b_0, $b_1 \neq 1$. The linear problem (3.1) together with the boundary conditions (1.2) and (1.3) has a unique solution for any continuous function σ, given by (3.5).*

3.1. **Fractional anti-periodic boundary conditions.** We point out that for b_0, $b_1 = -1$, the boundary conditions (1.2) and (1.3) reduce to the boundary conditions of anti-periodic type:

$$D^{\alpha-2}u(0^+) = -D^{\alpha-2}u(T^-) \quad , \quad D^{\alpha-1}u(0^+) = -D^{\alpha-1}u(T^-).$$

In this case the Green's function (3.6) takes the form

$$G(t,s)=\begin{cases}\dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)}+\dfrac{t^{\alpha-2}(2s-T)}{4\Gamma(\alpha-1)}+\dfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} & ,\ 0\le s<t\le T,\\[2ex] \dfrac{-t^{\alpha-1}}{2\Gamma(\alpha)}+\dfrac{t^{\alpha-2}(2s-T)}{4\Gamma(\alpha-1)} & ,\ 0\le t<s\le T.\end{cases}\tag{3.7}$$

For $\alpha=2$, we have the problem

$$D^2u(t)=\sigma(t),\quad t\in[0,T],\quad u(0)=-u(T),\quad u'(0)=-u'(T),$$

whose solution is given by

$$u(t)=\int_0^T G(t,s)\sigma(s)ds$$

with

$$G(t,s)=\begin{cases}\dfrac{1}{2}(t-s)-\dfrac{1}{4}T & ,\ 0\le s<t\le T,\\[2ex] \dfrac{1}{2}(s-t)-\dfrac{1}{4}T & ,\ 0\le t<s\le T.\end{cases}\tag{3.8}$$

Note that it is the Green's function obtained in [15]. Also, when $\alpha\to 2^-$ in (3.7), we obtain (3.8).

4. Nonlinear fractional boundary value problem

Let $C[0,T]$ denote the Banach space of all continuous real valued functions defined on $[0,T]$ with the norm $\|u\|=\sup\{|u(t)|:t\in[0,T]\}$. For $t\in[0,T]$, we define $u_r(t)=t^r u(t),\ r\ge 0$. Let $C_r[0,T]$ be the space of all functions u such that $u_r\in C[0,T]$ which turn out to be a Banach space when endowed with the norm $\|u\|_r=\sup\{t^r|u(t)|:t\in[0,T]\}$.

If u is a solution of (1.1) and (1.2)-(1.3), then

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds+\frac{b_1t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)}\int_0^T f(s,u(s))ds\\&+\frac{b_0t^{\alpha-2}}{(1-b_1)(1-b_0)\Gamma(\alpha-1)}\int_0^T\Big(T-(1-b_1)s\Big)f(s,u(s))ds.\end{aligned}\tag{4.1}$$

Define an operator $\mathcal{P}:C_{2-\alpha}[0,T]\to C_{2-\alpha}[0,T]$ as

$$\begin{aligned}(\mathcal{P}u)(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds+\frac{b_1t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)}\int_0^T f(s,u(s))ds\\&+\frac{b_0\,t^{\alpha-2}}{(1-b_1)(1-b_0)\Gamma(\alpha-1)}\int_0^T\Big(T-(1-b_1)s\Big)f(s,u(s))ds,t\in[0,T].\end{aligned}$$

Observe that the problem (1.1) and (1.2)-(1.3) has solutions if and only if the operator equation $\mathcal{P}u=u$ has fixed points.

To prove the existence of solutions, we need the following fixed point theorem.

Theorem 4.1. [14] *Let E be a Banach space. Assume that $T : E \to E$ be a completely continuous operator and the set $V = \{x \in E \mid x = \mu Tx, 0 < \mu < 1\}$ be bounded. Then T has a fixed point in E.*

Lemma 4.1. *Suppose that $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then the operator P is compact.*

Proof. ***(i)*** Let $\mathbf{B}$ be a bounded set in $C_{2-\alpha}[0, T]$. Hence $\mathbf{B}$ is bounded on $C[0, T]$ and there exists a constant M such that $|f(t, u(t))| \le M,\ \forall u \in \mathbf{B},\ t \in [0, T]$. Thus

$$t^{2-\alpha}|(\mathcal{P}u)(t)| \le \frac{Mt^{2-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \Big|\frac{b_1 Mt}{(1-b_1)\Gamma(\alpha)}\Big| \int_0^T ds$$

$$+ \Big|\frac{b_0 M}{(1-b_1)(1-b_0)\Gamma(\alpha-1)}\Big| \int_0^T |T - (1-b_1)s| ds$$

$$\le MT^2\Big(\frac{1}{\Gamma(\alpha+1)} + \Big|\frac{b_1}{(1-b_1)\Gamma(\alpha)}\Big| + \Big|\frac{b_0(1+|b_1|)}{2(1-b_1)(1-b_0)\Gamma(\alpha-1)}\Big|\Big),$$

which implies that

$$\|(\mathcal{P}u)\|_{2-\alpha} \le MT^2\Big(\frac{1}{\Gamma(\alpha+1)} + \Big|\frac{b_1}{(1-b_1)\Gamma(\alpha)}\Big| + \Big|\frac{b_0(1+|b_1|)}{2(1-b_1)(1-b_0)\Gamma(\alpha-1)}\Big|\Big).$$

Hence $\mathcal{P}(\mathbf{B})$ is uniformly bounded.

(ii) For any $t_1,\ t_2 \in [0, T], u \in \mathbf{B}$, we have

$$|t_1^{2-\alpha}(\mathcal{P}u)(t_1) - t_2^{2-\alpha}(\mathcal{P}u)(t_2)|$$

$$= \Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big[t_1^{2-\alpha}(t_1-s)^{\alpha-1} - t_2^{2-\alpha}(t_2-s)^{\alpha-1}\Big] f(s, u(s)) ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{2-\alpha}(t_2-s)^{\alpha-1} f(s, u(s)) ds + \frac{b_1(t_1-t_2)}{(1-b_1)\Gamma(\alpha-1)} \int_0^T f(s, u(s)) ds\Big|$$

$$\le M\Big(\Big|\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big[t_1^{2-\alpha}(t_1-s)^{\alpha-1} - t_2^{2-\alpha}(t_2-s)^{\alpha-1}\Big] ds$$

$$- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} t_2^{2-\alpha}(t_2-s)^{\alpha-1} ds\Big| + \Big|\frac{b_1(t_1-t_2)}{(1-b_1)\Gamma(\alpha-1)} \int_0^T ds\Big|\Big) \to 0 \text{ as } t_1 \to t_2.$$

Thus $t^{2-\alpha}\mathcal{P}(\mathbf{B})$ and hence $\mathcal{P}(\mathbf{B})$ is equicontinuous. Consequently, the operator $\mathcal{P}$ is compact. This completes the proof.

Theorem 4.2. *Assume that there exists a constant $M > 0$ such that*

$$|f(t, u)| \le M,\ \forall\ t \in [0, T],\ u \in \mathbb{R}.$$

Then the problem (1.1) and (1.2)-(1.3) has at least one solution in $C_{2-\alpha}[0, T]$.

Proof. Consider the set

$$V = \{u \in \mathbb{R} \mid u = \mu \mathcal{P} u,\ 0 < \mu < 1\},$$

and show that the set V is bounded. Let $u \in V$, then $u = \mu \mathcal{P} u$, $0 < \mu < 1$. For any $t \in [0, T]$, we have

$$|u(t)| \le \mu \Big[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s))| ds + \Big| \frac{b_1 t^{\alpha-1}}{(1-b_1)\Gamma(\alpha)} \Big| \int_0^T |f(s, u(s))| ds$$

$$+ \Big| \frac{b_0 t^{\alpha-2}}{(1-b_1)(1-b_0)\Gamma(\alpha-1)} \Big| \int_0^T |T - (1-b_1)s| |f(s, u(s))| ds \Big].$$

As in part (i) of Lemma 4.1, we have

$$\|(\mathcal{P}u)\|_{2-\alpha} \le M T^2 \Big(\frac{1}{\Gamma(\alpha+1)} + \Big| \frac{b_1}{(1-b_1)\Gamma(\alpha)} \Big| + \Big| \frac{b_0(1+|b_1|)}{2(1-b_1)(1-b_0)\Gamma(\alpha-1)} \Big| \Big).$$

This implies that the set V is bounded independently of $\mu \in (0, 1)$. Using Lemma 4.1 and Theorem 4.1, we obtain that the operator $\mathcal{P}$ has at least a fixed point, which implies that the problem (1.1) and (1.2)-(1.3) has at least one solution.

Theorem 4.3. *Assume that there exists a constant $L > 0$ such that*

$$|f(t, u) - f(t, v)| \le L|u - v|, \quad \forall t \in [0, T],\ u,\ v \in \mathbb{R}. \tag{4.2}$$

Then the problem (1.1) and (1.2)-(1.3) has a unique solution in $C_{2-\alpha}[0, T]$ if

$$L < \frac{1}{\nu}, \tag{4.3}$$

where

$$\nu = \frac{T^\alpha}{(\alpha-1)} \Big[\frac{1}{\Gamma(\alpha)} + \Big| \frac{b_1}{(1-b_1)\Gamma(\alpha)} \Big| + \Big| \frac{b_0(1+(\alpha-1)|b_1|)}{\alpha(1-b_1)(1-b_0)\Gamma(\alpha-1)} \Big| \Big].$$

Proof. In view of (4.2), for every $t \in [0, T]$, we have

$$\begin{aligned}
& t^{2-\alpha} |(\mathcal{P}u)(t) - (\mathcal{P}v)(t)| \\
& \le \frac{t^{2-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\
& + \Big| \frac{b_1 t}{(1-b_1)\Gamma(\alpha)} \Big| \int_0^T |f(s, u(s)) - f(s, v(s))| ds \\
& + \Big| \frac{b_0}{(1-b_1)(1-b_0)\Gamma(\alpha-1)} \Big| \int_0^T |T - (1-b_1)s| |f(s, u(s)) - f(s, v(s))| ds \\
& \le L \Big[\frac{t^{2-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s) - v(s)| ds + \Big| \frac{b_1 t}{(1-b_1)\Gamma(\alpha)} \Big| \int_0^T |u(s) - v(s)| ds \\
& + \Big| \frac{b_0}{(1-b_1)(1-b_0)\Gamma(\alpha-1)} \Big| \int_0^T |T - (1-b_1)s| |u(s) - v(s)| ds \Big].
\end{aligned}$$

By the definition of $\|.\|_{2-\alpha}$, we obtain

$$
\begin{aligned}
&\|(\mathcal{P}u)(t) - (\mathcal{P}v)(t)\|_{2-\alpha} \\
&\le L\Big[\frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-2}ds + \Big|\frac{b_1 t}{(1-b_1)\Gamma(\alpha)}\Big|\int_0^T s^{\alpha-2}ds \\
&+ \Big|\frac{b_0}{(1-b_1)(1-b_0)\Gamma(\alpha-1)}\Big|\int_0^T |T-(1-b_1)s|s^{\alpha-2}ds\Big]\|u-v\|_{2-\alpha} \\
&\le \frac{LT^\alpha}{(\alpha-1)}\Bigg[\frac{1}{\Gamma(\alpha)} + \Big|\frac{b_1}{(1-b_1)\Gamma(\alpha)}\Big| + \Big|\frac{b_0(1+(\alpha-1)|b_1|)}{\alpha(1-b_1)(1-b_0)\Gamma(\alpha-1)}\Big|\Bigg]\|u-v\|_{2-\alpha}.
\end{aligned}
$$

From the above estimate, it follows by the condition (4.3) that the operator $\mathcal{P}$ is a contraction. Hence, by Banach fixed point theorem, we deduce that $\mathcal{P}$ has a unique fixed point which in fact is a unique solution of problem (1.1) and (1.2)-(1.3). This completes the proof.

Example 4.1. *Consider the fractional boundary value problem*

$$
\begin{aligned}
&D^\alpha x(t) = \frac{e^{-|\sin x(t)|}[3\cos 2t + 7\ln(22+5\cos^2 x(t)]}{4+\cos x(t)}, \ 0<t<T, \\
&D^{\alpha-2}u(0^+) = b_0 D^{\alpha-2}u(T^-), \qquad D^{\alpha-1}u(0^+) = b_1 D^{\alpha-1}u(T^-).
\end{aligned} \tag{4.4}
$$

where $1<\alpha\le 2$.

Clearly

$$
|f(t,u(t)| = \Big|\frac{e^{-|\sin x(t)|}[3\cos 2t + 7\ln(22+5\cos^2 x(t)]}{4+\cos x(t)}\Big| \le 1 + 7\ln 3 = M.
$$

Thus, by Theorem 4.2, the problem (4.4) has at least one solution.

Example 4.2. *Consider the following fractional boundary value problem*

$$
\begin{aligned}
&D^{\frac{3}{2}}u(t) = L\Big(\sin t + \tan^{-1} u(t)\Big), \quad t\in[0,1], \\
&D^{-1/2}u(0^+) = \frac{1}{2}D^{-1/2}u(1^-), \qquad D^{1/2}u(0^+) = -D^{1/2}u(1^-).
\end{aligned} \tag{4.5}
$$

In this case, $\alpha = 3/2,\ T=1,\ b_0 = 1/2$, and $b_1 = -1$. Clearly,

$$
|f(t,u) - f(t,v)| \le L|\tan^{-1} u - \tan^{-1} v| \le L|u-v|.
$$

Further,

$$
\nu = \frac{T^\alpha}{(\alpha-1)}\Big[\frac{1}{\Gamma(\alpha)} + \Big|\frac{b_1}{(1-b_1)\Gamma(\alpha)}\Big| + \Big|\frac{b_0(1+(\alpha-1)|b_1|)}{\alpha(1-b_1)(1-b_0)\Gamma(\alpha-1)}\Big|\Big] = \frac{7}{\sqrt{\pi}}.
$$

With $L<\sqrt{\pi}/7$, all the assumptions of Theorem 4.3 are satisfied. Hence, there exists a unique solution for the fractional boundary value problem (4.5).

Acknowledgment. The authors thank the referee for his/her useful comments.

References

[1] S. Abbas, M. Benchohra, *Existence theory for impulsive partial hyperbolic diferential equations of fractional order at variable times*, Fixed Point Theory, **12**(2011), 3-16.

[2] R.P. Agarwal, B. Andrade, C. Cuevas, *Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations*, Nonlinear Anal. Real World Appl., **11**(2010), 3532-3554.

[3] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., **72**(2010), 2859-2862.

[4] B. Ahmad, J.J. Nieto, *Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions*, Bound. Value Probl., 2009, Art. ID 708576, 11 pp.

[5] B. Ahmad, J.J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl., **58**(2009), 1838-1843.

[6] B. Ahmad, J.J. Nieto, *Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory*, Topol. Methods Nonlinear Anal., **35**(2010), 295-304.

[7] Z. Bai, *On positive solutions of a nonlocal fractional boundary value problem*, Nonlinear Anal., **72**(2010), 916-924.

[8] M. Belmekki, J.J. Nieto, R. Rodríguez-López, *Existence of periodic solution for a nonlinear fractional differential equation*, Bound. Value Probl., (2009), Art. ID 324561, 18 pp.

[9] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[10] J.J. Nieto, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions*, Appl. Math. Lett., **23**(2010), 1248-1251.

[11] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[12] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[13] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[14] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1980.

[15] K. Wang, Y. Li, *A note on existence of (anti)-periodic and heteroclinic solutions for a class of second order odes*, Nonlinear Anal., **70**(2009), 1711-1724.

[16] S. Zhang, *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, *Comput. Math. Appl.*, **59**(2010), 1300-1309.

Received: February 2, 2011; Accepted: March 30, 2011.