Fixed Point Theory, 13(2012), No. 1, 237-246
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

CONVERGENCE THEOREMS FOR APPROXIMATION OF FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Let E be a uniformly smooth and uniformly convex real Banach space and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that either E admits weakly sequentially continuous duality mapping j or T is demicompact. Then, we introduce two approximation schemes (implicit and explicit) for finding a fixed point of a nonexpansive mapping and prove strong convergence of the schemes. Our results extend the recent results of Yao *et al.* [Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. volume 2009 (2009), Article ID 279058, 7 pages].
Key Words and Phrases: Strong convergence, nonexpansive mappings, uniformly smooth spaces, uniformly convex spaces.
2010 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25.

1. Introduction

Let E be a real Banach space and K a nonempty, closed and convex subset of E. A mapping $T : K \to K$ is said to be *nonexpansive* if for all $x, y \in K$, we have

$$||Tx - Ty|| \leq ||x - y||. \tag{1}$$

A point $x \in K$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in K : Tx = x\}$.
Construction of fixed points of nonexpansive mappings is an important subject in nonlinear operator theory and its applications; in particular, in image recovery and signal processing (see, for example, [3, 6, 12]). Many authors have worked extensively on the approximation of fixed points of nonexpansive mappings. For example, the reader can consult the recent monographs of Berinde [1] and Chidume [4].
Very recently, Yao *et al.* [11] proved path convergence for a nonexpansive mapping in a real Hilbert space. In particular, they proved the following theorem.

Theorem 1.1. *(Yao et al.,* [11]*) Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t \in (0,1)$, let the net $\{x_t\}$ be generated by $x_t = TP_K[(1-t)x_t]$, then as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point of T.*

Furthermore, they applied Theorem 1.1 to prove the following theorem.

Theorem 1.2. *(Yao et al.,* [11]*) Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} y_n = P_K[(1-\alpha_n)x_n] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \ n \geq 1, \end{cases} \tag{2}$$

Suppose the following conditions are satisfied:

(a) $\lim \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by* (2) *converges strongly to a fixed point of T.*

Motivated by the results of Yao *et al.* [11], we *introduce two approximation schemes (implicit and explicit) for finding a fixed point of a nonexpansive mapping and prove strong convergence of the schemes under the condition that E either admits weakly sequentially continuous duality map j or T is demicompact where E is uniformly smooth and uniformly convex real Banach space. Our results extend the results of Yao et al.* [11] *from real Hilbert spaces to Banach spaces considered here.*

2. Preliminaries

Let E be a real Banach space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Also, E is said to have a *Fréchet differentiable norm* if for all $x \in S$ the limit exists and is attainable uniformly in $y \in S$. In this case there exists an increasing function $b : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} b(t) = 0$ such that

$$\frac{1}{2}||x||^2 + \langle h, j(x)\rangle \leq \frac{1}{2}||x+h||^2 \leq \frac{1}{2}||x||^2 + \langle h, j(x)\rangle + b(||h||), \forall x, h \in E.$$

Furthermore, E is said to be *uniformly smooth* if the limit exists uniformly for $(x,y) \in S \times S$. It is well known that if E is uniformly smooth, then the norm of E is Fréchet differentiable. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\Big\};\ \tau > 0.$$

E is equivalently said to be *smooth* if $\rho_E(\tau) > 0$, $\forall\, \tau > 0$.

Let $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \Big|\Big|\frac{x+y}{2}\Big|\Big| : ||x|| = ||y||; \epsilon = ||x-y||\Big\}.$$

E is *uniformly convex* if for any $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \epsilon$, then $||\frac{1}{2}(x+y)|| \leq 1-\delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. E is called *strictly convex* if for all $x, y \in E$, $x \neq y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$, $\forall \lambda \in (0,1)$. It is known that every uniformly convex Banach space is reflexive.
Let E^* be the dual of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\},$$

where $\langle .,. \rangle$ denotes the generalized duality pairing between members of E and E^*. It is well known that if E is uniformly smooth then J is single-valued and norm-to-weak* uniformly continuous on bounded sets (see, e.g., [4, 9]). Also, it is known that a Banach space E is Fréchet differentiable if and only if the duality mapping J is single-valued and norm-to-norm continuous. In the sequel, we shall denote the single-valued normalized duality mapping by j.
Let $K \subset E$ be closed convex and Q be a mapping of E onto K. Then Q is said to be *sunny* if $Q(Q(x) + t(x - Q(x))) = Q(x)$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = (z)$ for every $z \in R(Q)$, *range of* Q. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H*. But this is not true in a general Banach spaces. We note that if E is smooth and Q is retraction of K onto $F(T)$, then Q is sunny and nonexpansive if and only if for each $x \in K$ and $z \in F(T)$ we have $\langle Qx - x, J(Qx - z)\rangle \leq 0$, (see [8] for more details).
A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at p if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx = p$.
A mapping $T : K \to E$ is said to be *demicompact* at h if for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in K such that $(x_n - Tx_n) \to h$ as $n \to \infty$, there exists a subsequence say $\{x_{n_j}\}$ of $\{x_n\}_{n=1}^{\infty}$ and $x^* \in K$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K and $x^* - Tx^* = h$.
We need the following lemmas in the sequel.
Lemma 2.1. *(Browder, [2]) Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E and $T : K \to K$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Then, $I - T$ is demiclosed at zero.*
Lemma 2.2. *(Suzuki, [7]) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in a Banach space X and let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence in [0,1] with* $0 < \liminf_{n\to\infty}\beta_n \leq$

$\limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 1$ *and* $\limsup_{n\to\infty}\Big(||y_{n+1}-y_n|| - ||x_{n+1}-x_n||\Big) \leq 0$. *Then,* $\lim_{n\to\infty}||y_n - x_n|| = 0$.

Lemma 2.3. *(Xu, [10]) Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n,\ n \geq 1,$$

where $\{a_n\}_{n=1}^{\infty} \subset [0,1]$ *and* $\{\sigma_n\}_{n=1}^{\infty}$ *is a sequence in* $\mathbb{R}$ *satisfying:*
(i) $\sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \leq 0$ *or* $\sum|\alpha_n\sigma_n| < \infty$.
Then, $a_n \to 0$ *as* $n \to \infty$.

Lemma 2.4. *(Cholamjiak and Suantai, [5]) Let E be a real Banach space with Fréchet differentiable norm. For* $x \in E$, *let* $\beta^*(t)$ *be defined for* $0 < t < \infty$ *by*

$$\beta^*(t) = \sup_{x\in S}\Big|\frac{||x+ty||^2 - ||x||^2}{t} - 2\langle y, j(x)\rangle\Big|. \tag{3}$$

Then, $\lim_{t\to 0^+}\beta^*(t) = 0$ *and*

$$||x+h||^2 \leq ||x||^2 + 2\langle h, j(x)\rangle + ||h||\beta^*(||h||)$$

for all $h \in E \setminus \{0\}$.

Remark 2.5. *In a real Hilbert space, we see that* $\beta^*(t) = t$ *for* $t > 0$.
In our more general setting, throughout this paper, we will assume that

$$\beta^*(t) \leq 2t,$$

where β^* *is the function appearing in* (3).

3. Main results

Let E be a uniformly smooth and uniformly convex real Banach space and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For $t_n \in (0,1),\ n \geq 1$, such that $\lim_{n\to\infty} t_n = 0$, we define a map $T_n : K \to K$ by

$$T_n x := TQ_K[(1-t_n)x],\ x \in K. \tag{4}$$

We show that T_n is a contraction.
For each $x, y \in K$, we have from (4) that

$$\begin{aligned} ||T_n x - T_n y|| &\leq ||Q_K(1-t_n)x - Q_K(1-t_n)y|| \\ &\leq (1-t_n)||x-y||. \end{aligned} \tag{5}$$

which implies that T_n is a contraction. Therefore, by the Banach contraction mapping principle, there exists a unique fixed point z_n of T_n in K. That is,

$$z_n = TQ_K[(1-t_n)z_n],\ \forall n \geq 1. \tag{6}$$

Next, we prove that $\{z_n\}_{n=1}^{\infty}$ is bounded. Let $x^* \in F(T)$, then using (6), we have

$$\begin{aligned}||z_n - x^*|| &= ||TQ_K(1-t_n)z_n - TQ_K x^*||\\ &\le ||Q_K(1-t_n)z_n - Q_K x^*||\\ &\le ||(1-t_n)z_n - t_n x^* + t_n x^* - x^*|| = ||(1-t_n)(z_n - x^*) - t_n x^*||\\ &\le (1-t_n)||z_n - x^*|| + t_n||x^*||.\end{aligned}$$

Hence, $||z_n - x^*|| \le ||x^*||$. This implies that $\{z_n\}_{n=1}^{\infty}$ is bounded.
We next show that $||z_n - Tz_n|| \to 0,\ n \to \infty$.

$$\begin{aligned}||z_n - Tz_n|| &= ||TQ_K(1-t_n)z_n - TQ_K z_n||\\ &\le ||Q_K(1-t_n)z_n - Q_K z_n|| \le ||(1-t_n)z_n - z_n||\\ &\le t_n||z_n|| \to 0,\ (\text{ since } t_n \to 0,\ n \to \ \infty).\end{aligned}$$

We now prove the following theorem in a uniformly smooth and uniformly convex real Banach space using (6).

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex real Banach space and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T: K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t_n \in (0,1),\ n \ge 1$, let $\{z_n\}_{n=1}^{\infty}$ be generated by (6) then as $\lim_{n\to\infty} t_n = 0$, $\{z_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T if E admits weak sequential continuous duality map j.*

Proof. Since $\{z_n\}_{n=1}^{\infty}$ is bounded, there exists a subsequence, say $\{z_{n_k}\}$ of $\{z_n\}_{n=1}^{\infty}$ that converges weakly to some point $z^* \in K$. Using the demiclosedness principle of $(I-T)$ at 0 (see Lemma 2.1) and the fact that $||z_{n_k} - Tz_{n_k}|| \to 0,\ \textit{as}\ k \to \infty$, we obtain that $z^* \in F(T)$. From (6), we get for $z^* \in F(T)$,

$$\begin{aligned}&||z_{n_k} - z^*||^2 = ||TQ_K[(1-t_{n_k})z_{n_k}] - TQ_K z^*||^2\\ \le\ & ||(1-t_{n_k})z_{n_k} - z^*||^2 = ||z_{n_k} - z^* - t_{n_k}z_{n_k}||^2\\ \le\ & ||z_{n_k} - z^*||^2 - 2t_{n_k}\langle z_{n_k}, j(z_{n_k} - z^*)\rangle + t_{n_k}||z_{n_k}||\beta^*(t_{n_k}||z_{n_k}||)\\ \le\ & ||z_{n_k} - z^*||^2 - 2t_{n_k}\langle z_{n_k}, j(z_{n_k} - z^*)\rangle + 2t_{n_k}^2||z_{n_k}||^2\\ =\ & ||z_{n_k} - z^*||^2 - 2t_{n_k}\langle z_{n_k} - z^*, j(z_{n_k} - z^*)\rangle - 2t_{n_k}\langle z^*, j(z_{n_k} - z^*)\rangle + 2t_{n_k}^2||z_{n_k}||^2\\ \le\ & ||z_{n_k} - z^*||^2 - 2t_{n_k}||z_{n_k} - z^*||^2 - 2t_{n_k}\langle z^*, j(z_{n_k} - z^*)\rangle + 2t_{n_k}^2||z_{n_k}||^2.\end{aligned}$$

This implies that

$$||z_{n_k} - z^*||^2 \le \langle z^*, j(z^* - z_{n_k})\rangle + t_{n_k}||z_{n_k}||^2.$$

Using the fact that j is weakly sequentially continuous, then from the last inequality, we have that $\{z_{n_k}\}$ converges strongly to z^*. We now show that $\{z_n\}_{n=1}^{\infty}$ actually converges to z^*. Suppose there is another subsequence $\{z_{n_j}\}$ of $\{z_n\}_{n=1}^{\infty}$ such that $z_{n_j} \to x^*, j \to \infty$. Then since $||z_{n_j} - Tz_{n_j}|| \to 0,\ \textit{as}\ j \to \infty$ and T is uniformly continuous, we have that $x^* \in F(T)$.

Claim. $z^* = x^*$

Suppose for contradiction that $x^* \neq z^*$. Using (6), we obtain using similar argument as above that

$$||z_{n_j} - z^*||^2 \leq \langle z^*, j(z^* - z_{n_j})\rangle + \frac{t_{n_j}}{2}||z_{n_j}||^2$$

Thus,

$$||x^* - z^*||^2 \leq \langle z^*, j(z^* - x^*)\rangle \tag{7}$$

Interchanging x^* and z^*, we obtain

$$||z^* - x^*||^2 \leq \langle x^*, j(x^* - z^*)\rangle \tag{8}$$

Adding (7) and (8) yields

$$2||x^* - z^*||^2 \leq ||x^* - z^*||^2.$$

This implies that $x^* = z^*$. This completes the proof.

Corollary 3.2. *Let $E := l_p,\ 1 < p < \infty$ and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t_n \in (0,1),\ n \geq 1$, let $\{z_n\}_{n=1}^{\infty}$ be generated by (6) then as $\lim_{n\to\infty} t_n = 0,\ \{z_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.*

Theorem 3.3. Let E be a uniformly smooth and uniformly convex real Banach space and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t_n \in (0,1),\ n \geq 1$, let $\{z_n\}_{n=1}^{\infty}$ be generated by (6) then as $\lim_{n\to\infty} t_n = 0,\ \{z_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T if T is demicompact.

Proof. Since T is demicompact and $\lim ||z_n - Tz_n|| = 0$, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}_{n=1}^{\infty}$ that converges strongly to some point $z^* \in K$. By continuity of T and the fact that $\lim ||z_{n_k} - Tz_{n_k}|| = 0$, we have that $z^* \in F(T)$.

Suppose there exists another subsequence $\{z_{n_j}\}$ of $\{z_n\}$ that converges strongly to u^*, say, then following the argument of the last part of Theorem 3.1, we get that $\{z_n\}_{n=1}^{\infty}$ converges strongly to $z^* \in F(T)$. This completes the proof.

Corollary 3.4. *Let E be a uniformly smooth and uniformly convex real Banach space and let K be a compact convex and nonempty sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For $t_n \in (0,1),\ n \geq 1$, let $\{z_n\}_{n=1}^{\infty}$ be generated by (6) then as $\lim_{n\to\infty} t_n = 0,\ \{z_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.*

Proof. Compactness of K implies that $\{z_n\}_{n=1}^{\infty}$ has a subsequence $\{z_{n_k}\}$ which converges strongly to some $z^* \in K$. The rest of the proof follows as in the proof of Theorem 3.3.

We now prove strong convergence theorem using an explicit iterative scheme in a uniformly smooth and uniformly convex real Banach space.

Theorem 3.5. *Let E be a uniformly smooth and uniformly convex real Banach space and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For* *arbitrary* $x_1 \in K$, *let the sequence* $\{x_n\}_{n=1}^{\infty}$ *be generated iteratively by*

$$\begin{cases} y_n = Q_K[(1-\alpha_n)x_n] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \ n \geq 1. \end{cases} \tag{9}$$

Suppose the following conditions are satisfied:
(a) $\lim \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T *if either*
(i) E *admits weakly sequentially continuous duality mapping* j *or*
(ii) T *is demicompact.*

Proof. First we show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $x^* \in F(T)$, we have from (9),

$$\begin{aligned} ||x_{n+1} - x^*|| &= ||(1-\beta_n)(x_n - x^*) + \beta_n(Ty_n - x^*)|| \\ &\leq (1-\beta_n)||x_n - x^*|| + \beta_n||Ty_n - x^*|| \\ &\leq (1-\beta_n)||x_n - x^*|| + \beta_n[(1-\alpha_n)||x_n - x^*|| + \alpha_n||x^*||] \\ &= (1-\alpha_n\beta_n)||x_n - x^*|| + \alpha_n\beta_n||x^*|| \\ &\leq \max\{||x_n - x^*||, ||x^*||\} \\ &\vdots \\ &\leq \max\{||x_1 - x^*||, ||x^*||\} \end{aligned}$$

Hence, $\{x_n\}_{n=1}^{\infty}$ is bounded and $\{Tx_n\}$ is bounded. Set $u_n = Ty_n, \ n \geq 1$. It follows that

$$\begin{aligned} ||u_{n+1} - u_n|| &= ||Ty_{n+1} - Ty_n|| \\ &\leq ||y_{n+1} - y_n|| \leq ||(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n|| \\ &\leq ||x_{n+1} - x_n|| + \alpha_{n+1}||x_{n+1}|| + \alpha_n||x_n||. \end{aligned}$$

Hence, $\limsup_{n\to\infty}(||u_{n+1} - u_n|| - ||x_{n+1} - x_n||) \leq 0$. This together with Lemma 2.2 imply that $\lim_{n\to\infty} ||u_n - x_n|| = 0$. Thus,

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} \beta_n ||x_n - u_n|| = 0.$$

$$\begin{aligned} ||x_n - Tx_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - Tx_n|| \\ &\leq ||x_n - x_{n+1}|| + (1-\beta_n)||x_n - Tx_n|| + \beta_n||Ty_n - Tx_n|| \\ &\leq ||x_n - x_{n+1}|| + (1-\beta_n)||x_n - Tx_n|| + \beta_n||y_n - x_n|| \\ &\leq ||x_n - x_{n+1}|| + (1-\beta_n)||x_n - Tx_n|| + \alpha_n||x_n||, \end{aligned}$$

that is,

$$||x_n - Tx_n|| \leq \frac{1}{\beta_n}\{||x_n - x_{n+1}|| + \alpha_n||x_n||\} \to 0, \ n \to \infty.$$

Let $\{z_n\}_{n=1}^{\infty}$ be defined by (6), then $z_n \to x^* \in F(T),\ n \to \infty$. (This is guaranteed either by condition (i) or condition (ii) above). Next, we show that

$$\limsup_{n\to\infty}\langle x^*, j(x^*-x_n)\rangle \leq 0.$$

For each integer $n \geq 1$, let $t_n \in (0,1)$ be such that

$$t_n \to 0,\ \frac{||Tx_n - x_n||}{t_n} \to 0,\ n \to \infty.$$

It then follows from (6) that

$$\begin{aligned}
||z_n - x_n||^2 &= ||z_n - Tx_n + Tx_n - x_n||^2 \\
&\leq ||z_n - Tx_n||^2 + 2\langle Tx_n - x_n, j(z_n - Tx_n)\rangle + ||Tx_n - x_n||\beta^*(||Tx_n - x_n||) \\
&\leq ||z_n - Tx_n||^2 + 2\langle Tx_n - x_n, j(z_n - Tx_n)\rangle + 2||Tx_n - x_n||^2 \\
&\leq ||z_n - Tx_n||^2 + 2||Tx_n - x_n||||z_n - Tx_n|| + 2||Tx_n - x_n||^2 \\
&\leq ||z_n - Tx_n||^2 + M||Tx_n - x_n|| \\
&\leq ||TQ_K(1-t_n)z_n - Tx_n||^2 + M||Tx_n - x_n|| \\
&\leq ||Q_K(1-t_n)z_n - x_n||^2 + M||Tx_n - x_n|| \\
&= ||(1-t_n)z_n - x_n||^2 + M||Tx_n - x_n|| \\
&= ||z_n - x_n - t_n z_n||^2 + M||Tx_n - x_n|| \\
&\leq ||z_n - x_n||^2 - 2t_n\langle z_n, j(z_n - x_n)\rangle + t_n||z_n||\beta^*(t_n||z_n||) + M||Tx_n - x_n|| \\
&\leq ||z_n - x_n||^2 - 2t_n\langle z_n, j(z_n - x_n)\rangle + 2t_n^2||z_n||^2 + M||Tx_n - x_n|| \\
&\leq ||z_n - x_n||^2 - 2t_n\langle z_n, j(z_n - x_n)\rangle + t_n^2 M_1 + M||Tx_n - x_n||
\end{aligned}$$

for some $M > 0$ and $M_1 > 0$. Thus, $\langle z_n, j(z_n - x_n)\rangle \leq \frac{M_1 t_n}{2} + \frac{M}{2t_n}||Tx_n - x_n||$. Therefore, $\limsup_{n\to\infty}\langle z_n, j(z_n - x_n)\rangle \leq 0$. Moreover,

$$\begin{aligned}
\langle -z_n, j(x_n - z_n)\rangle &= \langle -x^*, j(x_n - x^*)\rangle + \langle -x^*, j(x_n - z_n)\rangle \\
&\quad - \langle -x^*, j(x_n - x^*)\rangle + \langle x^* - z_n, j(x_n - z_n)\rangle \\
&= \langle -x^*, j(x_n - x^*)\rangle + \langle -x^*, j(x_n - z_n) - j(x_n - x^*)\rangle \\
&\quad + \langle x^* - z_n, j(x_n - z_n)\rangle,
\end{aligned}$$

and since j is norm-to-weak* uniformly continuous on bounded sets, we have $\limsup_{n\to\infty}\langle -x^*, j(x_n - x^*)\rangle \leq 0$.

From (9), we have

$$\begin{aligned}||y_n - x^*||^2 &= ||Q_K(1-t_n)x_n - x^*||^2 \le ||(1-\alpha_n)x_n - x^*||^2\\
&\le ||x_n - x^* - \alpha_n x_n||^2 \le ||x_n - x^*||^2 - 2\alpha_n\langle x_n, j(x_n - x^*)\rangle\\
&+ \alpha_n||x_n||\beta^*(\alpha_n||x_n||)\\
&\le ||x_n - x^*||^2 - 2\alpha_n\langle x_n, j(x_n - x^*)\rangle + \alpha_n||x_n||\beta^*(\alpha_n||x_n||)\\
&\le ||x_n - x^*||^2 - 2\alpha_n\langle x_n, j(x_n - x^*)\rangle + 2\alpha_n^2||x_n||^2\\
&= ||x_n - x^*||^2 + 2\alpha_n\langle x_n - x^* + x^*, j(x^* - x_n)\rangle + 2\alpha_n^2||x_n||^2\\
&= ||x_n - x^*||^2 + 2\alpha_n\langle x^*, j(x^* - x_n)\rangle - 2\alpha_n\langle x^* - x_n, j(x^* - x_n)\rangle\\
&+ 2\alpha_n^2||x_n||^2\\
&= ||x_n - x^*||^2 + 2\alpha_n\langle x^*, j(x^* - x_n)\rangle + \alpha_n^2||x_n||^2 - 2\alpha_n||x_n - x^*||^2\\
&= (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||x_n - x^*||^2 - 2\alpha_n||x_n - x^*||^2\\
&+ 2\alpha_n\langle x^*, j(x^* - x_n)\rangle + 2\alpha_n^2||x_n||^2\\
&\le (1-\alpha_n)||x_n - x^*||^2 + 2\alpha_n\langle x^*, j(x^* - x_n)\rangle\\
&+ 2\alpha_n^2||x_n||^2. \qquad (10)\end{aligned}$$

Furthermore, using (10) in (9), we obtain

$$\begin{aligned}||x_{n+1} - x^*||^2 &\le (1-\beta_n)||x_n - x^*||^2 + \beta_n||y_n - x^*||^2\\
&\le (1-\beta_n)||x_n - x^*||^2 + \beta_n[(1-\alpha_n)||x_n - x^*||^2\\
&\quad +2\alpha_n\langle x^*, j(x^* - x_n)\rangle + 2\alpha_n^2||x_n||^2]\\
&\le (1-\alpha_n\beta_n)||x_n - x^*||^2 + 2\alpha_n\beta_n\langle x^*, j(x^* - x_n)\rangle + \alpha_n^2 M_2\\
&= (1-\alpha_n\beta_n)||x_n - x^*||^2 + \alpha_n\beta_n[2\langle x^*, j(x^* - x_n)\rangle + \frac{\alpha_n}{\beta_n}M_2],\end{aligned}$$

where $M_2 := \sup\limits_{n\ge 1}||x_n||^2$. Using Lemma 2.3, we get that $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^* \in F(T)$. This completes the proof.

Corollary 3.6. *Let $E = l_p,\ 1 < p < \infty$ and let K be a nonempty, closed and convex sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \ne \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} y_n = Q_K[(1-\alpha_n)x_n] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n,\ n \ge 1. \end{cases} \qquad (11)$$

Suppose the following conditions are satisfied:

(a) $\lim \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty}\alpha_n = \infty$;

(b) $0 < \liminf\limits_{n\to\infty}\beta_n \le \limsup\limits_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Corollary 3.7. *Let E be a real Banach space which is uniformly smooth and also uniformly convex and let K be a compact convex and nonempty sunny nonexpansive retract of E with Q_K as the sunny nonexpansive retraction. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in $(0,1)$. For arbitrary $x_1 \in K$, let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by*

$$\begin{cases} y_n = Q_K[(1-\alpha_n)x_n] \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \; n \geq 1. \end{cases} \tag{12}$$

Suppose the following conditions are satisfied:

(a) $\lim \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Remark 3.8. *Our Corollary 3.2 extends the result of Yao et al. [10, Theorem 3.1] to l_p, $1 < p < \infty$ while our Corollary 3.6 extends extends the result of Yao et al. [10, Theorem 3.2] to l_p, $1 < p < \infty$.*

Acknowledgements. The authors would like to express their sincere thanks to the anonymous referee and the Editors for their valuable suggestions and comments which improved the original version of the manuscript greatly.

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Received: April 12, 2010; Accepted: June 30, 2011.