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FIXED POINT OF MULTIVALUED OPERATORS ON ORDERED GENERALIZED METRIC SPACES

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Abstract. Recently, Bucur, Guran and Petruşel presented some results on fixed point of multivalued operators on generalized metric spaces which extended some old fixed point theorems to the multivalued case ([1]). In this paper, we shall give some results on fixed points of multivalued operators on ordered generalized metric spaces by providing different conditions in respect to [1].
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1. Introduction

There are many works about fixed points of multivalued mappings (see for example, [2]–[4]) and weakly Picard maps (see for example, [6]–[8]). Let (X, ρ) be a metric space. We shall denote the set of all nonempty closed subsets of X by $P_{cl}(X)$. Also, we shall denote the set of fixed points of a multifunction T by $Fix(T)$. Let X be a nonempty set and consider the space $\mathbb{R}^m_+$ endowed with the usual component-wise partial order. The mapping $d : X \times X \to \mathbb{R}^m_+$ which satisfies all the usual axioms of the metric is called a generalized metric space in the sense of Perov ([1]). If $v, r \in \mathbb{R}^m$, $v := (v_1, v_2, \cdots, v_m)$ and $r := (r_1, r_2, \cdots, r_m)$, then by $v \le r$ we mean $v_i \le r_i$, for each $i \in \{1, 2, \cdots, m\}$, while $v < r$ stands for $v_i < r_i$, for each $i \in \{1, 2, \cdots, m\}$. Also, $|v| := (|v_1|, |v_2|, \cdots, |v_m|)$, $\max(v, r) := (\max(v_1, r_1), \cdots, \max(v_m, r_m))$, and if $c \in \mathbb{R}$, then $v \le c$ means $v_i \le c$, for each $i \in \{1, 2, \cdots, m\}$. In a generalized metric space in the sense of Perov, the concepts of Cauchy sequence, convergent sequence and completeness are similar defined as those in a metric space. We denote by $M_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. A matrix $A \in M_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero whenever $A^n \to 0$. We appeal next result in the following which has been proved in ([5]).

Theorem 1.1. *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following are equivalents:*
(i) $A^n \to 0$;
(ii) The eigenvalues of A are in the open unit disc, i.e. $|\lambda| < 1$, for all $\lambda \in C$ with $det(A - \lambda I) = 0$;
(iii) The matrix $I - A$ is non-singular and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$;

(iv) The matrix $I - A$ is non-singular and $(I - A)^{-1}$ has nonnegative elements;
(v) $A^n q \to 0$ and $qA^n \to 0$, for all $q \in \mathbb{R}^m$.

If $(X, \leq)$ is a partially ordered set, then we define

$$X_{\leq} = \{(x, y) \in X \times X : x \leq y \text{ or } y \leq x\}.$$

Let X be a nonempty set and $T : X \to P(X)$ be a multivalued operator. We set

$$(T \times T)(x, y) = \{(u, v) : u \in Tx,\ v \in Ty\},$$

for all $x, y \in X$. Note that, for each $x \in X$ there exists $b_x \in \mathbb{R}^m_+$ such that $b_x \leq d(x, y)$ for all $y \in Tx$. At least, we can set $b_x = 0$. Now, for each $x \in X$ we denote largest of these vectors by $d(x, Tx)$, that is, $d(x, Tx)$ is a vector in $\mathbb{R}^m_+$ such that $d(x, Tx) \leq d(x, y)$ for all $y \in Tx$ and $b_x \leq d(x, Tx)$ for all $b_x \in \mathbb{R}^m_+$ with $b_x \leq d(x, y)$ for all $y \in Tx$.

2. Main Results

We say that $(X, d, \leq)$ is an ordered generalized metric space whenever (X, d) is a generalized metric space in Perov' sense, and $(X, \leq)$ is a partially ordered set.

Theorem 2.1. *Let $(X, d, \leq)$ be a complete ordered generalized metric space, A a matrix in $M_{m,m}(\mathbb{R}_+)$ convergent to zero and $T : X \to P_{cl}(X)$ a multivalued operator. Suppose that $(T \times T)(X_{\leq}) \subseteq X_{\leq}$ and*
(i) For each $(x, y) \in X_{\leq}$ and $u \in T(x)$ there exist $v \in T(y)$ and $L(x, y) \in \mathcal{A}_{x,y}$ such that $d(u, v) \leq A\ L(x, y)$, where $\mathcal{A}_{x,y} = \{d(x, y), d(x, Tx), d(y, Ty)\}$,
(ii) For each sequence $\{x_n\}_{n\geq 1}$ in X with $x_n \to x$, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $(x_{n_k}, x) \in X_{\leq}$ for all $k \geq 1$,
(iii) There exist $x_0, x_1 \in X$ such that $(x_0, x_1) \in X_{\leq}$ and $x_1 \in Tx_0$.
Then T has a fixed point.

Proof. If $x_0 = x_1$, then x_0 is a fixed point of T. Let $x_1 \neq x_0$. By (i), there exist $x_2 \in Tx_1$ and $L(x_0, x_1) \in \mathcal{A}_{x_0,x_1}$ such that $d(x_1, x_2) \leq AL(x_0, x_1)$, where $\mathcal{A}_{x_0,x_1} = \{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1)\}$. If $L(x_0, x_1) = d(x_1, Tx_1)$, then

$$d(x_1, x_2) \leq Ad(x_1, Tx_1) \leq Ad(x_1, x_2)$$
$$\Rightarrow (I - A)d(x_1, x_2) \leq 0 \ \Rightarrow d(x_1, x_2) = 0 \ \Rightarrow x_1 = x_2.$$

If $L(x_0, x_1) = d(x_0, x_1)$ or $L(x_0, x_1) = d(x_0, Tx_0)$, then

$$d(x_1, x_2) \leq Ad(x_0, x_1). \tag{1}$$

Since $(x_0, x_1) \in X_{\leq}$, $x_1 \in Tx_0$, $(T \times T)(X_{\leq}) \subseteq X_{\leq}$ and $x_2 \in Tx_1$, $(x_1, x_2) \in X_{\leq}$. Now, by using (i) there exist $x_3 \in Tx_2$ and $L(x_1, x_2) \in \mathcal{A}_{x_1,x_2}$ such that

$$d(x_2, x_3) \leq AL(x_1, x_2).$$

If $L(x_1, x_2) = d(x_2, Tx_2)$, then

$$d(x_2, x_3) \leq Ad(x_2, Tx_2) \leq Ad(x_2, x_3) \ \Rightarrow x_2 = x_3.$$

If $L(x_1, x_2) = d(x_1, x_2)$ or $L(x_0, x_1) = d(x_1, Tx_1)$, then by using (1) we have

$$d(x_2, x_3) \leq Ad(x_1, x_2) \leq A^2 d(x_0, x_1).$$

Now by induction, we construct a sequence $\{x_n\}_{n\geq 0}$ in X which has the following properties:
(a) $x_{n+1} \in Tx_n$ for all $n \geq 0$,
(b) $(x_n, x_{n+1}) \in X_{\leq}$ for all $n \geq 0$,
(c) $d(x_n, x_{n+1}) \leq A^n d(x_0, x_1)$ for all $n \geq 0$.
Now, by using these properties and Theorem 1.1 we obtain

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq A^n d(x_0, x_1) + A^{n+1} d(x_0, x_1) + \cdots + A^{n+p-1} d(x_0, x_1) \\ &\leq A^n (I + A + A^2 + \cdots + A^{p-1}) d(x_0, x_1) \\ &\leq A^n (I - A)^{-1} d(x_0, x_1) \longrightarrow 0 \ (n \to \infty). \end{aligned}$$

Hence, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the complete metric space (X, d). Choose $x^* \in X$ such that $x_n \to x^*$. By (ii), there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 0}$ such that $(x_{n_k}, x^*) \in X_{\leq}$ for all $k \geq 1$. But, $x_{n_k} \in Tx_{n_k-1}$, $(x_{n_k-1}, x^*) \in X_{\leq}$ for all $n \geq 1$. Thus by using (i), for each $k \geq 1$ there exist $v_{n_k} \in Tx^*$ and $L(x_{n_k-1}, x^*) \in \mathcal{A}_{x_{n_k-1}, x^*}$ such that

$$d(v_{n_k}, x_{n_k}) \leq AL(x_{n_k-1}, x^*).$$

If $L(x_{n_k-1}, x^*) = d(x_{n_k-1}, x^*)$, then $d(v_{n_k}, x_{n_k}) \leq Ad(x_{n_k-1}, x^*)$. Hence,

$$d(v_{n_k}, x^*) \leq d(v_{n_k}, x_{n_k}) + d(x_{n_k}, x^*) \leq Ad(x_{n_k-1}, x^*) + d(x_{n_k-1}, x^*) \to 0 \ (k \to \infty).$$

If $L(x_{n_k-1}, x^*) = d(x_{n_k-1}, Tx_{n_k-1})$, then $d(v_{n_k}, x_{n_k}) \leq Ad(x_{n_k-1}, x_{n_k})$. Hence,

$$d(v_{n_k}, x^*) \leq d(v_{n_k}, x_{n_k}) + d(x_{n_k}, x^*) \leq Ad(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x^*) \to 0 \ (k \to \infty).$$

If $L(x_{n_k-1}, x^*) = d(x^*, Tx^*)$, then $d(v_{n_k}, x_{n_k}) \leq Ad(v_{n_k}, x^*)$. Hence,

$$\begin{aligned} d(v_{n_k}, x^*) &\leq d(v_{n_k}, x_{n_k}) + d(x_{n_k}, x^*) \leq Ad(v_{n_k}, x^*) + d(x_{n_k}, x^*) \\ &\Rightarrow (I - A) d(v_{n_k}, x^*) \leq d(x_{n_k}, x^*) \to 0 \ (k \to \infty). \end{aligned}$$

Therefore, $v_{n_k} \to x^*$ $(k \to \infty)$. Since $u_{n_k} \in Tx^*$ for all $k \geq 1$ and Tx^* is a closed subset of X, $x^* \in Tx^*$. □

Example 2.1. *Let $X = [-2, -1] \cup [1, 2] \cup \{0\}$, $r = \frac{4}{5}$, $A = rI_{2\times 2}$, $k > 0$ and $d : X \times X \to \mathbb{R}^2$ defined by $d(x, y) = (|x - y|, k|x - y|)$ for all $x, y \in X$. Then (X, d) is a generalized metric space. Define the multivalued mapping $T : X \to X$ by $Tx = [-\frac{x}{4} + 2, \frac{5}{2}]$ whenever $x \in [-2, -1)$, $Tx = \{0\}$ whenever $x \in \{-1, 0, 1\}$ and $Tx = [\frac{3}{2}, -\frac{x}{4} + 2]$ whenever $x \in (1, 2]$. We show that T satisfies the assumptions of Theorem 2.1 while it does not satisfy the assumptions of [7; Theorem 3.3]. In this way, note that if $x \in \{-1, 0, 1\}$, then $d(x, Tx) = (|x|, k|x|)$ and if $x \in [-2, -1)$ or $x \in (1, 2]$, then $d(x, Tx) = (|\frac{5x-8}{4}|, k|\frac{5x-8}{4}|)$. Let $x, y \in [-2, -1)$, $x \leq y$ and $u \in Tx$. Then, for each $v \in Ty$ we have $|u - v| \leq \frac{|y+2|}{4} \leq \frac{1}{5}\frac{|5y-8|}{4} \leq r\frac{|5y-8|}{4}$, and so $d(u, v) \leq Ad(y, Ty)$. Let $x \in [-2, -1)$, $y \in \{-1, 0, 1\}$ and $u \in Tx$. Then, for each $v \in Ty$ we have $|u - v| \leq \frac{5}{2} \leq \frac{13}{5} \leq r\frac{|5x-8|}{4}$, and so $d(u, v) \leq Ad(x, Tx)$. Let $x \in [-2, -1)$, $y \in (1, 2]$ and $u \in Tx$. Then, for each $v \in Ty$ we have $|u - v| \leq 1 \leq \frac{1}{2}|x - y| \leq \frac{4}{5}|x - y| = r|x - y|$, and so $d(u, v) \leq Ad(x, y)$. Therefore T satisfies the assumptions of Theorem 2.1. If $x = -\frac{3}{2}$ and $y = -1$, then $Tx = [\frac{19}{8}, \frac{5}{2}]$, $Ty = \{0\}$ and for each $u \in [\frac{19}{8}, \frac{5}{2}]$ and*

$v = 0$, we have $|u - v| \nleq r|x - y| \Rightarrow d(u, v) \nleq Ad(x, y)$. Hence, T does not satisfy the assumptions of [7; Theorem 3.3].

Theorem 2.2. *Let (X, d) be a complete generalized metric space, $\theta \in (0, 1)$ and $T : X \to P_{cl}(X)$ a multivalued operator. Suppose that $\varphi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is an increasing sublinear function such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t = (t_i)_{i=1}^m \in \mathbb{R}^m_{++}$. Also, suppose that for each $x, y \in X$ and $u \in T(x)$ there exist $v \in T(y)$ and $M(x, y) \in \mathcal{B}_{x,y}$ such that*

$$d(u, v) \leq \varphi(M(x, y)), \ (*)$$

where

$$\mathcal{B}_{x,y} = \{d(x, y), d(x, Tx), \theta d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ be arbitrary and take $x_1 \in Tx_0$. If $x_0 = x_1$, then x_0 is a fixed point of T. Let $x_1 \neq x_0$. By $(*)$, there exist $x_2 \in Tx_1$ and $M(x_0, x_1) \in \mathcal{B}_{x_0,x_1}$ such that $d(x_1, x_2) \leq \varphi(M(x_0, x_1))$. If $x_1 = x_2$, then x_1 is a fixed point of T. Let $x_1 \neq x_2$. We show that $d(x_1, x_2) \leq \varphi(d(x_0, x_1))$. If $M(x_0, x_1) = d(x_0, x_1)$, then

$$d(x_1, x_2) \leq \varphi(d(x_0, x_1)). \tag{2}$$

If $M(x_0, x_1) = d(x_0, Tx_0)$, then (2) holds because $x_1 \in Tx_0$. We claim that $M(x_0, x_1) \neq \theta d(x_1, Tx_1)$. In fact, if $M(x_0, x_1) = \theta d(x_1, Tx_1)$, then

$$d(x_1, x_2) \leq \varphi(\theta d(x_1, Tx_1)) \leq \varphi(\theta d(x_1, x_2)) < \theta d(x_1, x_2),$$

which is a contradiction. If $M(x_0, x_1) = \frac{d(x_0,Tx_0)+d(x_1,Tx_1)}{2}$, then

$$d(x_1, x_2) \leq \varphi(\frac{d(x_0, Tx_0) + d(x_1, Tx_1)}{2}) \leq \frac{1}{2}\varphi(d(x_0, x_1)) + \frac{1}{2}\varphi(d(x_1, x_2))$$
$$< \frac{1}{2}\varphi(d(x_0, x_1)) + \frac{1}{2}d(x_1, x_2),$$

because $x_1 \in Tx_0$, $x_2 \in Tx_1$ and φ is sublinear. Hence, $d(x_1, x_2) < \varphi(d(x_0, x_1))$. If $M(x_0, x_1) = \frac{d(x_0,Tx_1)+d(x_1,Tx_0)}{2} = \frac{d(x_0,Tx_1)}{2}$, then by a similar way we obtain $d(x_1, x_2) \leq \varphi(d(x_0, x_1))$. Thus, $d(x_1, x_2) \leq \varphi(d(x_0, x_1))$ holds. Now by $(*)$, there exists $x_3 \in Tx_2$ and $M(x_1, x_2) \in \mathcal{B}_{x_1,x_2}$ such that $d(x_2, x_3) \leq \varphi(M(x_1, x_2))$. If $x_2 = x_3$, then x_2 is a fixed point of T. Suppose that $x_2 \neq x_3$. Now, we show that $d(x_2, x_3) \leq \varphi^2(d(x_0, x_1))$. If $M(x_1, x_2) = d(x_1, x_2)$, then by using (2) we obtain

$$d(x_2, x_3) \leq \varphi(d(x_1, x_2)) \leq \varphi^2(d(x_0, x_1)). \tag{3}$$

If $M(x_1, x_2) = d(x_1, Tx_1)$, then (3) holds because $x_2 \in Tx_1$. We claim that $M(x_1, x_2) \neq \theta d(x_2, Tx_2)$. In fact, if $M(x_1, x_2) = \theta d(x_2, Tx_2)$, then

$$d(x_2, x_3) \leq \varphi(\theta d(x_2, Tx_2)) \leq \varphi(\theta d(x_2, x_3)) < \theta d(x_2, x_3),$$

which is a contradiction. If $M(x_1, x_2) = \frac{d(x_1,Tx_1)+d(x_2,Tx_2)}{2}$, then

$$d(x_2, x_3) \leq \varphi(\frac{d(x_1, Tx_1) + d(x_2, Tx_2)}{2}) \leq \frac{1}{2}\varphi(d(x_1, x_2)) + \frac{1}{2}\varphi(d(x_2, x_3))$$

$$< \frac{1}{2}\varphi(d(x_1, x_2)) + \frac{1}{2}d(x_2, x_3)$$

because $x_2 \in Tx_1$, $x_3 \in Tx_2$ and φ is sublinear. Hence,

$$d(x_2, x_3) < \varphi(d(x_1, x_2)) \leq \varphi^2(d(x_0, x_1)).$$

If $M(x_1, x_2) = \frac{d(x_1,Tx_2)+d(x_2,Tx_1)}{2}$, then by a similar way we obtain

$$d(x_2, x_3) \leq \varphi(d(x_1, x_2)) \leq \varphi^2(d(x_0, x_1)).$$

Thus, $d(x_2, x_3) \leq \varphi^2(d(x_0, x_1))$ holds. Now, by induction we construct a sequence $\{x_n\}_{n\geq 0}$ in X which has the following properties:
(a) $x_{n+1} \in Tx_n$ for all $n \geq 0$,
(b) $d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1))$ for all $n \geq 0$.
Now, for each natural number p we have

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p})$$

$$\leq \varphi^n(d(x_0, x_1)) + \varphi^{n+1}(d(x_0, x_1)) + \cdots + \varphi^{n+p-1}(d(x_0, x_1)) = \sum_{k=n}^{n+p-1} \varphi^k(d(x_0, x_1)).$$

Hence, $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in the complete metric space (X, d). Choose $x^* \in X$ such that $x_n \to x^*$. Let $n \geq 1$ be given. Since $x_n \in Tx_{n-1}$, by using $(*)$ there exist $u_n \in Tx^*$ and $M(x_{n-1}, x^*) \in \mathcal{B}_{x_{n-1},x^*}$ such that

$$d(u_n, x_n) \leq \varphi(M(x_{n-1}, x^*)).$$

If $u_n = x^*$ for some $n \geq 1$, then x^* is a fixed point of T. Suppose that $u_n \neq x^*$ for all $n \geq 1$. Now, we show that $\lim_{n\to\infty} d(u_n, x^*) = 0$. If $M(x_{n-1}, x^*) = d(x_{n-1}, x^*)$, then $d(u_n, x_n) \leq \varphi(d(x_{n-1}, x^*))$. Since

$$d(u_n, x^*) \leq d(u_n, x_n) + d(x_n, x^*) \leq \varphi(d(x_{n-1}, x^*)) + d(x_n, x^*),$$

$d(u_n, x^*) \to 0$. If $M(x_{n-1}, x^*) = d(x_{n-1}, Tx_{n-1})$, then

$$d(u_n, x_n) \leq \varphi(d(x_{n-1}, Tx_{n-1})) \leq \varphi(d(x_{n-1}, x_n)) \leq \varphi^{n-1}(d(x_0, x_1)).$$

Hence, $d(u_n, x^*) \leq \varphi^{n-1}(d(x_0, x_1)) + d(x_n, x^*)$ and so $d(u_n, x^*) \to 0$.

If $M(x_{n-1}, x^*) = \theta d(x^*, Tx^*)$, then

$$d(u_n, x_n) \leq \varphi(\theta d(x^*, Tx^*)) \leq \varphi(\theta d(x^*, u_n)) < \theta d(u_n, x^*).$$

Hence, $d(u_n, x^*) \leq \theta d(u_n, x^*) + d(x_n, x^*)$ and so $d(u_n, x^*) \leq (1-\theta)^{-1} d(x_n, x^*)$. Thus, $d(u_n, x^*) \to 0$.

If $M(x_{n-1}, x^*) = \frac{d(x_{n-1},Tx_{n-1})+d(x^*,Tx^*)}{2}$, then

$$d(u_n, x_n) \leq \varphi(\frac{d(x_{n-1}, Tx_{n-1}) + d(x^*, Tx^*)}{2}) \leq \frac{1}{2}\varphi(d(x_{n-1}, x_n)) + \frac{1}{2}\varphi(d(u_n, x^*))$$

$$< \frac{1}{2}\varphi(d(x_{n-1}, x_n)) + \frac{1}{2}d(u_n, x^*).$$

Hence,

$$d(u_n, x^*) \leq d(u_n, x_n) + d(x_n, x^*) < \frac{1}{2}\varphi(d(x_{n-1}, x_n)) + \frac{1}{2}d(u_n, x^*) + d(x_n, x^*).$$

Thus, $d(u_n, x^*) < \varphi(d(x_{n-1}, x_n)) + 2d(x_n, x^*)$ and so $d(u_n, x^*) \to 0$.

If $M(x_{n-1}, x^*) = \frac{d(x_{n-1},Tx^*)+d(x^*,Tx_{n-1})}{2}$, then

$$d(u_n, x_n) \leq \varphi(\frac{d(x_{n-1}, Tx^*) + d(x^*, Tx_{n-1})}{2}) \leq \frac{1}{2}\varphi(d(x_{n-1}, u_n)) + \frac{1}{2}\varphi(d(x_n, x^*))$$
$$\leq \frac{1}{2}\varphi(d(x_{n-1}, x^*)) + \frac{1}{2}\varphi(d(x^*, u_n)) + \frac{1}{2}\varphi(d(x_n, x^*))$$
$$< \frac{1}{2}\varphi(d(x_{n-1}, x^*)) + \frac{1}{2}\varphi(d(x^*, x_n)) + \frac{1}{2}d(u_n, x^*).$$

Hence,

$$d(u_n, x^*) \leq d(u_n, x_n) + d(x_n, x^*)$$
$$< \frac{1}{2}\varphi(d(x_{n-1}, x^*)) + \frac{1}{2}\varphi(d(x_n, x^*)) + \frac{1}{2}d(u_n, x^*) + d(x_n, x^*)$$

and so $d(u_n, x^*) \to 0$. Therefore, we proved that $\lim_{n\to\infty} d(u_n, x^*) = 0$.

Since $u_n \in Tx^*$ for all $n \geq 1$ and Tx^* is a closed subset of X, $x^* \in Tx^*$. □

Corollary 2.3. *Let (X, d) be a complete generalized metric space, $\theta, \alpha \in (0, 1)$ and $T : X \to P_{cl}(X)$ a multivalued operator. Suppose that each $x, y \in X$ and $u \in T(x)$ there exist $v \in T(y)$ and $M(x, y) \in \mathcal{B}_{x,y}$ such that*

$$d(u, v) \leq AM(x, y),$$

where

$$\mathcal{B}_{x,y} = \{d(x, y), d(x, Tx), \theta d(y, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\},$$

and $A \in M_{m\times m}(\mathbb{R}_+)$ is defined by $A = \alpha I$. Then T has a fixed point.

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References

[1] A. Bucur, L. Guran, A. Petruşel, *Fixed points for multivalued operators on a set endowed with vector-valued metrics and applications*, Fixed Point Theory, **10**(2009), no. 1, 19-34.

[2] S. Czerwik, *A fixed point theorem for a system of multivalued transformations*, Proc. Amer. Math. Soc., **55**(1976), 136–139.

[3] D. O'Regan, N. Shahzad, R.P. Agarwal, *Fixed point theory for generalized contractive maps on spaces with vector-valued metrics*, Fixed Point Theory and Appl., (Eds. Y. J. Cho, J. K. Kim, S. M. Kang), Vol. 6, Nova Science Publ., New York, 2007, 143-149.

[4] A. Petruşel, I.A. Rus, *Fixed point theory for multivalued operators on a set with two metrics*, Fixed Point Theory, **8**(2007), no. 1, 97–104.

[5] I.A. Rus, *Principles and Applications of Fixed Point Theory*, (in Romanian), Editura Dacia, Cluj-Napoca, 1979.

[6] I.A. Rus, *The theory of a metrical fixed point theory; theoretical and applicative relevances*, Fixed Point Theory, **9**(2008), no. 2, 541-559.

[7] I.A. Rus, A. Petruşel, A. Sântamărian, *Data dependence of the fixed point set of multivalued weakly Picard operators*, Nonlinear Analysis, **52**(2003), 1947-1959.

[8] I.A. Rus, A. Petruşel, M.A. Şerban, *Weakly Picard operators; equivalent definitions, applications and open problems*, Fixed Point Theory, **7**(2006), no. 1, 3-22.

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