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MINIMUM-NORM FIXED POINT OF NONEXPANSIVE NONSELF MAPPINGS IN HILBERT SPACES

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Abstract. Both implicit and explicit methods are introduced to find the minimum-norm fixed point of a nonexpansive nonself mapping from a closed convex subset C of a Hilbert space H into H and satisfying the weak inwardness condition. Our idea is to apply the nearest point projection P_C to the well-known Browder's implicit and Halpern's explicit methods.
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1. Introduction

Throughout this paper, it is assumed that H is a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to H$ a non-self nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$). We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$, and always assume that $F(T) \ne \emptyset$. Since now $F(T)$ is closed, convex and nonempty, there exists a unique point $x^\dagger \in F(T)$ satisfying the property:

$$\|x^\dagger\| = \min\{\|x\| : x \in F(T)\}. \tag{1.1}$$

Namely, $x^\dagger$ is the nearest point projection of the original onto the fixed point set $F(T)$.

In many occasions, it is of interest to find a particular solution of a problem (assume the problem has multiple solutions); in particular, the solution with least norm (e.g., in least-squares problems, the least-norm solutions are used to define the pseudoinverse of bounded linear operators).

In this paper we are concerned with the least-norm fixed point $x^\dagger$ of a nonexpansive nonself-mapping T. We will introduce two methods (one implicit and one explicit)

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to find $x^\dagger$. First let us review some literature in which iterative methods for finding fixed points of nonexpansive mappings are studied.

In the case where T is assumed to be a nonexpansive self-mapping of C with $F(T) \neq \emptyset$, Browder [1] and Halpern [6] introduced an implicit method and an explicit method, respectively.

Browder's implicit method generates a net $\{x_t\}$ in an implicit way: for each $t \in (0,1)$, $x_t \in C$ is the unique fixed point of the contraction

$$x \mapsto T_t x := tu + (1-t)Tx, \quad x \in C \tag{1.2}$$

where $u \in C$ is a fixed point. (See [26] for another implicit method.)

Halpern's explicit method generates a sequence $\{x_n\}$ explicitly by the recursive manner:

$$x_{n+1} = t_n u + (1-t_n)Tx_n, \quad n \geq 0 \tag{1.3}$$

where the initial guess $x_0 \in C$ is arbitrarily fixed, and where $\{t_n\}$ is a sequence in the unit interval $(0,1)$.

The convergence of the net $\{x_t\}$ and of the sequence $\{x_n\}$ is as follows.

Theorem 1.1. [1] *Suppose $T : C \to C$ is a nonexpansive self-mapping of C with $F(T) \neq \emptyset$. Then the net $\{x_t\}$ strongly converges as $t \to 0$ to the fixed point x^* of T that is closest to u from $F(T)$ (i.e., $P_{F(T)}(u)$).*

Theorem 1.2. [6, 18, 19, 20] *Suppose $T : C \to C$ is a nonexpansive self-mapping of C with $F(T) \neq \emptyset$. Assume the conditions:*

(C1) $\lim_{n\to\infty} t_n = 0$;
(C2) $\sum_{n=1}^{\infty} t_n = \infty$;
(C3) *either* $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$ *or* $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$.

Then the sequence $\{x_n\}$ generated by (1.3) strongly converges as $t \to 0$ to the fixed point x^* of T that is closest to u from $F(T)$ (i.e., $P_{F(T)}(u)$).

A number of authors made contributions to Theorem 1.2 on different choices of the parameters $\{t_n\}$; see [8, 15, 16, 18, 19, 20]. Related work can be found in [10, 11, 12, 14, 17, 21, 22, 23, 24, 25, 27]. A recent survey on Halpern's method can be found in [9].

Browder's implicit method is extended to the case where T is assumed to be a nonslf-mapping by Xu and Yin [28].

Note that if $0 \in C$ then indeed the limit in both Theorems 1.2 and 1.3 are the minimum-norm fixed point of T. However, if $0 \notin C$, then this is no longer true, and in this case, an additional projection is needed to apply to both Browder's and Halpern's methods. This has recently been done in [3]. In this paper we further investigate the case where the nonexpansive mapping T is nonself. We prove that if T satisfies the wean inwardness condition, then the results in [3] for self-mappings hold fully for the implicit method and partially for the explicit method.

It is observed that minimum-norm solutions of fixed point equations and variational inequalities have recently been paid attention (see the references [3, 7, 29, 30]).

We adopt the following notions as popularized in literature:

- $x_n \to x$ means that $\{x_n\}$ converges to x in norm;

- $x_n \rightharpoonup x$ means that $\{x_n\}$ converges to x in the weak topology;
- $\omega_w(x_n)$ is the weak ω-limit set of $\{x_n\}$; that is, the set of all those points x such that $x_{n_j} \rightharpoonup x$ as $j \to \infty$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}$.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be nonexpansive; namely, T satisfies the property:

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C.$$

The following result, the so-called demiclosedness principle for nonexpansive mappings, will play an important role in our argument in the subsequent sections.

Lemma 2.1. *(cf.* [4, 5, 13]*) If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

Our methods depend on (nearest point or metric) projections. Recall that the projection from H onto C is a mapping that assigns to each x the point $P_C x$ that is closest to x from C; that is, P_C is the unique point in C satisfying the property:

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

The proposition below collects some characterizations of projections.

Proposition 2.2. *The following hold.*

(i) *Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only there holds the inequality*

$$\langle x - z, y - z\rangle \leq 0, \quad y \in C.$$

(ii) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ *for all $x, y \in H$.*

(iii) $\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C x\|^2$ *for all $x \in H$ and $y \in C$.*

Since we deal with nonself-mappings, we need boundary conditions for the mappings. Recall that for a point $x \in C$, the inward set to C at x is the set

$$I_C(x) = \{y \in H : y = x + a(z - x) \quad \text{for some } a \geq 0 \text{ and } z \in C\}.$$

Recall also that a nonself-mapping $T : C \to H$ is said to satisfy the inwardness condition if $Tx \in I_C(x)$ for all $x \in C$, and the weak inwardness condition if $Tx \in \overline{I_C(x)}$ for all $x \in C$.

We need the following result which appeared implicitly in [28] (see also [2]) and which states the relationship between the fixed point sets of T and $P_C T$.

Lemma 2.3. *Let $T : C \to H$ be a nonexpansive nonself mapping satisfying the weak inwardness condition. Then the mappings $P_C T$ and T have the same fixed points; namely, $F(P_C T) = F(T)$.*

Proof. It is evident that $F(T) \subset F(P_C T)$. Conversely, we take $x \in F(P_C T)$; namely, $P_C(Tx) = x$. Since T satisfies the weak inwardness conditions, there exists a sequence $\{y_n\}$ converging to Tx strongly, where

$$y_n = x + a_n(z_n - x) \tag{2.1}$$

By Proposition 2.2(i), we have

$$\langle Tx - x, z_n - x\rangle \le 0.$$

This implies that

$$\langle Tx - x, y_n - x\rangle \le 0$$

which in turn implies that

$$\|Tx - x\|^2 = \lim_{n\to\infty} \langle Tx - x, y_n - x\rangle \le 0.$$

Therefore, $Tx = x$ and $x \in F(T)$. □

In our convergence argument for the explicit method, we need the following result.

Lemma 2.4. *(cf.* [19]*) Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) *either* $\sum_{n=1}^{\infty} \gamma_n|\delta_n| < \infty$ *or* $\limsup_{n\to\infty} \delta_n \le 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Methods for Finding Minimum-norm Fixed Point

3.1. **Implicit Method.** In this section we introduce an implicit method that can be used to find minimum-norm fixed point.

Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to H$ be a (possibly nonself) nonexpansive mapping such that $F(T) \ne \emptyset$. Let P_C be the projection from H onto C. For each $t \in (0,1)$, the mapping

$$x \mapsto T_t x := P_C((1-t)Tx), \quad x \in C$$

is a self-contraction of C; hence it has a unique fixed point which is denoted by $x_t \in C$. Consequently, x_t is the unique solution in C of the fixed point equation

$$x_t = P_C((1-t)Tx_t). \tag{3.1}$$

Theorem 3.1. *Assume in addition that* T *satisfies the weak inwardness condition. Then the net* $\{x_t\}$ *defined by (3.1) converges strongly as* $t \to 0$ *to the minimum-norm fixed point of* T.

Proof. We divide the proof into three steps.

(i) We prove that $\{x_t\}$ is bounded. As a matter of fact, taking any point $p \in F(T)$, we derive that

$$\begin{aligned} \|x_t - p\| &= \|P_C((1-t)Tx_t) - p\| \\ &\le \|(1-t)Tx_t - p\| \\ &= \|(1-t)(Tx_t - Tp) - tp\| \\ &\le \|(1-t)\|x_t - p\| + t\|p\|. \end{aligned}$$

This implies that, for all $t \in (0,1)$,

$$\|x_t - p\| \le \|p\|. \tag{3.2}$$

So $\{x_t\}$ is bounded. Let $M > 0$ satisfy $M \geq \max\{\|x_t\|, \|Tx_t\|\}$ for all $t \in (0,1)$.

(ii) We prove that $\omega_w(x_t) \subset F(T)$. Namely, if $\{t_n\}$ is a null sequence in $(0,1)$ such that $x_{t_n} \rightharpoonup \bar{x}$, then $\bar{x} \in F(T)$.

Since, for each $t \in (0,1)$,

$$\begin{aligned} \|x_t - P_C T x_t\| &= \|P_C((1-t)Tx_t) - P_C T x_t\| \\ &\leq \|(1-t)Tx_t - Tx_t\| \\ &= t\|Tx_t\| \leq tM \to 0 \quad \text{as } t \to 0. \end{aligned}$$

In particular, $\|x_{t_n} - P_C T x_{t_n}\| \to 0$ as $n \to \infty$. Therefore, by Lemma 2.1 and Lemma 2.3, we know that $\bar{x} \in F(T)$.

(iii) We prove that $x_t \to x^\dagger$ as $t \to 0$, where $x^\dagger$ is the minimum-norm fixed point of T; that is, $x^\dagger = \arg\min\{\|x\| : x \in F(T)\}$.

Set $y_t = (1-t)Tx_t$. Then we have $x_t = P_C y_t$ and for $\tilde{x} \in F(T)$ we deduce that

$$\begin{aligned} x_t - \tilde{x} &= P_C y_t - \tilde{x} = (y_t - \tilde{x}) + P_C y_t - y_t \\ &= (1-t)(Tx_t - \tilde{x}) + t(-\tilde{x}) + (P_C y_t - y_t). \end{aligned}$$

Using $x_t - \tilde{x}$ to make inner product from both sides of the above equation, we get

$$\begin{aligned} \|x_t - \tilde{x}\|^2 &= (1-t)\langle Tx_t - \tilde{x}, x_t - \tilde{x}\rangle + t\langle -\tilde{x}, x_t - \tilde{x}\rangle + \langle P_C y_t - y_t, x_t - \tilde{x}\rangle \\ &\leq (1-t)\|x_t - \tilde{x}\|^2 + t\langle -\tilde{x}, x_t - \tilde{x}\rangle + \langle P_C y_t - y_t, P_C y_t - \tilde{x}\rangle. \end{aligned} \tag{3.3}$$

However, $\langle P_C y_t - y_t, P_C y_t - \tilde{x}\rangle \leq 0$ by Proposition 2.2(i). It then follows from (3.3) that

$$\|x_t - \tilde{x}\|^2 \leq \langle -\tilde{x}, x_t - \tilde{x}\rangle. \tag{3.4}$$

Now if $\bar{x} \in \omega_w(x_t)$ and $x_{t_n} \rightharpoonup \bar{x}$ for some null sequence (t_n) in $(0,1)$. Then, from Step (ii), we get $\bar{x} \in F(T)$. We can therefore substitute $\bar{x}$ for $\tilde{x}$ and t_n for t in (3.4) to obtain that $x_{t_n} \to \bar{x}$. This shows that $\{x_t\}$ is indeed relatively compact (as $t \to 0$) in the norm topology.

Note that (3.4) is equivalent to

$$\|x_t\|^2 \leq \langle x_t, \tilde{x}\rangle. \tag{3.5}$$

Hence,

$$\|x_t\| \leq \|\tilde{x}\|, \quad t \in (0,1),\ \tilde{x} \in F(T). \tag{3.6}$$

This clearly implies that if $\bar{x} \in \omega_w(x_t) = \omega(x_t)$, then

$$\|\bar{x}\| \leq \|\tilde{x}\| \quad \forall \tilde{x} \in F(T).$$

Therefore, $\bar{x} = x^\dagger$, and $x^\dagger$ is the only limit point of the net $\{x_t\}$ as $t \to 0$. This is sufficient to conclude that $x_t \to x^\dagger$ as $t \to 0$. □

Corollary 3.2. *Let H be a real Hilbert space, C a nonempty closed convex subset of H , and $T : C \to C$ a nonexpansive self-mapping with $F(T) \neq \emptyset$. For each $t \in (0,1)$, let x_t be the unique fixed point in C of the contraction $P_C((1-t))T$ mapping C into C. Then $s - \lim_{t\downarrow 0} x_t = x^\dagger$.*

3.2. **Explicit Method.** In this section we introduce an explicit method that generates a sequence converging in norm to the minimum-fixed point of T. Our scheme is the discretization of the implicit method studied in the last section. Consider a sequence $\{t_n\}$ in $(0,1)$ and an arbitrary initial guess $x_0 \in C$, and define a sequence $\{x_n\}$ iteratively by the recursion:

$$x_{n+1} = P_C((1-t_n)Tx_n), \quad n \geq 0. \tag{3.7}$$

The convergence of $\{x_n\}$ depends on the choice of the parameters $\{t_n\}$.

Theorem 3.3. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to H$ a nonexpansive mapping such that $F(T) \neq \emptyset$ and satisfying the weak inwardness condition. Assume $\{t_n\}$ satisfies the following assumptions:*

- (A_1) $\lim_{n\to\infty} t_n = 0$;
- (A_2) $\sum_{n=1}^{\infty} t_n = \infty$;
- (A_3) *either* $\sum_{n=1}^{\infty} \frac{|t_{n+1}-t_n|}{t_n} < \infty$ *or* $\lim_{n\to\infty} \frac{|t_{n+1}-t_n|}{t_n^2} = 0$.

Then the sequence $\{x_n\}$ generated by the algorithm (3.7) converges strongly to the minimum-norm fixed point $x^\dagger$ of T.

Proof. We again divided the proof into three steps.

(i) We prove that (x_n) is bounded. Indeed, take a $p \in F(T)$ to deduce that

$$\begin{aligned} \|x_{n+1} - p\| &= \|P_C((1-t_n)Tx_n) - p\| \\ &\leq \|(1-t_n)Tx_n - p\| \\ &= \|(1-t_n)(Tx_n - p) - t_n p\| \\ &\leq (1-t_n)\|x_n - p\| + t_n\|p\| \\ &\leq \max\{\|x_n - p\|, \|p\|\}. \end{aligned}$$

By induction, we get

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \|p\|\}$$

for all $n \geq 0$. Hence (x_n) is bounded. Let $M > 0$ satisfy $M \geq \max\{\|x_n\|, \|Tx_n\|\}$ for all n.

(ii) We prove that $\|x_{n+1} - z_n\| \to 0$ as $n \to \infty$, where z_n is the unique fixed point in C of the contraction $z \mapsto P_C((1-t_n)Tz)$; that is, $z_n = P_C((1-t_n)Tz_n)$. To see this, we compute, using the fact that P_C is nonexpansive,

$$\begin{aligned} \|x_{n+1} - z_n\| &= \|P_C((1-t_n)Tx_n) - P_C((1-t_n)Tz_n)\| \\ &\leq \|(1-t_n)Tx_n - (1-t_n)Tz_n\| \\ &= \|(1-t_n)(Tx_n - Tz_n)\ \| \\ &\leq (1-t_n)\|x_n - z_n\| \\ &\leq \|x_n - z_{n-1}\| + \|z_n - z_{n-1}\|. \end{aligned} \tag{3.8}$$

However,

$$\begin{aligned}\|z_n - z_{n-1}\| &= \|P_C((1-t_n)Tz_n) - P_C((1-t_{n-1})Tz_{n-1})\| \\ &\le \|(1-t_n)Tz_n - (1-t_{n-1})Tz_{n-1}\| \\ &= \|(1-t_n)(Tz_n - Tz_{n-1}) + (t_{n-1}-t_n)Tz_{n-1}\| \\ &\le (1-t_n)\|z_n - z_{n-1}\| + |t_{n-1}-t_n|\|Tz_{n-1}\| \\ &\le (1-t_n)\|z_n - z_{n-1}\| + M|t_{n-1}-t_n|.\end{aligned}$$

It turns out that

$$\|z_n - z_{n-1}\| \le \frac{M|t_{n-1}-t_n|}{t_n}. \tag{3.9}$$

Substituting (3.9) into (3.8), we get

$$\begin{aligned}\|x_{n+1} - z_n\| &\le (1-t_n)\|x_n - z_{n-1}\| + \frac{M|t_{n-1}-t_n|}{t_n} && (3.10)\\ &= (1-t_n)\|x_n - z_{n-1}\| + t_n\delta_n. && (3.11)\end{aligned}$$

Where $\delta_n = (M|t_{n-1}-t_n|)/t_n^2$. Therefore, an application of Lemma 2.4 to either (3.10) or (3.11) and observing assumption (A_3) to get $\|x_{n+1} - z_n\| \to 0$.

(iii) We prove that $x_n \to x^\dagger$. First observe from Theorem 3.3 that $z_n \to x^\dagger$. This together with Step (ii) ensures that $x_n \to x^\dagger$. The proof is complete. □

Corollary 3.4. *Let H be a real Hilbert space, C a nonempty closed convex subset of H , and $T : C \to C$ a nonexpansive self-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0,1)$ satisfying assumptions $(A_1) - (A_3)$ in Theorem 3.1. starting an initial $x_0 \in C$, we define a sequence $\{x_n\}$ by the algorithm (3.7). Then $x_n \to x^\dagger$.*

Remark 3.5. It is interesting to know if assumption (A_3) in Theorem 3.1 can be weakened to condition $(C3)$ as introduced in the Introduction. For nonexpansive self-mappings, the answer is affirmative (see [3]). However, for nonexpansive nonself-mappings, the answer is still unknown.

Also, it is not hard to find that the choice

$$t_n = \frac{1}{(n+1)^\delta}, \quad n \ge 0$$

satisfies the assumptions (A_1), (A_2), and the second part of (A_3) in Theorem 3.1 provided $0 < \delta < 1$. Indeed, we have

$$\frac{|t_n - t_{n-1}|}{t_n^2} \sim \frac{1}{n^{1-\delta}} \to 0 \quad (\text{as } n \to \infty).$$

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