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ON ROLEWICZ-ZABCZYK TECHNIQUES IN THE STABILITY THEORY OF DYNAMICAL SYSTEMS

ADINA LUMINIŢA SASU, MIHAIL MEGAN AND BOGDAN SASU

Dedicated to Professor Ioan A. Rus on the occasion of his 75th birthday

West University of Timişoara, Faculty of Mathematics and Computer Science,
Department of Mathematics, V. Pârvan Blvd. No. 4, 300223 Timişoara, Romania
E-mails: sasu@math.uvt.ro, megan@math.uvt.ro, bsasu@math.uvt.ro

Abstract. The aim of this paper is to present a general overview concerning the Rolewicz-Zabczyk type techniques in the stability theory of dynamical systems. We discuss the main methods based on trajectories that may be used in order to characterize the uniform exponential stability of variational discrete systems and their applications to the case of skew-product flows. Beside our techniques used in the past decade on this topic, we also point out several new issues and analyze both their connections with previous results as well as some new characterizations for uniform exponential stability. Finally, motivated by the potential extension of the framework to dichotomy, we propose several open problems in the case of the exponential instability.

Key Words and Phrases: variational difference equation; exponential stability; skew-product flow; translation invariant sequence space.
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1. Introduction

We consider the difference equation

$$x_{k+1} = A_k x_k, \quad k \in \mathbb{N}.$$

The equilibrium solutions of this equation are defined by the common fixed points of the operators A_k (see [30], [31]). The stability of these solutions is a topic of large interest and was intensively studied in the passed decades. An important problem in this framework is the study of the exponential stability of difference equations and therefore our paper is devoted to the analysis of a class of techniques which led to several interesting characterizations for this property.

Some of the most popular techniques in the stability theory of difference and differential equations are those introduced by Przylusky and Rolewicz in the eighties (see [24]–[29]). Mainly, the idea was to express an asymptotic property of a system in terms of the convergence of certain associated series or integrals of scalar trajectories. A remarkable result for stability of systems of difference equations has been obtained in [24], where Przyluski and Rolewicz proved that a system of difference equations

$$x_{k+1} = A_k x_k, \quad k \geq k_0$$

on a Banach space X is uniformly exponentially stable if and only if there is $p \in [1, \infty)$ such that for every $x \in X$

$$\sup_{k \geq k_0} \sum_{n=k}^{\infty} ||(\prod_{i=k}^{n-1} A_i)x||^p < \infty.$$

This result may be regarded as the discrete-time version of the famous stability theorem of Datko (see [7], Theorem 1 and Remark 3). These methods have a history that goes back to the work of Zabzcyk, since the autonomous case was treated for the first time by Zabczyk in [46] (see Theorem 5.1):

Theorem 1.1. (Zabczyk) *Let X be a Banach space and let $T \in \mathcal{L}(X)$. If $N : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing convex function with $N(0) = 0$ such that for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\sum_{n=0}^{\infty} N(\alpha(x)||T^n x||) < \infty$$

then the spectral radius $r(T) < 1$.

The author presented in [46] a reasoning based on the Banach-Steinhaus Theorem and on the construction of an auxiliary sequence space associated with the function N. This result led to the formulation of an inedit characterization for the uniform exponential stability of semigroups given by:

Theorem 1.2. (Zabczyk) *A C_0-semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space X is uniformly exponentially stable if and only if there is a continuous strictly increasing convex function $N : [0, \infty) \to [0, \infty)$ with $N(0) = 0$ such that for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\sum_{n=0}^{\infty} N(\alpha(x)||T(n)x||) < \infty.$$

At that time it was clear that an asymptotic property, like stability, may be deduced from the convergence of a series of nonlinear trajectories, but the development of these techniques was only at the very beginning. A remarkable step was done by Rolewicz in [28], where the methods were diversified and the condition obtained by the author was one of the most general in the topic:

Theorem 1.3. (Rolewicz) *Let $N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function such that for every $t > 0$, $s \mapsto N(t, s)$ is continuous and non-decreasing with $N(t, 0) = 0$, $N(t, s) > 0$, for all $s > 0$ and for every $s \geq 0$, $t \mapsto N(t, s)$ is non-decreasing. If $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is a strongly continuous evolution family on the Banach space X such that for every $x \in X$, there is $\alpha(x) > 0$ with*

$$\sup_{s \geq 0} \int_s^{\infty} N(\alpha(x), ||U(t, s)x||)\, dt < \infty \qquad (1.1)$$

then $\mathcal{U}$ is uniformly exponentially stable.

The approach proposed by Rolewicz was completely distinct compared with the previous ones, being based on category arguments. The author noted that if (1.1) holds then the Banach space X, which is in particular a set of the second category, can be decomposed in a countable reunion of some auxiliary sets. Not only the proof but also the conclusions were starting points for the study of new classes of evolutionary processes as well as of new asymptotic properties, extending considerably the applicability area (see [12]–[16], [19]–[21], [29], [32], [34], [35], [37], [39], [41], [45], [47]). The Rolewicz type methods extended the framework developed by Datko and Pazy for linear differential equations (see [6], [7], [22], [23] and the references therein). It should be noted that the Datko-Pazy approach was generalized to the case of nonlinear operators by Ichikawa in [9], using a direct construction. We also refer here the work of Reghiş, where several interesting applications were pointed out (see [26], [27]).

A notable intervention on this subject is that of Neerven (see [19]) where he observed that the p-integrability of some associated trajectories of a semigroup, which became so familiar in the Datko-Pazy approach, can be generalized to a more advanced level, by replacing the classical L^p-spaces with arbitrary Banach function spaces:

Theorem 1.4. (Neerven) *A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if there is a Banach function space B with $\lim\limits_{t\to\infty} F_B(t)=\infty$ such that for every $x\in X$, the mapping $t\mapsto ||T(t)x||$ lies in B.*

The next chronological notable intervention was due to Neerven as well. More precisely, in [21] the author managed to connect the property of uniform exponential stability of a semigroup with a topological property of an associated subset defined by means of a functional:

Theorem 1.5. (Neerven) *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let $J : C^+[0,\infty)\to[0,\infty]$ be a lower semi-continuous and nondecreasing functional on $C^+[0,\infty)$ (the positive cone of $C[0,\infty)$), satisfying $J(c\chi_{\mathbb{R}_+})=\infty$, for all $c>0$. If $\mathbf{T}$ is not uniformly exponentially stable, then the set $\{x\in X : J(||T(\cdot)x||)=\infty\}$ is residual in X.*

The increasing interest in this subject directed the attention on the variational case. First attempts in characterizing the stability of skew-products flows in terms of the membership of some associated orbits to certain Banach sequence spaces and Banach function spaces, respectively, were done in [12], where we proposed a unified treatment for these problems both in discrete and continuous time for the general case of dynamical systems modeled by skew-product flows. After that, the techniques were gradually improved and extended: from stability to instability (see [14], [15]), from Datko-type methods to Zabczyk and Przyluski-Rolewicz type characterizations (see [11], [13], [16], [32], [34]– [37]), from stability investigations to extensive studies on dichotomy or trichotomy (see [39], [41]).

In this paper we bring together conclusions on stability issues published over the past decade. We point out those methods which are specific for the variational case and deduce several interesting new conclusions that facilitate the future development of the subject to the more general setting of dichotomy or trichotomy. The survey will present the most useful technical requirements in this topic, proposing an overview in the framework of Banach sequence spaces and their applications in the asymptotic theory of variational equations. Beside recalling already known results and techniques, our aim is to present a step-by-step construction and to draw out those ideas that lead to results with the widest range of application. The central ideas will be pointed out in three main stages: variational difference equations with arbitrary coefficients, variational difference equations with bounded coefficients and skew-product flows, with detailed comments on each case and pointing out several new interesting situations. Finally we will present open problems and motivate their connections with previous results in this topic.

2. Banach sequence spaces

The theory of sequence spaces was intensively used in our papers in the last few years in order to investigate the asymptotic properties of dynamical systems (see [33], [35], [38], [40], [42] and the references therein). In this section, for the sake of clarity, we present several basic definitions and properties of Banach sequences spaces. We will recall here only those properties that are indeed necessary for the presentation that follows in the next sections and we briefly discuss only the proofs which bring into the attention some technical aspects that make the difference between the methods used in previous works and those on which we will insist in this paper. For more examples we refer to [2] and[18].

Let $\mathbb{Z}$ denote the set of the integers, let $\mathbb{N}$ denote the set of all non negative integers, let $\mathbb{R}$ denote the set of all real numbers and let $\mathcal{S}(\mathbb{N},\mathbb{R})$ be the linear space of all sequences $s:\mathbb{N}\to\mathbb{R}$. We denote $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. For every set $A\subset\mathbb{N}$ let χ_A denote the characteristic function of the set A.

For every $s\in\mathcal{S}(\mathbb{N},\mathbb{R})$ we define the sequence

$$s_+:\mathbb{N}\to\mathbb{R},\quad s_+(n)=\begin{cases}0, & n=0\\ s(n-1), & n\in\mathbb{N}^*.\end{cases}$$

Definition 2.1. A linear space $B\subset\mathcal{S}(\mathbb{N},\mathbb{R})$ is called a *normed sequence space* if there is a norm $|\cdot|_B:B\to\mathbb{R}_+$ with the property that if $s,\gamma\in\mathcal{S}(\mathbb{N},\mathbb{R}),|s(j)|\le|\gamma(j)|$, for all $j\in\mathbb{N}$ and $\gamma\in B$, then $s\in B$ and $|s|_B\le|\gamma|_B$.

If, moreover, $(B,|\cdot|_B)$ is complete, then B is called *Banach sequence space*.

Definition 2.2. A Banach sequence space $(B,|\cdot|_B)$ is said to be *invariant under translations* if for every $s\in B$, $s_+\in B$ and $|s_+|_B=|s|_B$.

Notation We denote by $\mathcal{Q}(\mathbb{N})$ the class of all Banach sequence spaces B which are invariant under translations and have the following properties:
(i) $\chi_{\{0\}}\in B$;
(ii) if $s\in\mathcal{S}(\mathbb{N},\mathbb{R})$ and there is $M>0$ such that $|s\cdot\chi_{\{0,\dots,n\}}|_B\le M$, for all $n\in\mathbb{N}$, then $s\in B$ and $|s|_B\le M$.

Example 2.1. (i) For every $p \in [1, \infty)$, $\ell^p(\mathbb{N}, \mathbb{R})$ with $||s||_p = \left(\sum_{k=0}^{\infty} ||s(k)||^p\right)^{1/p}$ is a Banach sequence space which belongs to $\mathcal{Q}(\mathbb{N})$;

(ii) $\ell^\infty(\mathbb{N}, \mathbb{R})$ with $||s||_\infty = \sup_{n\in\mathbb{N}} |s(n)|$ is also a Banach sequence space in the class $\mathcal{Q}(\mathbb{N})$.

(iii) $c_0(\mathbb{N}, \mathbb{R}) := \{s \in \mathcal{S}(\mathbb{N}, \mathbb{R}) : \lim\limits_{n\to\infty} s(n) = 0\}$ with respect to the norm $|| \cdot ||_\infty$ is a Banach sequence space which belongs to class $\mathcal{Q}(\mathbb{N})$ as well. $\diamond$

Example 2.2. (*Orlicz sequence spaces*) Let $\varphi : \mathbb{R}_+ \to [0, \infty]$ be a nondecreasing left continuous function which is not identically 0 or ∞ on $(0, \infty)$. We consider the associated *Young function*:

$$Y_\varphi : \mathbb{R}_+ \to [0, \infty], \quad Y_\varphi(t) := \int_0^t \varphi(s)\ ds.$$

Then Y_φ is a nondecreasing convex function. For every $s \in \mathcal{S}(\mathbb{N}, \mathbb{R})$, let $M_\varphi(s) := \sum_{k=0}^{\infty} Y_\varphi(|s(k)|)$. Then $\ell_\varphi(\mathbb{N}, \mathbb{R}) := \{s \in \mathcal{S}(\mathbb{N}, \mathbb{R}) : \exists\ c > 0$ such that $M_\varphi(cs) < \infty\}$ is a Banach space with respect to the norm $|s|_\varphi := \inf\{c > 0 : M_\varphi(s/c) \leq 1\}$. The space $\ell_\varphi(\mathbb{N}, \mathbb{R})$ is called the *Orlicz sequence space* associated to φ. $\diamond$

Lemma 2.1. *If $\ell_\varphi(\mathbb{N}, \mathbb{R})$ is an Orlicz sequence space, then $\ell_\varphi(\mathbb{N}, \mathbb{R}) \in \mathcal{Q}(\mathbb{N})$.*

Proof. Let $s \in \mathcal{S}(\mathbb{N}, \mathbb{R})$. We observe that

$$M_\varphi(s/c) = M_\varphi(s_+/c), \quad \forall c > 0,$$

which yields that $\ell_\varphi(\mathbb{N}, \mathbb{R})$ is invariant under translations. Since φ is not indentically 0 or ∞ on $(0, \infty)$ there exists $\delta > 0$ with $Y_\varphi(\delta) < \infty$. Then, taking into account that $M_\varphi(\delta\chi_{\{0\}}) = Y_\varphi(\delta)$ we deduce that $\chi_{\{0\}} \in \ell_\varphi(\mathbb{N}, \mathbb{R})$.

Let $s \in \mathcal{S}(\mathbb{N}, \mathbb{R})$ and let $M > 0$ be such that

$$|s \cdot \chi_{\{0,\ldots,n\}}|_\varphi \leq M, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

Let $\varepsilon > 0$. From relation (2.1) it follows that

$$M_\varphi\left(\frac{s \cdot \chi_{\{0,\ldots,n\}}}{M + \varepsilon}\right) \leq 1, \quad \forall n \in \mathbb{N}$$

which means that

$$\sum_{k=0}^{n} Y_\varphi\left(\frac{|s(k)|}{M + \varepsilon}\right) \leq 1, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

From relation (2.2) we deduce that

$$\sum_{k=0}^{\infty} Y_\varphi\left(\frac{|s(k)|}{M + \varepsilon}\right) \leq 1.$$

This implies that $s \in \ell_\varphi(\mathbb{N}, \mathbb{R})$ and $|s|_\varphi \leq M + \varepsilon$. Since $\varepsilon > 0$ was arbitrary it follows that $|s|_\varphi \leq M$.

In conclusion, $\ell_\varphi(\mathbb{N}, \mathbb{R})$ belongs to the class $\mathcal{Q}(\mathbb{N})$. $\square$

Remark 2.1. For every $p \in [1, \infty]$, the space $\ell^p(\mathbb{N}, \mathbb{R})$ is a particular Orlicz sequence space (see e.g. [33]).

Remark 2.2. If $B \in \mathcal{Q}(\mathbb{N})$, then the following properties hold:
(i) for every $A \subset \mathbb{N}, \chi_A \in B$;
(ii) $\ell^1(\mathbb{N}, \mathbb{R}) \subset B \subset \ell^\infty(\mathbb{N}, \mathbb{R})$ (see e.g. [33], Lemma 2.1).

Definition 2.3. If $(B, |\cdot|_B)$ is a Banach sequence space with $B \in \mathcal{Q}(\mathbb{N})$ then $F_B : \mathbb{N}^* \to \mathbb{R}_+, F_B(n) = |\chi_{\{0,\dots,n-1\}}|_B$ is called *the fundamental function* of B.

Notation We denote by $\mathcal{V}(\mathbb{N})$ the class of all Banach sequence spaces $B \in \mathcal{Q}(\mathbb{N})$ with the property that $\sup_{n\in\mathbb{N}} F_B(n) = \infty$.

Lemma 2.2. *If $B \in \mathcal{Q}(\mathbb{N})$, then $B \in \mathcal{Q}(\mathbb{N}) \setminus \mathcal{V}(\mathbb{N})$ if and only if $c_0(\mathbb{N}, \mathbb{R}) \subset B$.*

Proof. See Lemma 2.8 in [40]. □

Remark 2.3. According to Remark 2.2 and Lemma 2.2 we deduce that $B \in \mathcal{Q}(\mathbb{N}) \setminus \mathcal{V}(\mathbb{N})$ if and only if $c_0(\mathbb{N}, \mathbb{R}) \subset B \subset \ell^\infty(\mathbb{N}, \mathbb{R})$.

A technical property of the class $\mathcal{V}(\mathbb{N})$ is the following (see also [40]):

Lemma 2.3. *Let $\ell_\varphi(\mathbb{N}, \mathbb{R})$ be an Orlicz space. Then either $\ell_\varphi(\mathbb{N}, \mathbb{R}) \in \mathcal{V}(\mathbb{N})$ or $\ell_\varphi(\mathbb{N}, \mathbb{R}) = \ell^\infty(\mathbb{N}, \mathbb{R})$.*

Proof. If $\ell_\varphi(\mathbb{N}, \mathbb{R}) \notin \mathcal{V}(\mathbb{N})$ then $a_\varphi := \sup_{n\in\mathbb{N}} F_{\ell_\varphi}(n) < \infty$. Since $(n+1)Y_\varphi(1/a_\varphi) = M_\varphi(\chi_{\{0,\dots,n\}}/a_\varphi) \leq 1$, for all $n \in \mathbb{N}$, we deduce that $Y_\varphi(1/a_\varphi) = 0$.

Let $s \in \ell^\infty(\mathbb{N}, \mathbb{R})$ and let $\tilde{s} := s/[a_\varphi(1+||s||_\infty)]$. Since $|\tilde{s}(k)| < 1/a_\varphi$, it follows that $Y_\varphi(|\tilde{s}(k)|) = 0$, for every $k \in \mathbb{N}$. Then $M_\varphi(\tilde{s}) = 0$, so $\tilde{s} \in \ell_\varphi(\mathbb{N}, \mathbb{R})$ which implies that $s \in \ell_\varphi(\mathbb{N}, \mathbb{R})$. Hence, by applying Remark 2.2 (ii) we obtain that $\ell_\varphi(\mathbb{N}, \mathbb{R}) = \ell^\infty(\mathbb{N}, \mathbb{R})$, which completes the proof. □

Remark 2.4. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a left-continuous nondecreasing function with $\varphi(t) > 0$, for all $t > 0$, then, from Lemma 2.3 it follows that $\ell_\varphi(\mathbb{N}, \mathbb{R})$ belongs to the class $\mathcal{V}(\mathbb{N})$.

3. Stability of variational difference equations

Let X be a real or complex Banach space and let I_d denote the identity operator on X. The norm on X and on $\mathcal{L}(X)$ - the Banach algebra of all bounded linear operators on X will be denoted by $||\cdot||$. For every $x \in X$ and every $r > 0$ let $D(x, r) := \{y \in X : ||y - x|| \leq r\}$.

Let (Θ, d) be a metric space, let $J \in \{\mathbb{N}, \mathbb{Z}\}$ and let $\sigma : \Theta \times J \to \Theta$ be a discrete flow on Θ, i.e. $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, m+n) = \sigma(\sigma(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times J^2$.

Let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the variational discrete dynamical system

$$(A) \qquad x(\theta)(n+1) = A(\sigma(\theta, n))x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}.$$

The discrete cocycle associated with the system (A) is $\Phi : \Theta \times \mathbb{N} \to \mathcal{L}(X)$ where

$$\Phi(\theta, n) = \begin{cases} A(\sigma(\theta, n-1)) \dots A(\theta) & , \quad n \in \mathbb{N}^* \\ I_d & , \quad n = 0 \end{cases} , \quad \forall(\theta, n) \in \Theta \times \mathbb{N}.$$

Remark 3.1. The discrete cocycle associated with the system (A) has the property that $\Phi(\theta, m+n) = \Phi(\sigma(\theta, m), n)\Phi(\theta, m)$, for all $(\theta, m, n) \in \Theta \times \mathbb{N}^2$.

Example 3.1. Let $\Theta \in \{\mathbb{N}, \mathbb{Z}\}$, $J = \mathbb{N}$ and let $\sigma(\theta, n) = \theta + n$ be the translation flow. Then

$$\Phi(\theta, n) = \begin{cases} A(\theta+n-1)\dots A(\theta) & , \quad n \in \mathbb{N}^* \\ I_d & , \quad n = 0 \end{cases} , \quad \forall(\theta, n) \in \Theta \times \mathbb{N},$$

which shows that there exists a discrete evolution family $\{U(m,n)\}_{m\geq n, m,n\in\Theta}$ such that

$$\Phi(\theta, n) = U(\theta + n, \theta), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}.$$

Moreover, it follows that difference equations (see [1], [8]) are particular cases of variational discrete dynamical systems. ◇

Example 3.2. Let X be a Banach space and let $L > 0$. Let

$$\Theta := \{T = \{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X) : \sup_{n\in\mathbb{N}} ||T_n|| \leq L\}$$

endowed with the metric

$$d(T, S) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{\sup_{k\leq n} ||T_k - S_k||}{1 + \sup_{k\leq n} ||T_k - S_k||}.$$

We define

$$\sigma : \Theta \times \mathbb{N} \to \Theta, \quad \sigma(T, m_0) := \{T_{n+m_0}\}_{n\in\mathbb{N}}, \quad \forall T = \{T_n\}_{n\in\mathbb{N}}, \forall m_0 \in \mathbb{N}$$

and it is easy to see that σ is a discrete flow. Moreover

$$\Phi(T, n_0) := \begin{cases} T_{n_0-1}\dots T_1 T_0, & n_0 \in \mathbb{N}^* \\ I_d, & n_0 = 0 \end{cases} , \quad \forall T = \{T_n\}_{n\in\mathbb{N}}$$

is a discrete cocycle over the flow σ. ◇

Definition 3.1. The system (A) is said to be *uniformly exponentially stable* if there are $K, \nu > 0$ such that

$$||\Phi(\theta, n)|| \leq K e^{-\nu n}, \quad \forall(\theta, n) \in \Theta \times \mathbb{N}.$$

For every $(x, \theta) \in X \times \Theta$ we consider the trajectory

$$s_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad s_{x,\theta}(n) = ||\Phi(\theta, n)x||.$$

In what follows we shall see that the class of Banach function spaces $\mathcal{V}(\mathbb{N})$ introduced in the previous section has a significant role in the characterization of the uniform exponential stability of variational difference equations in terms of the associated trajectories.

Theorem 3.1. *Let $B \in \mathcal{V}(\mathbb{N})$. Then the system (A) is uniformly exponentially stable if and only if there exist $x_0 \in X$ and $L, r > 0$ such that*

$$\sup_{\theta\in\Theta} |s_{x,\theta}|_B \leq L, \quad \forall x \in D(x_0, r). \tag{3.1}$$

Proof. Necessity. Let $K, \nu > 0$ be two constants given by Definition 3.1. We consider the sequence

$$e_\nu : \mathbb{N} \to \mathbb{R}_+, \quad e_\nu(n) = e^{-\nu n}$$

and obviously $e_\nu \in \ell^1(\mathbb{N}, \mathbb{R})$. From Remark 2.2 (ii) it follows that $e_\nu \in B$.

Let $x_0 = 0$ and let $r > 0$. Then

$$||s_{x,\theta}(n)|| \leq Ke_\nu(n)||x|| \leq Kre_\nu(n), \quad \forall n \in \mathbb{N}, \forall x \in D(0, r). \tag{3.2}$$

From (3.2) it follows that $s_{x,\theta} \in B$ and

$$|s_{x,\theta}|_B \leq Kr|e_\nu|_B, \quad \forall x \in D(0, r)$$

which completes the proof.

Sufficiency. Let $q = |\chi_{\{0\}}|_B$. Let $x_0 \in X$ and $L, r > 0$ be such that relation (3.1) holds.

Let $\theta \in \Theta$ and let $(n, x) \in \mathbb{N} \times D(x_0, r)$. From

$$\chi_{\{n\}}(j)||\Phi(\theta, n)x|| \leq s_{x,\theta}(j), \quad \forall j \in \mathbb{N}$$

we have that

$$q \; ||\Phi(\theta, n)x|| \leq |s_{x,\theta}|_B \leq L.$$

This shows that

$$||\Phi(\theta, n)x|| \leq \frac{L}{q}, \quad \forall n \in \mathbb{N}, \forall x \in D(x_0, r), \forall \theta \in \Theta. \tag{3.3}$$

Let $\theta \in \Theta$ and let $x \in X \setminus \{0\}$. Then using (3.3) we successively deduce that

$$||\Phi(\theta, n)\frac{rx}{||x||}|| \leq ||\Phi(\theta, n)(x_0 + \frac{rx}{||x||})|| + ||\Phi(\theta, n)x_0|| \leq \frac{2L}{q}.$$

This implies that

$$||\Phi(\theta, n)x|| \leq \frac{2L}{qr}||x||, \quad \forall x \in X, \forall n \in \mathbb{N}, \forall \theta \in \Theta. \tag{3.4}$$

Setting $M = (2L)/(qr)$ from (3.4) it follows that

$$||\Phi(\theta, n)|| \leq M, \quad \forall (\theta, n) \in \Theta \times \mathbb{N}. \tag{3.5}$$

Since $B \in \mathcal{V}(\mathbb{N})$ there is $p \in \mathbb{N}^*$ such that

$$F_B(p + 1) \geq \frac{2MLe}{r}. \tag{3.6}$$

Let $\theta \in \Theta$ and let $x \in D(x_0, r)$. Taking into account that

$$||\Phi(\theta, p)x|| \leq M||\Phi(\theta, n)x||, \quad \forall n \in \{0, \ldots, p\}$$

we deduce that

$$\chi_{\{0,\ldots,p\}}(n)||\Phi(\theta, p)x|| \leq Ms_{x,\theta}(n), \quad \forall n \in \mathbb{N}.$$

This implies that

$$F_B(p + 1) \; ||\Phi(\theta, p)x|| \leq ML. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$||\Phi(\theta,p)x|| \leq \frac{r}{2e}, \quad \forall x \in D(x_0,r), \forall \theta \in \Theta. \tag{3.8}$$

Let $\theta \in \Theta$ and let $x \in X \setminus \{0\}$. Using relation (3.8) we successively have that

$$||\Phi(\theta,p)\frac{rx}{||x||}|| \leq ||\Phi(\theta,p)(x_0 + \frac{rx}{||x||})|| + ||\Phi(\theta,p)x_0|| \leq \frac{r}{e}$$

which shows that

$$||\Phi(\theta,p)x|| \leq \frac{1}{e}||x||, \quad \forall (x,\theta) \in X \times \Theta.$$

This implies that

$$||\Phi(\theta,p)|| \leq \frac{1}{e}, \quad \forall \theta \in \Theta. \tag{3.9}$$

Let $\nu = 1/p$ and $K = Me$. Let $\theta \in \Theta$ and let $n \in \mathbb{N}$. Then there are $k \in \mathbb{N}$ and $j \in \{0,\ldots,p-1\}$ such that $n = kp+j$. Using relations (3.5) and (3.9) we obtain that

$$||\Phi(\theta,n)|| \leq M||\Phi(\theta,kp)|| \leq Me^{-k} \leq Ke^{-\nu n}.$$

So, the system (A) is uniformly exponentially stable. □

Remark 3.2. The above result shows that in the stability theory of variational discrete systems it is sufficient to analyze the behavior of the trajectories corresponding to vectors from a closed disk.

Next, we point out an inedit property of the Banach sequence spaces in the general class $\mathcal{Q}(\mathbb{N})$ with respect to the trajectories of a discrete dynamical system.

Proposition 3.1. *If $B \in \mathcal{Q}(\mathbb{N})$, then for every $r > 0$ the set*

$$A_r = \{x \in X : \sup_{\theta\in\Theta} |s_{x,\theta}|_B \leq r\}$$

is closed.

Proof. Let $r > 0$. If $A_r \neq \emptyset$, then for every $\theta \in \Theta$ and every $h \in \mathbb{N}$ we consider the set

$$F_r^{\theta,h} = \{x \in X : \ |s_{x,\theta} \cdot \chi_{\{0,\ldots,h\}}|_B \leq r\}.$$

Since $B \in \mathcal{Q}(\mathbb{N})$ we deduce that

$$A_r = \bigcap_{\theta\in\Theta} \bigcap_{h\in\mathbb{N}} F_r^{\theta,h}. \tag{3.10}$$

Let $(\theta,h) \in \Theta \times \mathbb{N}$. From $A_r \neq \emptyset$ and (3.10) we have that $F_r^{\theta,h} \neq \emptyset$. Let $x \in \overline{F_r^{\theta,h}}$. Then there is a sequence $(x_n) \subset F_r^{\theta,h}$ with $x_n \to x$ as $n \to \infty$.

Let $M_{\theta,h} := \max\{||\Phi(\theta,j)|| : j \in \{0,\ldots,h\}\}$. From

$$||\Phi(\theta,j)x|| \leq ||\Phi(\theta,j)x_n|| + M_{\theta,h}||x - x_n||, \quad \forall j \in \{0,\ldots,h\}, \forall n \in \mathbb{N}$$

we deduce that

$$s_{x,\theta}(j)\chi_{\{0,\ldots,h\}}(j) \leq s_{x_n,\theta}(j)\chi_{\{0,\ldots,h\}}(j) + \\ + M_{\theta,h}||x - x_n||\chi_{\{0,\ldots,h\}}(j), \quad \forall j \in \mathbb{N}, \forall n \in \mathbb{N}. \tag{3.11}$$

Relation (3.11) implies

$$|s_{x,\theta}\chi_{\{0,\dots,h\}}|_B \le |s_{x_n,\theta}\chi_{\{0,\dots,h\}}|_B + M_{\theta,h}F_B(h+1)||x-x_n||, \quad \forall n \in \mathbb{N}. \tag{3.12}$$

For $n \to \infty$ in (3.12) it follows that $|s_{x,\theta}\chi_{\{0,\dots,h\}}|_B \le r$, so $x \in F_r^{\theta,h}$. This shows that $F_r^{\theta,h}$ is closed, for all $(\theta,h) \in \Theta \times \mathbb{N}$. Then, from relation (3.10) we obtain the conclusion. □

Remark 3.3. It is interesting to point out that the above proposition may be proved for any family $\{\Phi(\theta,n)\}_{\theta\in\Theta,n\in\mathbb{N}}$ of bounded linear operators. In the proof, we didn't use any other property of the cocycle, excepting the fact that each $\Phi(\theta,n)$ is a bounded linear operator.

As a consequence of the above property we deduce the following main result:

Theorem 3.2. *Let $B \in \mathcal{V}(\mathbb{N})$. Then the system (A) is uniformly exponentially stable if and only if the set*

$$\mathcal{S} = \{x \in X : \sup_{\theta\in\Theta} |s_{x,\theta}|_B < \infty\}$$

is of the second category.

Proof. Necessity. If the system (A) is uniformly exponentially stable, then from Theorem 3.1 we have that there are $x_0 \in X$ and $r > 0$ such that $D(x_0,r) \subset \mathcal{S}$. This implies that the set $\mathcal{S}$ is of the second category.

Sufficiency. For every $n \in \mathbb{N}^*$ we consider the set

$$A_n = \{x \in X : \sup_{\theta\in\Theta} |s_{x,\theta}|_B \le n\}$$

and we have that

$$\mathcal{S} = \bigcup_{n=1}^{\infty} A_n. \tag{3.13}$$

From Proposition 3.1 we obtain that A_n is closed, for all $n \in \mathbb{N}^*$. Since $\mathcal{S}$ is a set of the second category, from relation (3.13) it follows that there is $h \in \mathbb{N}^*$ such that the interior of the set A_h is not empty, so there are $x_0 \in X$ and $r > 0$ such that $D(x_0,r) \subset A_h$. Then, we have that

$$\sup_{\theta\in\Theta} |s_{x,\theta}|_B \le h, \quad \forall x \in D(x_0,r). \tag{3.14}$$

From relation (3.14), by applying Theorem 3.1 we deduce that (A) is uniformly exponentially stable. □

Remark 3.4. The result given by the above theorem was obtained for the first time in [35] (see Theorem 2.1 in [35]).

Notation We denote by $\mathcal{F}$ the set of all nondecreasing functions $N : \mathbb{R}_+ \to \mathbb{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$.

Theorem 3.3. *The system (A) is uniformly exponentially stable if and only if there are an unbounded function $N \in \mathcal{F}$, $x_0 \in X$ and $L, r > 0$ such that*

$$\sup_{\theta\in\Theta}\sum_{n=0}^{\infty} N(||\Phi(\theta,n)x||) \leq L, \quad \forall x \in D(x_0, r).$$

Proof. Necessity. This is immediate for $N(t) = t$, for all $t \geq 0$, $x_0 = 0$ and $r > 0$.

Sufficiency. Let $N \in \mathcal{F}$ with $\lim\limits_{t\to\infty} N(t) = \infty$, $x_0 \in X$ and $L, r > 0$ be such that

$$\sup_{\theta\in\Theta}\sum_{n=0}^{\infty} N(||\Phi(\theta,n)x||) \leq L, \quad \forall x \in D(x_0, r). \tag{3.15}$$

Since N is unbounded there is $q > 0$ such that $N(q) > L$. Then from (3.15) we deduce that

$$||\Phi(\theta,n)x|| \leq q, \quad \forall x \in D(x_0, r), \quad \forall(\theta,n) \in \Theta\times\mathbb{N}. \tag{3.16}$$

Let $(\theta, n) \in \Theta \times \mathbb{N}$ and let $x \in X \setminus \{0\}$. Then, using (3.16) we have that

$$||\Phi(\theta,n)\frac{rx}{||x||}|| \leq ||\Phi(\theta,n)(x_0 + \frac{rx}{||x||})|| + ||\Phi(\theta,n)x_0|| \leq 2q.$$

This shows that

$$||\Phi(\theta,n)|| \leq \frac{2q}{r}, \quad \forall(\theta,n) \in \Theta\times\mathbb{N}. \tag{3.17}$$

We consider the function

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(t) = \begin{cases} 0, & t = 0 \\ \lim\limits_{s\nearrow t} N(s), & t > 0 \end{cases}$$

and we have that $\varphi \in \mathcal{F}$ and φ is also left-continuous. Since N is nondecreasing we have that $\varphi(t) \leq N(t)$, for all $t \geq 0$. This implies that

$$\sup_{\theta\in\Theta}\sum_{n=0}^{\infty} \varphi(||\Phi(\theta,n)x||) \leq L, \quad \forall x \in D(x_0, r). \tag{3.18}$$

Let $\ell_\varphi(\mathbb{N}, \mathbb{R})$ be the Orlicz space associated with φ and let Y_φ be the corresponding Young function. For every $(x, \theta) \in X \times \Theta$, we consider the sequence

$$s_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad s_{x,\theta}(n) = ||\Phi(\theta,n)x||.$$

Let $K := \max\{1, (2qL/r)(||x_0|| + r)\}$. Then, for every $x \in D(x_0, r)$ and $\theta \in \Theta$ we have that

$$\begin{aligned} Y_\varphi(\frac{1}{K}s_{x,\theta}(n)) &\leq \frac{1}{K}\ ||\Phi(\theta,n)x||\varphi(\frac{1}{K}\ s_{x,\theta}(n)) \leq \\ &\leq \frac{1}{L}\ \varphi(||\Phi(\theta,n)x||), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.19}$$

From relations (3.18) and (3.19) it follows that

$$M_\varphi(\frac{1}{K}s_{x,\theta}) \leq \frac{1}{L}\sum_{n=0}^{\infty}\varphi(||\Phi(\theta,n)x||) \leq 1. \tag{3.20}$$

From (3.20) we deduce that $s_{x,\theta} \in \ell_\varphi(\mathbb{N}, \mathbb{R})$ and

$$|s_{x,\theta}|_\varphi \leq K, \quad \forall\theta \in \Theta, \forall x \in D(x_0, r)$$

which means that

$$\sup_{\theta\in\Theta}|s_{x,\theta}|_\varphi \le K, \quad \forall x\in D(x_0,r). \tag{3.21}$$

According to Remark 2.4 we deduce that $\ell_\varphi(\mathbb{N},\mathbb{R})\in\mathcal{V}(\mathbb{N})$. Then, using relation (3.21) and Theorem 3.1 we obtain that the system (A) is uniformly exponentially stable. □

A version of Proposition 3.1 for the special class of continuous functions is given by the following:

Proposition 3.2. *If $N:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function, then for every $r>0$ the set*

$$B_r=\{x\in X:\sup_{\theta\in\Theta}\sum_{n=0}^{\infty}N(||\Phi(\theta,n)x||)\le r\}$$

is closed.

Proof. Let $r>0$. If $B_r\neq\emptyset$, for every $\theta\in\Theta$ and $n\in\mathbb{N}$ we consider the set

$$G_r^{\theta,n}=\{x\in X:\ \sum_{j=0}^{n}N(||\Phi(\theta,j)x||)\le r\}.$$

Let $(\theta,n)\in\Theta\times\mathbb{N}$. Let $(x_k)\subset G_r^{\theta,n}$ with $x_k\to x$.

Let $\alpha_{\theta,n}:=\max\{||\Phi(\theta,j)||:j\in\{0,\dots,n\}\}$ and let $m:=\sup_{k\in\mathbb{N}}||x_k||$. Since N is continuous on $[0,m\alpha_{\theta,n}]$, N is uniformly continuous. Let $\varepsilon>0$. Then there is $\delta>0$ such that for every $t,s\in[0,m\alpha_{\theta,n}]$ with $|t-s|<\delta$ we have

$$|N(t)-N(s)|<\frac{\varepsilon}{n+1}. \tag{3.22}$$

Since $x_k\to x$ as $k\to\infty$ there is $h\in\mathbb{N}^*$ such that

$$||x_h-x||<\frac{\delta}{\alpha_{\theta,n}}. \tag{3.23}$$

Then, using (3.23) we deduce that

$$|\ ||\Phi(\theta,j)x||-||\Phi(\theta,j)x_h||\ |\le||\Phi(\theta,j)(x-x_h)||<\delta, \quad \forall j\in\{0,\dots,n\}. \tag{3.24}$$

From (3.22) and (3.24) we obtain that

$$N(||\Phi(\theta,j)x||)\le N(||\Phi(\theta,j)x_h||)+\frac{\varepsilon}{n+1}, \quad \forall j\in\{0,\dots,n\}$$

which implies that

$$\sum_{j=0}^{n}N(||\Phi(\theta,j)x||)\le\sum_{j=0}^{n}N(||\Phi(\theta,j)x_h||)+\varepsilon. \tag{3.25}$$

Since $x_h\in G_r^{\theta,n}$ from (3.25) it follows that

$$\sum_{j=0}^{n}N(||\Phi(\theta,j)x||)\le r+\varepsilon, \quad \forall\varepsilon>0.$$

This implies that $x \in G_r^{\theta,n}$, so the set $G_r^{\theta,n}$ is closed. Hence, observing that

$$B_r = \bigcap_{\theta\in\Theta}\bigcap_{n\in\mathbb{N}} G_r^{\theta,n}$$

we deduce the conclusion. □

Notation We denote by $\mathcal{C}$ the set of all continuous functions $N \in \mathcal{F}$.

Corollary 3.1. *The system (A) is uniformly exponentially stable if and only if there is an unbounded function $N \in \mathcal{C}$ such that the set*

$$\Gamma = \{x \in X : \sup_{\theta\in\Theta}\sum_{n=0}^{\infty} N(||\Phi(\theta,n)x||) < \infty\}$$

is of the second category.

Proof. Necessity. Taking $N(t) = t$, for all $t \geq 0$ we obtain that $\Gamma = X$.

Sufficiency. For every $j \in \mathbb{N}^*$ we consider the set

$$B_j = \{x \in X : \sup_{\theta\in\Theta}\sum_{n=0}^{\infty} N(||\Phi(\theta,n)x||) \leq j\}$$

and we have that

$$\Gamma = \bigcup_{j=1}^{\infty} B_j. \tag{3.26}$$

From Proposition 3.2 we have that B_j is closed, for all $j \in \mathbb{N}^*$. Since Γ is a set of the second category, from (3.26) it follows that there are $p \in \mathbb{N}^*$, $x_0 \in X$ and $r > 0$ such that $D(x_0, r) \subset B_p$. Then by applying Theorem 3.3 we obtain that (A) is uniformly exponentially stable. □

Remark 3.5. The result given by Corollary 3.1 was proved for the first time in [37] (see Theorem 3.4 in [37]).

An interesting consequence in the nonlinear setting is the following:

Corollary 3.2. *The system (A) is uniformly exponentially stable if and only if there is a function $f : \mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}_+$ with $f(n,\cdot) \in \mathcal{C}$ and $\lim_{t\to\infty} f(n,t) = \infty$, for all $n \in \mathbb{N}$ such that the set*

$$\Delta = \{x \in X : (\exists)\alpha(x) \in \mathbb{N} \ \ \textit{such that} \ \ \sup_{\theta\in\Theta}\sum_{n=0}^{\infty} f(\alpha(x), ||\Phi(\theta,n)x||) < \infty\}$$

is of the second category.

Proof. Necessity. Taking $f(n,t) = t$, for all $(n,t) \in \mathbb{N}\times\mathbb{R}_+$ it follows that $\Delta = X$.

Sufficiency. This immediately follows from Corollary 3.1 using the same idea like in the proof of Theorem 3.5 in [37]. □

Remark 3.6. It worth mentioning that all the results presented above are obtained *without any assumption concerning the cocycle* Φ excepting the fact that this satisfies the cocycle identity (see Remark 3.1). In fact, it is interesting to observe that the cocycle was not supposed to have a uniform or nonuniform exponential growth. But, for all that, the conditions considered in our study imply the existence of a *uniform* concept of exponential stability.

4. Exponential stability of variational difference equations with bounded coefficients

As we have already mentioned, the results presented in the previous section were obtained for the most general case of variational difference equations without any requirement concerning the coefficients. But, many natural phenomenons are modeled by systems with uniformly bounded coefficients and often, for difference equations, this working hypothesis appears as a natural one (see [8]). In what follows we shall consider this case and discuss several notable situations.

Let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the linear system of variational difference equations

$$(A) \qquad x(\theta)(n+1) = A(\sigma(\theta,n))x(\theta)(n), \quad \forall(\theta,n) \in \Theta \times \mathbb{N}.$$

In what follows we will work in the hypotheses that

$$\sup_{\theta\in\Theta} ||A(\theta)|| < \infty \tag{4.1}$$

i.e. the system (A) has uniformly bounded coefficients.

Remark 4.1. Relation (4.1) is equivalent with the existence of two constants $M, \omega > 0$ such that

$$||\Phi(\theta,n)|| \leq Me^{\omega n}, \quad \forall(\theta,n) \in \Theta \times \mathbb{N}.$$

Theorem 4.1. *Let $B \in \mathcal{V}(\mathbb{N})$ be a Banach sequence space. Then the following assertions are equivalent:*

(i) the system (A) is uniformly exponentially stable;

(ii) there is a sequence $(k_n) \subset \mathbb{N}$ with $\sup_{n\in\mathbb{N}} |k_{n+1} - k_n| < \infty$, $x_0 \in X$ and $L, r > 0$ such that

$$\sup_{\theta\in\Theta} |u_{x,\theta}|_B \leq L, \quad \forall x \in D(x_0, r),$$

where for every $(x,\theta) \in X \times \Theta$

$$u_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad u_{x,\theta}(n) = ||\Phi(\theta, k_n)x||.$$

Proof. $(i) \Longrightarrow (ii)$ This follows from Theorem 3.1 for $k_n = n$, for all $n \in \mathbb{N}$.

$(ii) \Longrightarrow (i)$ Let $M, \omega > 0$ be given by Remark 4.1. Then

$$||\Phi(\theta,n)|| \leq Me^{\omega n}, \quad \forall(\theta,n) \in \Theta \times \mathbb{N}.$$

Case 1. If (k_n) is bounded we set $h = \sup_{n\in\mathbb{N}} k_n$.

Let $\theta \in \Theta$ and $x \in D(x_0, r)$. Since

$$||\Phi(\theta, h)x|| \leq Me^{\omega h}||\Phi(\theta, k_j)x||, \quad \forall j \in \mathbb{N}$$

we deduce that

$$\chi_{\{0,\ldots,n\}}(j)||\Phi(\theta, h)\frac{x}{\lambda}|| \leq u_{x,\theta}(j), \quad \forall j \in \mathbb{N} \tag{4.2}$$

where $\lambda = Me^{\omega h}$. From relation (4.2) we obtain that

$$F_B(n+1)||\Phi(\theta, h)\frac{x}{\lambda}|| \leq |u_{x,\theta}|_B \leq L, \quad \forall n \in \mathbb{N}.$$

Taking into account that $B \in \mathcal{V}(\mathbb{N})$, this implies that

$$\Phi(\theta, h)x = 0, \quad \forall x \in D(x_0, r), \forall \theta \in \Theta. \tag{4.3}$$

Let $\theta \in \Theta$ and let $x \in X \setminus \{0\}$. Using relation (4.3) we deduce that

$$\Phi(\theta, h)\frac{rx}{||x||} = \Phi(\theta, h)\left(x_0 + \frac{rx}{||x||}\right) - \Phi(\theta, h)x_0 = 0.$$

It follows that $\Phi(\theta, h) = 0$, for all $\theta \in \Theta$. This shows that the system (A) is uniformly exponentially stable.

Case 2. If (k_n) is unbounded then without loss of generality we may assume that (k_n) is a nondecreasing sequence (if not, we consider a subsequence with this property and the proof is analogous).

Let $q = |\chi_{\{0\}}|_B$ and let $l = \sup_{n\in\mathbb{N}}(k_{n+1} - k_n)$. Using the hypothesis and similar arguments as in the proof of Theorem 3.1 we obtain that

$$||\Phi(\theta, k_n)|| \leq \frac{2L}{qr}, \quad \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{4.4}$$

Let $n \in \mathbb{N}$. If $n \in \{0, \ldots, k_0\}$ then using relation (4.1) we have that

$$||\Phi(\theta, n)|| \leq Me^{\omega n} \leq Me^{\omega k_0}. \tag{4.5}$$

If $n \geq k_0$ then there is $j \in \mathbb{N}$ such that $k_j \leq n \leq k_{j+1}$. Then

$$||\Phi(\theta, n)|| \leq Me^{\omega(n-k_j)}||\Phi(\theta, k_j)|| \leq Me^{\omega l}\ \frac{2L}{qr}. \tag{4.6}$$

Setting $\gamma := \max\{Me^{\omega k_0}, Me^{\omega l}(2L/qr)\}$, from relations (4.5) and (4.6) it follows that

$$||\Phi(\theta, n)|| \leq \gamma, \quad \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{4.7}$$

Using relation (4.7) and similar arguments as in the proof of Theorem 3.1 we obtain that there is $p \in \mathbb{N}$ such that

$$||\Phi(\theta, k_p)|| \leq \frac{1}{e}, \quad \forall \theta \in \Theta. \tag{4.8}$$

From relations (4.7) and (4.8) and using analogous arguments with those in the proof of Theorem 3.1 we deduce that (A) is uniformly exponentially stable.

□

Proposition 4.1. *Let $B \in \mathcal{Q}(\mathbb{N})$ and let $(k_n) \subset \mathbb{N}$. For every $(x,\theta) \in X \times \Theta$, we consider the sequence*

$$u_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad u_{x,\theta}(n) = ||\Phi(\theta, k_n)x||.$$

Then for every $r > 0$ the set

$$D_r = \{x \in X : \sup_{\theta\in\Theta} |u_{x,\theta}|_B \le r\}$$

is closed.

Proof. This follows using similar arguments with those in the proof of Proposition 3.1. □

Corollary 4.1. *Let $B \in \mathcal{V}(\mathbb{N})$ be a Banach sequence space. The system (A) is uniformly exponentially stable if and only if there is a sequence $(k_n) \subset \mathbb{N}$ with $\sup_{n\in\mathbb{N}} |k_{n+1} - k_n| < \infty$ such that*

$$\mathcal{U} = \{x \in X : \sup_{\theta\in\Theta} |u_{x,\theta}|_B < \infty\}$$

is a set of the second category, where for every $(x,\theta) \in X \times \Theta$

$$u_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad u_{x,\theta}(n) = ||\Phi(\theta, k_n)x||.$$

Proof. Necessity is immediate from Theorem 3.2, taking $k_n = n$, for all $n \in \mathbb{N}$.

Sufficiency follows from Theorem 4.1 and Proposition 4.1, using similar arguments with those used in the proof of Theorem 3.2. □

Remark 4.2. A distinct proof of Corollary 4.1 was presented in [35] (see Corollary 2.1 in [35]).

Let $\mathcal{F}$ be the set of all non-decreasing functions $N : \mathbb{R}_+ \to \mathbb{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$.

Theorem 4.2. *The system (A) is uniformly exponentially stable if and only if there are $(k_n) \subset \mathbb{N}$ with $\sup_{n\in\mathbb{N}} |k_{n+1} - k_n| < \infty$, a function $N \in \mathcal{F}$, $x_0 \in X$ and $L, r > 0$ such that*

$$\sup_{\theta\in\Theta} \sum_{n=0}^{\infty} N(||\Phi(\theta, k_n)x||) \le L, \quad \forall x \in D(x_0, r).$$

Proof. Necessity is immediate for $x_0 = 0$, $r > 0$, $N(t) = t$, for all $t \ge 0$ and $k_n = n$, for all $n \in \mathbb{N}$.

Sufficiency. Let $M, \omega > 0$ be given by Remark 4.1. Then

$$||\Phi(\theta, n)|| \le Me^{\omega n}, \quad \forall(\theta, n) \in \Theta \times \mathbb{N}.$$

Case 1. If (k_n) is bounded, let $h = \sup_{n\in\mathbb{N}} k_n$ and let $q = Me^{\omega h}$. Then, for every $x \in D(x_0, r)$ we have that

$$(n+1)N(||\Phi(\theta, h)\frac{x}{q}||) \le \sum_{k=0}^{n} N(||\Phi(\theta, k_n)x||) \le L, \quad \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{4.9}$$

From (4.9) it follows that $\Phi(\theta, h)x = 0$, for all $x \in D(x_0, r)$.

Let $x \in X \setminus \{0\}$ and let $x_1 = rx/||x||$. Then $\Phi(\theta, h)x_1 = \Phi(\theta, h)(x_0 + x_1) - \Phi(\theta, h)x_0 = 0$. This implies that $\Phi(\theta, h) = 0$, for all $\theta \in \Theta$, so (A) is uniformly exponentially stable.

Case 2. If $\sup_{n\in\mathbb{N}} k_n = \infty$ without loss of generality, we may assume that (k_n) is nondecreasing (if not we consider a subsequence with this property and the proof is analogous).

Let $l = \sup_{n\in\mathbb{N}}(k_{n+1} - k_n)$ and let $n_0 \in \mathbb{N}^*$ be such that $L < n_0 N(1)$. Let $\lambda = Me^{\omega n_0 l}$.

Let $\theta \in \Theta$ and let $x \in D(x_0, r)$. For every $n \geq n_0$ we have that

$$\begin{aligned} ||\Phi(\theta, k_n)\frac{x}{\lambda}|| &\leq \frac{1}{\lambda}\ Me^{\omega(k_n - k_j)}||\Phi(\theta, k_j)x|| \leq \\ &\leq ||\Phi(\theta, k_j)x||, \quad \forall j \in \{n - n_0 + 1, \ldots, n\}. \end{aligned}$$

This implies that

$$n_0 N(||\Phi(\theta, k_n)\frac{x}{\lambda}||) \leq \sum_{j=n-n_0+1}^{n} N(||\Phi(\theta, k_j)x||) \leq L. \tag{4.10}$$

From relation (4.10) it follows that

$$||\Phi(\theta, k_n)x|| \leq \lambda, \quad \forall x \in D(x_0, r), \forall \theta \in \Theta, \forall n \geq n_0. \tag{4.11}$$

Let $\theta \in \Theta$ and let $n \geq n_0$. Let $x \in X \setminus \{0\}$. Then using (4.11) we deduce that

$$||\Phi(\theta, k_n)\frac{rx}{||x||}|| \leq ||\Phi(\theta, k_n)(x_0 + \frac{rx}{||x||})|| + ||\Phi(\theta, k_n)x_0|| \leq 2\lambda.$$

This implies that

$$||\Phi(\theta, k_n)|| \leq \frac{2\lambda}{r}, \quad \forall n \geq n_0, \forall \theta \in \Theta. \tag{4.12}$$

Let $j \in \mathbb{N}$. If $j \geq k_{n_0}$, then there is $n \geq n_0$ such that $k_n \leq j \leq k_{n+1}$. Using (4.12) we have that

$$||\Phi(\theta, j)|| \leq Me^{\omega(j-k_n)}||\Phi(\theta, k_n)|| \leq Me^{\omega l}\frac{2\lambda}{r}. \tag{4.13}$$

If $j \in \{0, \ldots, k_{n_0}\}$ then

$$||\Phi(\theta, j)|| \leq Me^{\omega j} \leq Me^{\omega k_{n_0}}. \tag{4.14}$$

From (4.17) and (4.18) it follows that there is $\gamma > 0$ such that

$$||\Phi(\theta, n)|| \leq \gamma, \quad \forall (\theta, n) \in \Theta \times \mathbb{N}.$$

Let

$$\varphi : \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(t) = \begin{cases} 0, & t = 0 \\ \lim\limits_{s \nearrow t} N(s), & t > 0 \end{cases}.$$

Then we have that $\varphi \in \mathcal{F}$, φ is left-continuous and

$$\sup_{\theta\in\Theta} \sum_{n=0}^{\infty} \varphi(||\Phi(\theta, k_n)x||) \leq L, \quad \forall x \in D(x_0, r).$$

For every $(x, \theta) \in X \times \Theta$ let

$$u_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad u_{x,\theta}(n) = ||\Phi(\theta, k_n)x||.$$

Let $\ell_\varphi(\mathbb{N}, \mathbb{R})$ be the Orlicz space associated to φ. Using similar arguments with those in the proof of Theorem 3.3 we deduce that there is $\alpha > 0$ such that

$$\sup_{\theta\in\Theta} |u_{x,\theta}|_\varphi \leq \alpha, \quad \forall x \in D(x_0, r). \tag{4.15}$$

Since $\varphi \in \mathcal{F}$ using Remark 2.4 we have that $\ell_\varphi(\mathbb{N}, \mathbb{R}) \in \mathcal{V}(\mathbb{N})$. Then from (4.15) and by applying Theorem 4.1 it follows that the system (A) is uniformly exponentially stable. □

Remark 4.3. We note that the proof line of the above theorem develops for the case of discrete dynamical systems the method used in [32] for a real-time characterization of the exponential stability of skew-product flows.

In what follows we denote by $\mathcal{C}$ the set of all continuous functions $N \in \mathcal{F}$.

Theorem 4.3. *The system (A) is uniformly exponentially stable if and only if there are a sequence $(k_n) \subset \mathbb{N}$ with $\sup_{n\in\mathbb{N}} |k_{n+1} - k_n| < \infty$ and a function $N \in \mathcal{C}$ such that the set*

$$\Lambda = \{x \in X : \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} N(||\Phi(\theta, k_j)x||) < \infty\}$$

is of the second category.

Proof. *Necessity* is immediate.

Sufficiency. Using the continuity of N and similar arguments with those used in the proof of Proposition 3.2 we have that the set

$$K_r = \{x \in X : \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} N(||\Phi(\theta, k_j)x||) \leq r\}$$

is closed. Using this fact, similar arguments like in the proof of Corollary 3.1 and by applying Theorem 4.2 we obtain the conclusion. □

Remark 4.4. Theorem 4.3 extends Theorem 3.6 in [37]. Moreover, as an immediate consequence of Theorem 4.3 we may deduce Theorem 3.7 in [37].

5. Applications: exponential stability of skew-product flows

In this section we will present several consequences of the main results to the case of skew-product flows. On the one hand we will point out some new methods in order to deduce previously obtained results and on the other hand we deduce some new and interesting conclusions.

Let X be a real or complex Banach space, let (Θ, d) be a metric space, let $J \in \{\mathbb{R}_+, \mathbb{R}\}$. We denote by I_d the identity operator on X.

Definition 5.1. A continuous mapping $\sigma : \Theta \times J \to \Theta$ is called a *flow* on Θ if $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, s+t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times J^2$.

Definition 5.2. A dynamical system $\pi = (\Phi, \sigma)$ is called a *skew-product flow* on $\mathcal{E} = X \times \Theta$ if σ is a flow on Θ and the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$;

(ii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, s), t)\Phi(\theta, s)$, for all $(\theta, t, s) \in \Theta \times \mathbf{R}^2_+$ (*the cocycle identity*);

(iii) there are $M \geq 1$ and $\omega > 0$ such that $||\Phi(\theta, t)|| \leq Me^{\omega t}$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

The mapping Φ is called a *cocycle* over the flow σ.

Example 5.1. Let $J \in \{\mathbb{R}_+, \mathbb{R}\}$ and let $\Theta = J$. We consider the translation flow $\sigma : \Theta \times J \to \Theta, \sigma(\theta, t) = \theta + t$. If X is a Banach space and $\mathcal{U} = \{U(t,s)\}_{t \geq s, t, s \in J}$ is a strongly continuous evolution family on X, then

$$\Phi : \mathbb{R} \times \mathbb{R}_+ \to \mathcal{L}(X), \quad \Phi_{\mathcal{U}}(\theta, t) = U(\theta + t, \theta)$$

is a cocycle over the flow σ. Usually, $\pi_{\mathcal{U}} = (\Phi_{\mathcal{U}}, \sigma)$ is called *the skew-product flow associated to $\mathcal{U}$*. $\diamond$

Example 5.2. (*The variational equation*) Let Θ be a locally compact metric space, let σ be a flow on Θ and let $\{A(\theta)\}_{\theta \in \Theta}$ be a family of densely defined closed operators on a Banach space X. We consider the variational equation

$$(A) \qquad \dot{x}(t) = A(\sigma(\theta, t))x(t), \quad (\theta, t) \in \Theta \times \mathbb{R}_+.$$

A cocycle $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X)$ over the flow σ is *a solution of the equation* (A) if for every $\theta \in \Theta$, there is a dense subset $D_\theta \subset D(A(\theta))$ such that for every $x_\theta \in D_\theta$ the function $t \mapsto x(t) := \Phi(\theta, t)x_\theta$ is differentiable on $\mathbb{R}_+$, $x(t) \in D(A(\sigma(\theta, t)))$, for every $t \in \mathbb{R}_+$, and the mapping $t \mapsto x(t)$ satisfies the equation (A). $\diamond$

For other examples of skew-product flows we refer to [3]–[5] and [43].

Definition 5.3. A skew-product flow is said to be *uniformly exponentially stable* if there are $K \geq 1$ and $\nu > 0$ such that

$$||\Phi(\theta, t)|| \leq Ke^{-\nu t}, \quad \forall t \geq 0, \forall \theta \in \Theta.$$

Example 5.3. Let $\beta > \alpha > 0$ and let $a : \mathbb{R} \to [\alpha, \beta]$ be a continuous function and for every $s \in \mathbb{R}$, let $a_s : \mathbb{R} \to [\alpha, \beta], a_s(t) = a(t+s)$. Let $\Theta := \{a_s : s \in \mathbb{R}\}$. On Θ we consider the metric

$$d(\theta, \tilde{\theta}) = \sup_{s \in \mathbb{R}} |\theta(s) - \tilde{\theta}(s)|.$$

Then the mapping

$$\sigma : \Theta \times \mathbb{R} \to \Theta, \quad \sigma(\theta, t)(s) := \theta(t+s)$$

is a flow on Θ.

Let X be a Banach space and let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X with the infinitesimal generator $A : D(A) \subset X \to X$. For every $\theta \in \Theta$ let $A(\theta) := \theta(0)A$. We consider the variational equation

$$(A; x_0) \qquad \begin{cases} \dot{x}(t) = A(\sigma(\theta,t))x(t), & t \geq 0 \\ x(0) = x_0 \end{cases}$$

with $x_0 \in D(A)$. Let

$$\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{L}(X), \quad \Phi(\theta,t)x = T(\int_0^t \theta(s)\ ds)x$$

which is a cocycle over the flow σ. It is easy to see that for every $x_0 \in D(A)$, the function $x(t) := \Phi(\theta,t)x_0$, for all $t \geq 0$, is the strong solution of the equation $(A; x_0)$.

Then $\pi = (\Phi, \sigma)$ is a skew-product flow on $\mathcal{E} = X \times \Theta$. Moreover, if $\mathbf{T}$ is uniformly exponentially stable, then π is uniformly exponentially stable. $\diamond$

Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. For every $\theta \in \Theta$ we consider the operator $A(\theta) = \Phi(\theta, 1)$. If $M, \omega > 0$ are given by Definition 5.2 (iii), then we have that

$$\sup_{\theta\in\Theta} ||A(\theta)|| \leq Me^{\omega}.$$

We consider the variational discrete dynamical system

$$(A_\pi) \qquad x(\theta)(n+1) = A(\sigma(\theta,n))x(\theta)(n), \quad \forall(\theta,n) \in \Theta \times \mathbb{N}.$$

If $\{\Phi_{A_\pi}(\theta,n)\}_{(\theta,n)\in\Theta\times\mathbb{N}}$ is the discrete cocycle associated with (A_π) we observe that

$$\Phi_{A_\pi}(\theta,n) = \Phi(\theta,n), \quad \forall(\theta,n).$$

Remark 5.1. A skew-product $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if the system (A_π) is uniformly exponentially stable.

Using this remark and the results obtained in the previous section, we deduce the following characterizations for the exponential stability of skew-product flows:

Theorem 5.1. *Let B be a Banach sequence space with $B \in \mathcal{V}(\mathbb{N})$ and let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there are a sequence $(t_n) \subset \mathbb{R}_+$ with $\sup_{n\in\mathbb{N}} |t_{n+1} - t_n| < \infty$, $x_0 \in X$ and $L, r > 0$ such that*

$$\sup_{\theta\in\Theta} |v_{x,\theta}|_B \leq L, \quad \forall x \in D(x_0, r),$$

where for every $(x,\theta) \in X \times \Theta$

$$v_{x,\theta} : \mathbb{N} \to \mathbb{R}, \quad v_{x,\theta}(n) = ||\Phi(\theta, t_n)x||.$$

Proof. Necessity is immediate.

Sufficiency. Let $k_n = [t_n] + 1$, for all $n \in \mathbb{N}$. By applying Theorem 4.1 and Remark 5.1 we obtain that π is uniformly exponentially stable. $\square$

Theorem 5.2. *Let B be a Banach sequence space with $B \in \mathcal{V}(\mathbb{N})$ and let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there is a sequence $(t_n) \subset \mathbb{R}_+$ with $\sup_{n\in\mathbb{N}} |t_{n+1} - t_n| < \infty$ such that the set*

$$\mathcal{V} = \{x \in X : \sup_{\theta\in\Theta} |v_{x,\theta}|_B < \infty\}$$

is of the second category, where for every $(x, \theta) \in X \times \Theta$

$$v_{x,\theta} : \mathbb{N} \to \mathbb{R}, \quad v_{x,\theta}(n) = ||\Phi(\theta, t_n)x||.$$

Proof. *Necessity* is immediate.

Sufficiency. Let $k_n = [t_n]+1$, for all $n \in \mathbb{N}$. By applying Corollary 4.1 and Remark 5.1 we obtain the conclusion. $\square$

Remark 5.2. A distinct approach for Theorem 5.2 was given in [35] (see Theorem 3.1 in [35]).

Let $\mathcal{F}$ denote the set of all non-decreasing functions $N : \mathbb{R}_+ \to \mathbb{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$.

Theorem 5.3. *A skew-product flow $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there are a sequence $(t_n) \subset \mathbb{R}_+$ with $\sup_{n\in\mathbb{N}} |t_{n+1} - t_n| < \infty$, a function $N \in \mathcal{F}$, $x_0 \in X$ and $L, r > 0$ such that*

$$\sup_{\theta\in\Theta} \sum_{n=0}^{\infty} N(||\Phi(\theta, t_n)x||) \le L, \quad \forall x \in D(x_0, r).$$

Proof. *Necessity* is immediate.

Sufficiency. Let $M, \omega > 0$ be given by Definition 5.2 (iii). For every $n \in \mathbb{N}$, let $k_n = [t_n] + 1$. Then

$$||\Phi(\theta, k_n)x|| \le Me^{\omega}||\Phi(\theta, t_n)x||, \quad \forall(x, \theta) \in X \times \Theta, \forall n \in \mathbb{N}. \tag{5.1}$$

We consider the function

$$\tilde{N} : \mathbb{R}_+ \to \mathbb{R}_+, \quad \tilde{N}(t) = N(\frac{t}{Me^{\omega}}).$$

Then $\tilde{N} \in \mathcal{F}$ and using relation (5.1) we deduce that

$$\sup_{\theta\in\Theta} \sum_{n=0}^{\infty} \tilde{N}(||\Phi(\theta, k_n)x||) \le L, \quad \forall x \in D(x_0, r).$$

By applying Theorem 4.2 and Remark 5.1 it follows that π is uniformly exponentially stable. $\square$

Remark 5.3. A direct proof of Theorem 5.3 was presented in [32] (see Theorem 3.1 in [32]).

We denote by $\mathcal{C}$ the set of all continuous functions $N \in \mathcal{F}$.

Theorem 5.4. *A skew-product flow $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there are a sequence $(t_n) \subset \mathbb{R}_+$ with $\sup_{n\in\mathbb{N}} |t_{n+1} - t_n| < \infty$ and a function $N \in \mathcal{C}$ such that the set*

$$\mathcal{Z} = \{x \in X : \sup_{\theta\in\Theta} \sum_{j=0}^{\infty} N(||\Phi(\theta, t_j)x||) < \infty\}$$

is of the second category.

Proof. Necessity is immediate.

Sufficiency. This follows from Theorem 4.3 and Remark 5.1, using similar arguments with those in the proof of Theorem 5.3. □

Remark 5.4. Theorem 5.4 extends the main idea from Theorem 4.3 in [37].

Theorem 5.5. *Let $N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function such that for every $t > 0, N(t, \cdot) \in \mathcal{C}$ and for every $s \geq 0, N(\cdot, s)$ is nondecreasing. For every $m \in \mathbb{N}^*$ let $(t_n^m)_{n\in\mathbb{N}}$ be a sequence such that*

$$\sup_{n\in\mathbb{N}} |t_{n+1}^m - t_n^m| < \infty.$$

Let $\pi = (\Phi, \sigma)$ be a skew-product flow. If for every $x \in X$ there is $\alpha(x) > 0$ and $m_x \in \mathbb{N}^*$ such that

$$\sup_{\theta\in\Theta} \sum_{n=0}^{\infty} N(\alpha(x), ||\Phi(\theta, t_n^{m_x})x||) < \infty$$

then π is uniformly exponentially stable.

Proof. This follows from Theorem 5.3 using similar arguments with those in the proof of Theorem 3.3 in [32]. □

Corollary 5.1. *Let $N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function such that for every $t > 0, N(t, \cdot) \in \mathcal{C}$ and for every $s \geq 0, N(\cdot, s)$ is nondecreasing. Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E}$. If for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\sup_{\theta\in\Theta} \sum_{n=0}^{\infty} N(\alpha(x), ||\Phi(\theta, n)x||) < \infty$$

then π is uniformly exponentially stable.

Proof. This immediately follows from Theorem 5.5 for $t_n^m = n$, for all $(m, n) \in \mathbb{N}^* \times \mathbb{N}$. □

Definition 5.4. A skew-product flow is said to be *strongly continuous* if for every $(\theta, x) \in \Theta \times X$, the mapping $t \mapsto \Phi(\theta, t)x$ is continuous.

Corollary 5.2. *Let $N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ be a function such that for every $t > 0, N(t, \cdot) \in \mathcal{C}$ and for every $s \geq 0, N(\cdot, s)$ is nondecreasing. Let $\pi = (\Phi, \sigma)$ be a strongly continuous skew-product flow. If for every $x \in X$ there is $\alpha(x) > 0$ such that*

$$\sup_{\theta\in\Theta} \int_0^{\infty} N(\alpha(x), ||\Phi(\theta, t)x||)dt < \infty$$

then π is uniformly exponentially stable.

Proof. This follows a relatively standard idea of passing from a discrete-time characterization to a continuous-type one (see also [32], [35], [37]). We present here the idea from [32]. Indeed, let $M, \omega > 0$ be given by Definition 5.2 (iii) and let

$$\tilde{N} : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}_+, \quad \tilde{N}(t, \tau) = N(t, \frac{\tau}{Me^{\omega}}).$$

Then $\tilde{N}$ preserves the qualitative properties of the function N. Let $x \in X$. Then it is easy to see that

$$\tilde{N}(\alpha(x), ||\Phi(\theta, n)x||) \leq N(\alpha(x), ||\Phi(\theta, s)x||), \quad \forall s \in [n-1, n], \forall n \in \mathbb{N}^*, \forall \theta \in \Theta.$$

which yields

$$\tilde{N}(\alpha(x), ||\Phi(\theta, n)x||) \leq \int_{n-1}^{n} N(\alpha(x), ||\Phi(\theta, s)x||) ds, \quad \forall n \in \mathbb{N}^*, \forall \theta \in \Theta.$$

It follows that

$$\sup_{\theta \in \Theta} \sum_{n=0}^{\infty} \tilde{N}(\alpha(x), ||\Phi(\theta, n)x||) \leq \tilde{N}(\alpha(x), ||x||) + \sup_{\theta \in \Theta} \int_0^{\infty} N(\alpha(x), ||\Phi(\theta, t)x||) dt.$$

Hence according to our hypothesis and by applying Corollary 5.1 we deduce the conclusion. □

Remark 5.5. A different proof of Corollary 5.2 was given in [34] (see Theorem 3.3 in [34]). The results given by Corollaries 5.1 and 5.2 were obtained for the first time in [32] (see Corollary 3.1 and Theorem 3.4 in [32]). A version of Corollary 5.2 was deduced in [37] (see Theorem 4.5 in [37]). It worth mentioning that Corollary 5.1 generalizes Theorem 3.2 in [12] and Corollary 5.2 extends to a more general situation the result obtained in Theorem 3.4 in [12].

Remark 5.6. Corollary 5.2 represents the generalization of the theorem of Rolewicz (Theorem 1.3) to the more general case of skew-product flows. In fact, by applying Corollary 5.2 for the skew-product flow $\pi_{\mathcal{U}} = (\Phi_{\mathcal{U}}, \sigma)$ introduced in Example 5.1, we obtain as particular case the theorem of Rolewicz.

6. A direct proof for the Rolewicz type theorem on the stability of skew-product flows

In contrast with the directions promoted in the papers [11], [12] and [32], [34], [35], where exponential stability was characterized in terms of various classes of Banach function spaces and consequently the Rolewicz type theorems were deduced by adapting the conclusions of the main results to the case of Orlicz spaces or, respectively, by employing discrete-time arguments, in the present section we propose a direct procedure for the study of the exponential stability of skew-product flows, without requiring additional properties of the associated trajectories. In fact, we do not even work with trajectories or with function spaces, our aim being to provide a constructive and elegant method for the stability theorems of Rolewicz type in the variational case. We present a natural approach for this type of theorems in the continuous-time setting, improving and clarifying the proof lines from our previous works. We discuss the technical tools involved with the advantages of each method as well as possible directions of generalization to other classes of systems.

Let X be a real or complex Banach space and let (Θ, d) be a metric space. Denote by $\mathcal{E} = X \times \Theta$. In what follows we will formulate characterizations for uniform exponential stability of skew-product flows on $\mathcal{E}$.

Theorem 6.1. *A strongly continuous skew-product flow $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there exist a function $N \in \mathcal{F}, x_0 \in X$ and $r > 0$ such that*

$$\sup_{\theta \in \Theta} \sup_{x \in D(x_0, r)} \int_0^\infty N(||\Phi(\theta, t)x||)\ dt < \infty.$$

Proof. Necessity. This follows for $N(t) = t$, for all $t \geq 0$, $x_0 = 0$ and $r > 0$.

Sufficiency. Let $N \in \mathcal{F}$, $x_0 \in X$ and $r > 0$ be given by our hypothesis and let $K > 0$ be such that

$$\int_0^\infty N(||\Phi(\theta, t)x||)\ dt \leq K, \quad \forall (x, \theta) \in D(x_0, r) \times \Theta. \tag{6.1}$$

Step 1. Let $M, \omega > 0$ be given by Definition 5.2 (iii). Let $h > 0$ be such that $h > (K/N(1))$ and let $\gamma = Me^{\omega h}$. Let $\theta \in \Theta$ and let $t \geq h$. Let $x \in D(x_0, r)$. Taking into account that

$$||\Phi(\theta, t)x|| \leq \gamma\ ||\Phi(\theta, s)x||, \quad \forall s \in [t - h, t] \tag{6.2}$$

and that N is nondecreasing, from relations (6.2) and respectively (6.1) we obtain that

$$N(\frac{||\Phi(\theta, t)x||}{\gamma}) \leq \frac{1}{h} \int_{t-h}^t N(||\Phi(\theta, s)x||)\ ds \leq \frac{K}{h} < N(1). \tag{6.3}$$

Since N is nondecreasing from (6.3) we have that

$$||\Phi(\theta, t)x|| \leq \gamma, \quad \forall x \in D(x_0, r). \tag{6.4}$$

Let $x \in X \setminus \{0\}$. Then, using (6.4) we deduce that

$$||\Phi(\theta, t)\frac{rx}{||x||}|| \leq ||\Phi(\theta, t)(x_0 + \frac{rx}{||x||})|| + ||\Phi(\theta, t)x_0|| \leq 2\gamma.$$

This implies that

$$||\Phi(\theta, t)x|| \leq \frac{2\gamma}{r}\ ||x||, \quad \forall x \in X, \forall t \geq h, \forall \theta \in \Theta$$

which yields

$$||\Phi(\theta, t)|| \leq \frac{2\gamma}{r}, \quad \forall t \geq h, \forall \theta \in \Theta. \tag{6.5}$$

Since

$$||\Phi(\theta, t)|| \leq \gamma, \quad \forall t \in [0, h], \forall \theta \in \Theta$$

setting $L := \max\{\gamma, (2\gamma/r)\}$, from relation (6.5) and the above estimation it follows that

$$||\Phi(\theta, t)|| \leq L, \quad \forall t \geq 0, \forall \theta \in \Theta. \tag{6.6}$$

Step 2. Let $p > 0$ be such that $p > K/N(r/2eL)$. Let $\theta \in \Theta$ and let $x \in D(\frac{x_0}{L}, \frac{r}{L})$. Using relation (6.6) we have that

$$||\Phi(\theta, p)x|| \leq ||\Phi(\theta, t)Lx||, \quad \forall t \in [0, p]$$

which implies that

$$pN(||\Phi(\theta, p)x||) \leq \int_0^p N(||\Phi(\theta, t)Lx||)\, dt \leq K. \tag{6.7}$$

Using relation (6.7) and the fact that N is nondecreasing we obtain that

$$||\Phi(\theta, p)x|| \leq \frac{r}{2eL}, \quad \forall x \in D(\frac{x_0}{L}, \frac{r}{L}). \tag{6.8}$$

Let $x \in X \setminus \{0\}$. Then using (6.8) we have that

$$||\Phi(\theta, p)\frac{rx}{L||x||}|| \leq ||\Phi(\theta, p)(\frac{x_0}{L} + \frac{rx}{L||x||})|| + ||\Phi(\theta, p)\frac{x_0}{L}|| \leq \frac{r}{eL}. \tag{6.9}$$

From relation (6.9) it follows that

$$||\Phi(\theta, p)x|| \leq \frac{1}{e}\, ||x||, \quad \forall x \in X, \forall \theta \in \Theta. \tag{6.10}$$

Let $\nu = 1/p$ and let $K = Le$. Let $\theta \in \Theta$ and let $t \geq 0$. Then there are $k \in \mathbb{N}$ and $s \in [0, p)$ such that $t = kp + s$. Using relations (6.6) and (6.10) we obtain that

$$||\Phi(\theta, t)|| \leq L\, ||\Phi(\theta, kp)|| \leq Le^{-k} \leq Ke^{-\nu t}. \qquad \square$$

Remark 6.1. The above result was obtained for the first time in [34] using the theory of Banach function spaces and the method of the membership of the associated trajectories to certain function spaces. The arguments used in [34] strongly relayed on the main result in [12] and required a more complicated mathematical structure, involving both the characterization of the stability in terms of Banach function spaces and also the properties of the Orlicz space associated to the function N.

Remark 6.2. We recall that the Rolewicz type theorem was deduced in [34] based on an intermediary main result expressed in terms of a sequence of functions from the class $\mathcal{C}$. In what follows we shall see that this step may be also avoided using an elegant proof line, which on the one side will clarify the minimal set of properties required by a Rolewicz type method for stability and on the other side may be adapted to other more general dynamical systems like those described in [17] and [44].

In order to support a first step in the direction of a possible study concerning the individual or pointwise stability we establish the following:

Proposition 6.1. *If $\pi = (\Phi, \sigma)$ is a strongly continuous skew-product flow and $N : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, then the set*

$$A_r = \{x \in X : \sup_{\theta \in \Theta} \int_0^\infty N(||\Phi(\theta, t)x||)\, dt \leq r\}$$

is closed, for each $r > 0$.

Proof. Let $r > 0$. For every $\theta \in \Theta$ and $h > 0$ we consider the set

$$B_{\theta,h} = \{x \in X : \int_0^h N(||\Phi(\theta,t)x||)\, dt \leq r\}.$$

We prove that $B_{\theta,h}$ is closed, for each $(\theta,h) \in \Theta \times (0,\infty)$.

Indeed, let $\theta \in \Theta$ and let $h > 0$. Let $M, \omega > 0$ such that $||\Phi(\theta,t)|| \leq Me^{\omega t}$, for all $t \geq 0$.

Let $(x_n) \subset B_{\theta,h}$ with $x_n \to x$ as $n \to \infty$. We set $q := \sup_{n\in\mathbb{N}} ||x_n||$. Then

$$||\Phi(\theta,t)x_n|| \leq qMe^{\omega h}, \quad \forall n \in \mathbb{N}, \forall t \in [0,h] \tag{6.11}$$

which implies also that

$$||\Phi(\theta,t)x|| \leq qMe^{\omega h}, \quad \forall t \in [0,h]. \tag{6.12}$$

Let $\varepsilon > 0$. Since N is continuous on $[0, qMe^{\omega h}]$ there is $\delta > 0$ such that for every $s, s' \in [0, qMe^{\omega h}]$ with $|s - s'| < \delta$ we have that

$$|N(s) - N(s')| < \frac{\varepsilon}{h}. \tag{6.13}$$

Let $n_0 \in \mathbb{N}$ be such that

$$||x_{n_0} - x|| < \frac{\delta}{Me^{\omega h}}. \tag{6.14}$$

For every $t \in [0,h]$, using relations (6.11) and (6.12) we deduce that $||\Phi(\theta,t)x_{n_0}||$, $||\Phi(\theta,t)x|| \in [0, qMe^{\omega h}]$. In addition, using relation (6.14) we have that

$$|\ ||\Phi(\theta,t)x|| - ||\Phi(\theta,t)x_{n_0}||\ | \leq ||\Phi(\theta,t)(x - x_{n_0})|| < \delta, \quad \forall t \in [0,h]. \tag{6.15}$$

From (6.13) and (6.15) it follows that

$$|N(||\Phi(\theta,t)x||) - N(||\Phi(\theta,t)x_{n_0}||)| < \frac{\varepsilon}{h}, \quad \forall t \in [0,h]$$

which implies that

$$\int_0^h N(||\Phi(\theta,t)x||)\, dt \leq \int_0^h N(||\Phi(\theta,t)x_{n_0}||)\, dt + \varepsilon \leq r + \varepsilon. \tag{6.16}$$

Since $\varepsilon > 0$ was arbitrary we have that relation (6.16) holds for every $\varepsilon > 0$. This shows that $x \in B_{\theta,h}$, so $B_{\theta,h}$ is closed. Taking into account that

$$A_r = \bigcap_{\theta\in\Theta} \bigcap_{h>0} B_{\theta,h}$$

we finally obtain that A_r is closed. $\square$

We may now get back to the main theme, presenting a fluent and more natural proof for the Rolewicz type result.

Theorem 6.2. *Let $\pi = (\Phi, \sigma)$ be a strongly continuous skew-product flow on $\mathcal{E} = X \times \Theta$. Then π is uniformly exponentially stable if and only if there exist a function*

$N : \mathbf{R}_+^* \times \mathbf{R}_+ \to \mathbf{R}_+$ that for every $t > 0, N(t, \cdot) \in \mathcal{C}$ and for every $s \geq 0, N(\cdot, s)$ is nondecreasing and for every $x \in X$ there is $\alpha(x) > 0$ such that

$$\sup_{\theta\in\Theta} \int_0^\infty N(\alpha(x), ||\Phi(\theta, t)x||)dt < \infty.$$

Proof. Necessity follows by choosing the function

$$N : \mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}_+, \quad N(t, s) = s.$$

Sufficiency. For every $p \in \mathbb{N}^*$ we consider the function

$$N_p : \mathbb{R}_+ \to \mathbb{R}_+, \quad N_p(t) = N(\frac{1}{p}, t)$$

and the associated set

$$A_p = \{x \in X : \sup_{\theta\in\Theta} \int_0^\infty N_p(||\Phi(\theta, t)x||) \, dt \leq p\}$$

According to our hypothesis the function N_p is continuous and then, from Proposition 6.1 we obtain that A_p is closed.

Let $x \in X$. Then there is $\alpha(x) > 0$ such that

$$\sup_{\theta\in\Theta} \int_0^\infty N(\alpha(x), ||\Phi(\theta, t)x||) \, dt < \infty.$$

Let $k_x \in \mathbb{N}^*$ be such that

$$\int_0^\infty N(\alpha(x), ||\Phi(\theta, t)x||) \, dt \leq k_x, \quad \forall\theta \in \Theta. \tag{6.18}$$

If $m_x = [1/\alpha(x)] + 1$ then $1/m_x < \alpha(x)$. We set $p_x = \max\{m_x, k_x\}$. From the hypothesis we have that for every $s \geq 0$, $N(\cdot, s)$ is nondecreasing, so

$$N(\frac{1}{p_x}, ||\Phi(\theta, t)x||) \leq N(\alpha(x), ||\Phi(\theta, t)x||), \quad \forall t \geq 0. \tag{6.19}$$

Using relations (6.18) and (6.19) we deduce that

$$\int_0^\infty N(\frac{1}{p_x}, ||\Phi(\theta, t)x||) \, dt \leq p_x, \quad \forall\theta \in \Theta,$$

so $x \in A_{p_x}$.

It follows that

$$X = \bigcup_{j\in\mathbb{N}^*} A_j. \tag{6.20}$$

Using relation (6.20) and the theorem of Baire we obtain that there are $j \in \mathbb{N}^*$, $x_0 \in A_j$ and $r > 0$ such that $D(x_0, r) \subset A_j$. By applying Theorem 6.1 for N_j we obtain that π is uniformly exponentially stable. □

7. Open Problems

It was always the next natural step in any investigation concerning the stability of a dynamical system to analyze if the methods may be also applied to the study of the instability (see e.g. [14], [36]). This question for the case of skew-product flows was considered and answered in [14], by applying the techniques developed in [12] for stability to the new case of instability. The conclusions were interesting and extended the topic to a new direction. But, for all that, at first sight it was not possible to provide a sufficient condition for exponential instability employing an integral condition which is nonuniform with respect to $x \in X$.

The main techniques described in the previous sections relied on the behavior of some associated trajectories and it was clear that starting with [12] where we identified for the first time the potential methods in the investigation of the exponential stability of skew-product flows and moreover the study was later extended for the concept of exponential instability. The open problems arising in this framework will be presented in what follows both for the discrete-time case as well as in the continuous-time setting. In order to formulate these problems, we recall first the basic concepts.

Let X be a real or complex Banach space. For every $x \in X$ and every $r > 0$ let $C(x, r) := \{y \in X : ||y - x|| = r\}$ and let $C(0, 1) =: C$.

Let (Θ, d) be a metric space, let $J \in \{\mathbb{N}, \mathbb{Z}\}$ and let $\sigma : \Theta \times J \to \Theta$ be a discrete flow on Θ, i.e. $\sigma(\theta, 0) = \theta$ and $\sigma(\theta, m + n) = \sigma(\sigma(\theta, m), n)$, for all $(\theta, m, n) \in \Theta \times J^2$.

Let $\{A(\theta)\}_{\theta\in\Theta} \subset \mathcal{L}(X)$. We consider the variational discrete dynamical system

$$(A) \qquad x(\theta)(n + 1) = A(\sigma(\theta, n))x(\theta)(n), \quad \forall(\theta, n) \in \Theta \times \mathbb{N}$$

and the associated discrete cocycle $\{\Phi(\theta, n)\}_{\theta\in\Theta,n\in\mathbb{N}}$.

Definition 7.1. We say that the system (A) is *uniformly exponentially unstable* if there are $K, \nu > 0$ such that

$$||\Phi(\theta, n)x|| \geq Ke^{\nu n}||x||, \quad \forall x \in X, \forall(\theta, n) \in \Theta \times \mathbb{N}. \tag{7.1}$$

Remark 7.1. It is easy to see that relation (7.1) is sufficient to hold on the circle $C(0, r)$.

Remark 7.2. Let $x \in C$ and $\theta \in \Theta$. If there is $n_0 \in \mathbb{N}$ such that $\Phi(\theta, n_0)x = 0$, then using the cocycle identity it follows that $\Phi(\theta, n)x = 0$, for all $n \geq n_0$. In this case it is obvious that the system (A) is not uniformly exponentially unstable.

Definition 7.2. We say that the discrete cocycle Φ is *injective* if $\Phi(\theta, n)$ is an injective operator, for all $(\theta, n) \in \Theta \times \mathbb{N}$.

In what follows we suppose that the cocycle associated with the system (A) is injective. Then for every $(x, \theta) \in X \times \Theta, x \neq 0$ we define the sequence

$$\gamma_{x,\theta} : \mathbb{N} \to \mathbb{R}_+, \quad \gamma_{x,\theta}(n) = \frac{1}{||\Phi(\theta, n)x||}.$$

Remark 7.3. Following the same idea like in [14] (see Theorem 3.1) we can deduce the following:

Theorem 7.1. *The system (A) is uniformly exponentially unstable if and only if the associated cocycle is injective and there is a Banach sequence space $B \in \mathcal{V}(\mathbb{N})$ such that for every $(x, \theta) \in C \times \Theta$, $\gamma_{x,\theta} \in B$ and there is $L > 0$ such that $|\gamma_{x,\theta}|_B \leq L$, for all $(x, \theta) \in C \times \Theta$.*

Taking into account the methods presented in the case of stability, for a space $B \in \mathcal{Q}(\mathbb{N})$ we associate the set

$$\mathcal{U}_B = \{x \in X \setminus \{0\} : \sup_{\theta \in \Theta} |\gamma_{x,\theta}|_B < \infty\}.$$

Problem 7.1. The first open question is to identify a subclass of Banach sequence spaces in $\mathcal{Q}(\mathbb{N})$ such that the uniform exponential instability of a variational discrete dynamical system (A) may be characterized in terms of some topological properties of the set

$$\mathcal{U}_B = \{x \in X \setminus \{0\} : \sup_{\theta \in \Theta} |\gamma_{x,\theta}|_B < \infty\}$$

for a given sequence space B in this subclass.

Similarly, the question arises as well in the case of skew-product flows.

Definition 7.3. A skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly exponentially unstable* if there are $K, \nu > 0$ such that

$$||\Phi(\theta, t)x|| \geq K e^{\nu t} ||x||, \quad \forall x \in X, \forall (\theta, t) \in \Theta \times \mathbb{R}_+. \tag{7.2}$$

Remark 7.4. It is easy to see that relation (7.2) is sufficient to take place on the circle $C(0, r)$.

Definition 7.4. We say that the cocycle Φ is *injective* if $\Phi(\theta, t)$ is an injective operator, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$.

Remark 7.5. If $\pi = (\Phi, \sigma)$ is a skew-product flow with the property that Φ is injective, then for every $(x, \theta) \in X \times \Theta, x \neq 0$ we define the function

$$\varphi_{x,\theta} : \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi_{x,\theta}(t) = \frac{1}{||\Phi(\theta, t)x||}.$$

In [14], based on the discrete result given by Theorem 7.1, we proved that the uniform exponential instability of skew-product flows may be expressed in terms of the ownership of these orbits to certain Banach function spaces (see [14], Theorem 3.2) and consequently we deduced the first information of Rolewicz and Datko type for the instability case (see Corollary 3.1 and Theorem 3.4 in [14]).

Problem 7.2. Another question concerning the exponential instability of (injective) cocycles is whether one may deduce this asymptotic property from a condition in terms of the existence of a function $N(\cdot, \cdot)$ such that for every $x \in X, x \neq 0$ there exists $\alpha(x) > 0$ with

$$\sup_{\theta \in \Theta} \sum_{n=0}^{\infty} N(\alpha(x), \frac{1}{||\Phi(\theta, n)x||} < \infty. \tag{7.3}$$

We note that a particular case of (7.3) was considered in [14] (see Theorem 3.3) but that was a direct consequence of the characterization described in Theorem 7.1.

Problem 7.3. For strongly continuous skew-product flows $\pi = (\Phi, \sigma)$ with the associated cocycle Φ injective it remains an open question to determine the appropriate properties of a function $N(\cdot, \cdot)$ such that the property that for each $x \in X, x \neq 0$ there is $\alpha(x) > 0$ with

$$\sup_{\theta \in \Theta} \int_0^\infty N(\alpha(x), \frac{1}{||\Phi(\theta, t)x||})\; dt < \infty \tag{7.4}$$

is a sufficient condition for the uniform exponential instability of π. A preliminary answer was formulated in [14], but that was more a generalization of Datko's type result (see [14], Theorem 3.4) i.e. we have shown that if there exists a function $N \in \mathcal{F}$ such that

$$\sup_{\theta \in \Theta \;\; x \in C} \int_0^\infty N\left(\frac{1}{||\Phi(\theta, t)x||}\right)\; dt < \infty.$$

then the injective skew-product flow π is uniformly exponentially unstable. But the main question remains if one may employ the non-uniformity with respect to $x \in X$ via a two variables function as we did in the theorems of Rolewicz type for stability, working with a condition like (7.4).

Remark 7.6. The Zabzcyk type techniques were recently extended to the case of the exponential dichotomy of variational difference equations in [39], providing several interesting consequences concerning the dichotomy of difference equations as well as of skew-product flows.

Remark 7.7. The most complex asymptotic concept represented by the exponential trichotomy was treated for the first time from the perspective of the Rolewicz type techniques in [41], where we have also pointed out the applications for the study of the trichotomy of evolution families on the real line.

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