Fixed Point Theory, 13(2012), No. 1, 259-265
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A DUALITY FIXED POINT THEOREM AND APPLICATIONS

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Abstract. Let E be a $2-$uniformly convex Banach space with the $2-$uniformly convex constant $1/c$, let $T : E \to E^*$ be a $L-$Lipschitz mapping with condition $0 < \frac{2L}{c^2} < 1$. Then T has a unique duality fixed point $x^* \in E$ ($Tx^* = Jx^*$) and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}Tx_n$ converges strongly to this duality fixed point x^*. If $0 < \frac{2L}{c^2} \leq 1$ and the duality fixed point set of T is nonempty, let $\{\alpha_n\} \subset [0,1]$ be a real sequence which satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$, then for any guess $x_0 \in E$, the iterative sequence $x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^{-1}Tx_n$ converges weakly to a duality fixed point. This main result can be used for solving the variational inequalities and optimal problems.

Key Words and Phrases: 2-uniformly smooth Banach space, dual space, fixed point, contraction mapping principle, application.
2010 Mathematics Subject Classification: 47H05, 47H09, 47H10.

1. Introduction and preliminaries

Let E be a real Banach space with the dual E^*, let T be an operator from E into E^*. Firstly, we consider the variational inequality problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^* - x\rangle \geq 0, \quad \forall\ \|x\| \leq \|x^*\|. \tag{1.1}$$

Secondly, we consider the optimal problem of finding an element $x^* \in E$ such that

$$(\|x^*\| - \|Tx^*\|)^2 = \min_{x\in E}(\|x\| - \|Tx\|)^2. \tag{1.2}$$

Thirdly, we consider the operator equation problem of finding an element $x^* \in E$ such that

$$\langle Tx^*, x^*\rangle = \|Tx^*\|^2 = \|x^*\|^2. \tag{1.3}$$

Let E be a real Banach space with the dual E^*. Let p be a given real number with $p > 1$. The generalized duality mapping J_p from E into 2^{E^*} is defined by

$$J_p(x) = \{f \in E^* : \langle x, f\rangle = \|f\|^p, \|f\| = \|x\|^{p-1}\}, \quad \forall\ x \in E, \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular, $J = J_2$ is called the normalized duality mapping and $J_p(x) = \|x\|^{p-2}J(x)$ for all $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. The duality mapping J has the following properties:

- if E is smooth, then J is single-valued;
- if E is strictly convex, then J is one-to-one;
- if E is reflexive, then J is surjective;
- if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.
- if E^* is uniformly convex, then J is uniformly continuous on each bounded subsets of E and J is singe-valued and also one-to-one.

For more details, see [1].

In this paper, we firstly present the definition of duality fixed point for a mapping T from E into its dual E^* as follows.

Let E be a Banach space with a single valued generalized duality mapping $J_p : E \to E^*$. Let $T : E \to E^*$. An element $x^* \in E$ is said to be a generalized duality fixed point of T if $Tx^* = J_p x^*$. An element $x^* \in E$ is said to be a duality fixed point of T if $Tx^* = Jx^*$.

Example 1. Let E be a smooth Banach space with the dual E^*, $A : E \to E^*$ be an operator, then an element $x^* \in E$ is a zero point of A if and only if x^* is a duality fixed point of $J + \lambda A$ for any $\lambda > 0$. Namely, the x^* is a duality fixed point of $J + \lambda A$ for any $\lambda > 0$ if and only if x^* is a fixed point of $J_\lambda = (J + \lambda A)^{-1}J : E \to E$ (if A is maximal monotone, then J_λ is namely the resolvent of A).

Example 2. In Hilbert space, the fixed point of an operator is always duality fixed point.

Example 3. Let E be a smooth Banach space with the dual E^*, then any element of E must be the duality fixed point of the normalized duality mapping J.

Conclusion 1.1. *If x^* is a duality fixed point of T, then x^* must be a solution of variational inequality problem (1.1).*

Proof. Suppose x^* is a duality fixed point of T. Then $\langle Tx^*, x^* \rangle = \langle Jx^*, x^* \rangle = \|Jx^*\|^2 = \|Tx^*\|^2 = \|x^*\|^2$. Observe that

$$\begin{aligned} \langle Tx^*, x^* - x \rangle &= \langle Tx^*, x^* \rangle - \langle Tx^*, x \rangle \geq \|Tx^*\|^2 - \|Tx^*\|\|x\| \\ &= \|Tx^*\|(\|Tx^*\| - \|x\|) = \|Tx^*\|(\|x^*\| - \|x\|) \geq 0 \end{aligned}$$

for all $\|x\| \leq \|x^*\|$. □

Conclusion 1.2. *If x^* is a duality fixed point of T, then x^* must be a solution of the optimal problem (1.2). Therefore, x^* is also a solution of operator equation problem (1.3).*

Proof. If x^* is a duality fixed point of T, then $Tx^* = Jx^*$, so that

$$\langle Tx^*, x^* \rangle = \langle Jx^*, x^* \rangle = \|Jx^*\|^2 = \|Tx^*\|^2 = \|x^*\|^2.$$

The all conclusions are obvious. □

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in U$, $x \neq y$ implies $\|\frac{x+y}{2}\| < 1$. It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| < 1 - \delta$. It is well known that a uniformly convex Banach space is reflexive and strictly convex. We define now a function $\delta : [0, 2] \to [0, 1]$ called the modulus of convexity of E as follows

$$\delta(\varepsilon) = \{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

It is well known that E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. Then E is said to be $p-$uniformly convex if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$. For example, see [2,3] for more details. The constant $\frac{1}{c}$ is said to be uniformly convexity constant of E.

A Banach space E is said to be smooth if the limit $\lim_{t \to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists for all $x, y \in U$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U$. One should note that no Banach space is $p-$uniformly convex for $1 < p < 2$; see [6] for more details. It is well known that the Hilbert and the Lebesgue $L^q (1 < q \leq 2)$ spaces are $2-$uniformly convex and uniformly smooth. Let X be a Banach space and let $L^q(X) = \{\Omega, \Sigma, \mu; X\}, 1 < q \leq \infty$ be the Lebesgue-Bochner space on an arbitrary measure space (Ω, Σ, μ). Let $2 \leq p < \infty$ and let $1 < q \leq p$. Then $L^q(X)$ is $p-$uniformly convex if and only if X is $p-$uniformly convex; see [3].

Lemma 1.3. ([4,5]). *Let E be a $p-$uniformly convex Banach space with $p \geq 2$. Then, for all $x, y \in E$, $j(x) \in J_p(x)$ and $j(y) \in J_p(y)$,*

$$\langle x - y, j(x) - j(y)\rangle \geq \frac{c^p}{c^{p-2}p}\|x - y\|^p, \tag{1.5}$$

where J_p is the generalized duality mapping from E into E^* and $1/c$ is the $p-$uniformly convexity constant of E.

Lemma 1.4. *Let E be a $p-$uniformly convex Banach space with $p \geq 2$. Then J_p is one-to-one from E onto $J_p(E) \subset E^*$ and for all $x, y \in E$,*

$$\|x - y\| \leq (\frac{p}{c^2})^{\frac{1}{p-1}} \|J_p(x) - J_p(y)\|^{\frac{1}{p-1}}. \tag{1.6}$$

where J_p is the generalized duality mapping from E into E^* with range $J_p(E)$, and $1/c$ is the $p-$uniformly convexity constant of E.

Proof. Let E be a $p-$uniformly convex Banach space with $p \geq 2$, then $J = J_2$ is one-to-one from E onto E^*. Since $J_p(x) = \|x\|^{p-2}J(x)$, then $J_p(x)$ is single-valued. From (1.5) we have

$$\langle x - y, J_p(x) - J_p(y)\rangle \geq \frac{c^p}{c^{p-2}p}\|x - y\|^p,$$

which implies that

$$\|x-y\|\|J_p(x)-J_p(y)\| \geq \frac{c^p}{c^{p-2}p}\|x-y\|^p.$$

That is

$$\|J_p(x)-J_p(y)\| \geq \frac{c^p}{c^{p-2}p}\|x-y\|^{p-1}.$$

Hence

$$\|x-y\| \leq (\frac{p}{c^2})^{\frac{1}{p-1}}\|J_p(x)-J_p(y)\|^{\frac{1}{p-1}}.$$

Then (1.6) has been proved. Therefore, from (1.6) we can see, for any $x, y \in E$, $J_p(x) = J_p(y)$ implies that $x = y$. □

2. Duality contraction mapping principle and applications

Let E be a Banach space with the dual E^*. An operator $T : E \to E^*$ is said to be $L-$Lipschitz, if

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \forall\, x, y \in E,$$

where $L \in (0, +\infty)$ is a constant.

Theorem 2.1. (Duality contraction mapping principle) *Let E be a $2-$uniformly convex Banach space, let $T : E \to E^*$ be a $L-$Lipschitz mapping with condition $0 < \frac{2L}{c^2} < 1$. Then T has a unique duality fixed point $x^* \in E$ and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}Tx_n$ converges strongly to this duality fixed point x^*.*

Proof. Let $A = J^{-1}T$, then A is a mapping from E into it-self. By using Lemma 1.4, we have

$$\begin{aligned}\|Ax - Ay\| &= \|J^{-1}Tx - J^{-1}Ty\| \\ &\leq (\frac{2}{c^2})^{\frac{1}{2-1}}\|Tx-Ty\|^{\frac{1}{2-1}} \leq \frac{2}{c^2}\|Tx-Ty\| \leq \frac{2L}{c^2}\|x-y\|,\end{aligned}$$

for all $x, y \in E$. By using Banach's contraction mapping principle, there exists a unique element $x^* \in E$ such that $Ax^* = x^*$. That is, $Tx^* = Jx^*$, so p is a unique duality fixed point of T. Further, the Picard iterative sequence $x_{n+1} = Ax_n = J_p^{-1}Tx_n$ (n=0,1,2,...) converges strongly to this duality fixed point x^*. □

From Conclusions 1.1-1.2 and Theorem 2.1, we have the following results for solving the variational inequality problem (1.1), the optimal problem (1.2) and the operator equation problem (1.3).

Theorem 2.2. *Let E be a $2-$uniformly convex Banach space and let $T : E \to E^*$ be a $L-$Lipschitz mapping with condition $0 < \frac{2L}{c^2} < 1$. Then the variational inequality problem (1.1) (the optimal problem (1.2) and operator equation problem (1.3)) has solutions and for any given guess $x_0 \in E$, the iterative sequence $x_{n+1} = J^{-1}Tx_n$ converges strongly to a solution of the variational inequality problem (1.1) (the optimal problem (1.2) and the operator equation problem (1.3)).*

Theorem 2.3. (Duality Mann weak convergence theorem) *Let E be a 2-uniformly convex Banach space which satisfying Opial's condition and let $T : E \to E^*$ be a*

$L-$Lipschitz mapping with nonempty duality fixed point set. Assume $0 < \frac{2L}{c^2} \leq 1$, and the real sequence $\{\alpha_n\} \subset [0,1]$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = +\infty$. Then for any given guess $x_0 \in E$, the generalized Mann iterative sequence

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^{-1}Tx_n$$

converges weakly to a duality fixed point of T.

Proof. Let $A = J^{-1}T$, by using Lemma 1.4, we have

$$\|Ax - Ay\| = \|J^{-1}Tx - J^{-1}Ty\| \leq \frac{2}{c^2}\|Tx - Ty\| \leq \frac{2L}{c^2}\|x-y\| \leq \|x-y\|,$$

for all $x, y \in E$. Hence A is a nonexpansive mapping from E into it-self. In addition, we have

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J^{-1}Tx_n = (1-\alpha_n)x_n + \alpha_n Ax_n.$$

By using a well known result, we know that the sequence $\{x_n\}$ converges weakly to a fixed point x^* of A, which is also a duality fixed point of T $(Tx^* = Jx^*)$. □

Next, we prove a more general weak convergence theorem for a finite family of Lipschitz mappings from a Banach space E into its dual E^*. Therefore, we give the applications to solve the system of variational inequalities, the system of optimal problem and the system of operator equations.

Let E be a real Banach space with the dual E^* and let $\{T_i\}_{i=1}^N : E \to E^*$ be N L_i-Lipschitz mappings with the Lipschitz constants L_i respectively.

Firstly, we consider the system of variational inequalities problem of finding an element $x^* \in E$ such that

$$\langle T_i x^*, x^* - x\rangle \geq 0, \quad \forall\ \|x\| \leq \|x^*\|. \tag{2.1}$$

for all $i = 1, 2, 3, ..., N$.

Secondly, we consider the system of optimal problem of finding an element $x^* \in E$ such that

$$(\|x^*\| - \|T_i x^*\|)^2 = \min_{x \in E}(\|x\| - \|T_i x\|)^2. \tag{2.2}$$

for all $i = 1, 2, 3, ..., N$.

Thirdly, we consider the system of operator equations problem of finding an element $x^* \in E$ such that

$$\langle T_i x^*, x^*\rangle = \|T_i x^*\|^2 = \|x^*\|^2. \tag{2.3}$$

for all $i = 1, 2, 3, ..., N$.

Theorem 2.4. *Let E be a $2-$uniformly convex Banach space which satisfying Opial's condition, $\{T_i\}_{i=1}^N : E \to E^*$ be N L_i-Lipschitz mappings with nonempty common duality fixed points set. Assume $0 < \frac{2L_i}{c^2} \leq 1$ for all $i = 1, 2, 3, ..., N$. Let $\{\alpha_n\}, \{r_n\}, \{s_n\}, \{t_n\}, \{w_n\}$ be five real sequences in [0,1] satisfying $0 < a \leq \alpha_n \leq b < 1$ and $t_n + w_n \leq b < 1$, where a, b are some constants. For any guess $x_0 \in E$,* *define a iterative sequence* $\{x_n\}$ *by*

$$\begin{cases} x_n = \alpha_n x_{n-1} + (1-\alpha_n)J^{-1}T_n y_n \\ y_n = r_n x_n + s_n x_{n-1} + t_n J^{-1}T_n x_n + w_n J^{-1}T_n x_{n-1}, \\ r_n + s_n + t_n + w_n = 1, \end{cases} \tag{2.4}$$

where $T_n = T_{nmodN}$. *Then* $\{x_n\}$ *converges weakly to a common duality fixed point of* $\{T_i\}_{i=1}^N$ *(the solution of the system of variational inequalities problem (2.1), the solution of system of optimal problem (2.2) and the solution of system of operator equations problem (2.3)) .*

Proof. Let $A_i = J^{-1}T_i$ for $i = 1, 2, 3, \cdots, N$, by using Lemma 1.4, we have

$$\|A_i x - A_i y\| = \|J^{-1}T_i x - J^{-1}T_i y\| \leq \frac{2}{c^2}\|T_i x - T_i y\| \leq \frac{2L}{c^2}\|x - y\| \leq \|x - y\|,$$

for all $x, y \in E$. Hence $\{A_i\}_{i=1}^N$ is a finite family of nonexpansive mappings from E into it-self. In addition, we can rewrite the iterative scheme (2.4) as follows

$$\begin{cases} x_n = \alpha_n x_{n-1} + (1-\alpha_n)A_n y_n \\ y_n = r_n x_n + s_n x_{n-1} + t_n A_n x_n + w_n A_n x_{n-1}, \\ r_n + s_n + t_n + w_n = 1. \end{cases} \tag{2.5}$$

By using the Su and Qin's result (see [6, Theorem 2.1]), we know the iterative sequence $\{x_n\}$ converges weakly to a common fixed point of $\{A_i\}_{i=1}^N$. Hence the sequence $\{x_n\}$ converges weakly to a common duality fixed point of $\{T_i\}_{i=1}^N$. □

Theorem 2.5. (Duality Halpren strong convergence theorem) *Let* E *be a 2-uniformly convex and uniformly smooth Banach space with the dual* E^*, *let* $T : E \to E^*$ *be a* $L-$*Lipschitz mapping with nonempty duality fixed point set. Assume* $0 < \frac{2L}{c^2} \leq 1$. *Let* u, x_0 *be given. Assume real sequence* $\{\alpha_n\} \subset [0, 1]$ *satisfies the following conditions*

$(C_1) : \lim_{n\to\infty} \alpha_n = 0$

$(C_2) : \sum_{n=0}^{\infty} \alpha_n = \infty$

$(C_3) : \lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}} = 0$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1.$

Then iterative sequence

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J^{-1}Tx_n, \tag{2.6}$$

converges strongly to a duality fixed point of T.

Proof. Let $A = J^{-1}T$, by using Lemma 1.4, we have

$$\|Ax - Ay\| = \|J^{-1}Tx - J^{-1}Ty\| \leq \frac{2}{c^2}\|Tx - Ty\| \leq \frac{2L}{c^2}\|x - y\| \leq \|x - y\|,$$

for all $x, y \in E$. Hence A is a nonexpansive mapping from E into it-self. In addition, we have

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J^{-1}Tx_n = \alpha_n u + (1-\alpha_n)Ax_n.$$

By using the well-known result of Xu [7, Theorem 2.3], we know the iterative sequence $\{x_n\}$ converges strongly to a fixed point of nonexpansive mapping A. Hence the sequence $\{x_n\}$ converges strongly to a duality fixed point of T. □

Theorem 2.6. *Let H be a Hilbert space, then its uniformly convexity constant $\frac{1}{c} \geq \frac{\sqrt{2}}{2}$, that is $c \leq \sqrt{2}$.*

Proof. If $c > \sqrt{2}$. For any $x \neq y$, by using Lemma 1.4, we have

$$\|x - y\| = \|J^{-1}x - J^{-1}y\| \leq \frac{2}{c^2}\|x - y\| < \|x - y\|.$$

This is a contradiction. □

Acknowledgments. This project is supported by the National Natural Science Foundation of China under grant (11071279). The research of H.-K. Xu was supported in part by NSC 100-2115-M-110-003-MY2 (Taiwan).

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Received: January 15, 2011; Accepted: June 23, 2011.