Fixed Point Theory, 13(2012), No. 1, 165-172
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ASYMPTOTIC BEHAVIOR OF PERTURBED ITERATES OF SET-VALUED MAPPINGS

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Abstract. It has recently been shown that if for any initial point there exists a trajectory of a nonexpansive set-valued mapping attracted by a given set, then this property is stable under small perturbations of the mapping. In the present paper we show that the same conclusion continues to hold under the weaker condition that for any initial point there exists a trajectory of the nonexpansive set-valued mapping with a subsequence which is attracted by the attractor.
Key Words and Phrases: Attractor, Hausdorff distance, metric space, nonexpansive set-valued mapping, trajectory.
2010 Mathematics Subject Classification: 47H04, 47H09, 47H10, 54E35.

1. Introduction and Preliminaries

The study of iterations of contractive mappings has been an important topic in Nonlinear Functional Analysis since Banach's seminal paper [1] on the existence of a unique fixed point for a strict contraction acting on a complete metric space. It is well known that Banach's fixed point theorem also yields convergence of iterates to the unique fixed point. Many developments have taken place in this area in recent decades. Interesting results have also been obtained regarding set-valued mappings, where the situation is more difficult and less understood. See, for example, [2, 6, 9-14] and the references mentioned therein.

As we have already mentioned, one of the methods used for proving the classical Banach theorem is to show the convergence of Picard iterations, which holds for any initial point. In the case of set-valued mappings, not all trajectories of the dynamical system induced by the given mapping converge. Therefore convergent trajectories have to be constructed in a special way. For instance, in [6], if at the moment $t = 0, 1, \dots$ one has reached a point x_t, then one chooses an element of $T(x_t)$ (here T is the given mapping) such that x_{t+1} approximates the best approximation of x_t from $T(x_t)$. Since the given mapping acts on an arbitrary complete metric space, one cannot, in general, choose x_{t+1} to be the best approximation of x_t by elements of $T(x_t)$. Instead, one chooses x_{t+1} so that it approximates the best approximation up to a positive number ϵ_t, such that the sequence $\{\epsilon_t\}_{t=0}^{\infty}$ is summable. This method allowed Nadler [6] to obtain the existence of a fixed point of a strictly contractive set-valued mapping and the authors of [2] to establish more general results. Thus it is important to study convergence of the iterates of both single- and set-valued mappings in the presence of inexact data.

In particular, it is natural to ask if convergence of the iterates of nonexpansive mappings will be preserved in the presence of computational errors. Affirmative answers to this question are given in [3]. Related results can be found, for example, in [4, 5, 7, 8].

In a recent paper [13] it has been shown (Theorem 1.1 below) that if for any initial point there exists a trajectory of a nonexpansive set-valued mapping attracted by a given set, then this property is stable under small perturbations of the mapping.

In the present paper we show (see Theorems 1.5 and 1.6 below) that the same conclusions continue to hold under the weaker condition that for any initial point there exists a trajectory of the nonexpansive set-valued mapping with a subsequence which is attracted by the attractor.

Let (X, ρ) be a metric space. For each $x \in X$ and each nonempty set $A \subset X$, put

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

For each pair of nonempty sets $A, B \subset X$, set

$$H(A, B) = \max\{\sup_{x \in A} \rho(x, B), \ \sup_{y \in B} \rho(y, A)\}.$$

Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy

$$H(T(x), T(y)) \leq \rho(x, y) \text{ for all } x, y \in X. \tag{1.1}$$

The following result has recently been established in [13].

Theorem 1.1 *Assume that F is a nonempty subset of X and that for each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, $x_{i+1} \in T(x_i)$ for all integers $i \geq 0$, and*

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Assume that

$$\{\epsilon_i\}_{i=0}^{\infty} \subset (0, \infty), \ \sum_{i=0}^{\infty} \epsilon_i < \infty.$$

For each integer $i \geq 0$, let $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfy

$$H(T_i(x), T(x)) \leq \epsilon_i, \ x \in X.$$

Then the following two assertions hold.

1. Let $\delta > 0$. For each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_0 = x$, for each integer $i \geq 0$,

$$x_{i+1} \in T_i(x_i),$$

and

$$\rho(x_i, F) \leq \delta \text{ for all sufficiently large integers } i \geq 0.$$

2. For each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_0 = x$, for each integer $i \geq 0$,

$$x_{i+1} \in T_i(x_i),$$

and

$$\liminf_{i\to\infty} \rho(x_i, F) = 0.$$

The main assumption of Theorem 1.1 is that for any initial state x_0, there is a trajectory $\{x_i\}_{i=0}^{\infty}$ of the dynamical system induced by the mapping T such that $\lim_{i\to\infty} \rho(x_i, F) = 0$. In the present paper we show that Assertions 1 and 2 of Theorem 1.1 continue to hold when instead of $\lim_{i\to\infty} \rho(x_i, F) = 0$ we only assume that $\liminf_{i\to\infty} \rho(x_i, F) = 0$. Therefore Theorems 1.5 and 1.6 improve upon Assertions 1 and 2 of Theorem 1.1, respectively. Note, however, that in Theorem 1.5 we introduce an additional hypothesis. Namely, we assume that for any initial point from F there is a trajectory of the dynamical system induced by the given mapping T which stays in F (see (1.5)).

Since in this paper we deal with subsequences of trajectories which converge to the attractor, the following three propositions turn out to be useful.

The proof of the first one is obvious.

Proposition 1.2 *Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy (1.1) and let F be a nonempty subset of X. Then the following properties are equivalent.*

1. For each $\epsilon > 0$ and each $x \in X$, there is a natural number k and a sequence $\{x_i\}_{i=0}^{k} \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i),\ i = 0, \dots, k-1, \text{ and } \rho(x_k, F) < \epsilon.$$

2. For each $\epsilon > 0$ and each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i) \text{ for all integers } i \geq 0, \tag{1.2}$$

and

$$\liminf_{i\to\infty} \rho(x_i, F) < \epsilon.$$

3. For each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, (1.2) holds and $\liminf_{i\to\infty} \rho(x_i, F) = 0$.

The next proposition is proved in Section 2.

Proposition 1.3 *Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy (1.1). Assume that F is a nonempty subset of X such that*

$$T(y) \cap F \neq \emptyset \text{ for each } y \in F.$$

Assume further that for each $x \in X$ and each $\epsilon > 0$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, (1.2) holds and

$$\liminf_{i\to\infty} \rho(x_i, F) < \epsilon.$$

Then for each $\epsilon > 0$ and each $x \in X$, there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$, (1.2) holds and

$$\rho(x_i, F) < \epsilon \text{ for all sufficiently large natural numbers } i.$$

These two propositions, when taken together, imply the third one.

Proposition 1.4 *Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy (1.1). Assume that F is a nonempty subset of X such that*

$$T(y) \cap F \neq \emptyset \text{ for each } y \in F.$$

Then the following properties are equivalent.

1. For each $\epsilon > 0$ and each $x \in X$, there is a natural number k and a sequence $\{x_i\}_{i=0}^k \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i),\ i = 0, \dots, k-1, \text{ and } \rho(x_k, F) < \epsilon.$$

2. For each $\epsilon > 0$ and each $x \in X$, there is a sequence $\{x_i\}_{i=0}^\infty \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i) \text{ for all integers } i \geq 0,$$

and

$$\liminf_{i \to \infty} \rho(x_i, F) < \epsilon.$$

3. For each $x \in X$, there is a sequence $\{x_i\}_{i=0}^\infty \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i) \text{ for all integers } i \geq 0,$$

and

$$\liminf_{i \to \infty} \rho(x_i, F) = 0.$$

4. For each $\epsilon > 0$ and each $x \in X$, there is a sequence $\{x_i\}_{i=0}^\infty \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i) \text{ for all integers } i \geq 0,$$

and

$$\rho(x_i, F) < \epsilon \text{ for all sufficiently large natural numbers } i.$$

The following example shows that the assumption

$$T(y) \cap F \neq \emptyset \text{ for each } y \in F$$

is essential in both Propositions 1.3 and 1.4.

Example. *Let $X = \{0, 1\}$, $F = \{0\}$, $T(0) = \{1\}$, $T(1) = \{0\}$. Clearly, properties (1)-(3) of Proposition 1.4 hold while property (4) does not.*

Now assume that F is a nonempty subset of X and that the following property holds:

(P) For each $x \in X$, there is a sequence $\{x_i\}_{i=0}^\infty \subset X$ such that $x_0 = x$,

$$x_{i+1} \in T(x_i) \text{ for all integers } i \geq 0, \text{ and } \liminf_{i \to \infty} \rho(x_i, F) = 0.$$

Let

$$\{\epsilon_i\}_{i=0}^\infty \subset (0, \infty),\ \sum_{i=0}^\infty \epsilon_i < \infty. \tag{1.3}$$

For each integer $i \geq 0$, let $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfy

$$H(T_i(x), T(x)) \leq \epsilon_i,\ x \in X. \tag{1.4}$$

The following two theorems are the main results of our paper.

Theorem 1.5 *Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy (1.1). Assume that property (P) holds and that for each integer $i \geq 0$, the mapping $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfies (1.4). Assume further that*

$$T(y) \cap F \neq \emptyset \text{ for each } y \in F. \tag{1.5}$$

Let $x \in X$ and $\epsilon > 0$. Then there exists a sequence $\{x_i\}_{i=0}^{\infty}$ in X such that $x_0 = x$, $x_{i+1} \in T_i(x_i)$ for all natural numbers i and $\rho(x_i, F) < \epsilon$ for all sufficiently large natural numbers i.

Theorem 1.6 *Let $T : X \to 2^X \setminus \{\emptyset\}$ satisfy (1.1). Assume that property (P) holds and that for each integer $i \geq 0$, the mapping $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfies (1.4). Let $x \in X$. Then there exists a sequence $\{x_i\}_{i=0}^{\infty}$ such that $x_0 = x$, $x_{i+1} \in T_i(x_i)$ for all natural numbers i and $\liminf_{i\to\infty} \rho(x_i, F) = 0$.*

Our paper is organized as follows. The next section is devoted to the proof of Proposition 1.3. In the third section we recall an auxiliary result from [13]. Theorems 1.5 and 1.6 are proved in Sections 4 and 5, respectively.

2. Proof of Proposition 1.3

Let $\epsilon > 0$ and $x \in X$. By the assumptions of the proposition, there is a natural number k and a sequence $\{x_i\}_{i=0}^{k} \subset X$ such that

$$x_0 = x, \ x_{i+1} \in T(x_i) \text{ for all integers } i = 0, \ldots, k-1, \tag{2.1}$$

and

$$\rho(x_k, F) < \epsilon/2. \tag{2.2}$$

Using induction, we now show that there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$\rho(x_i, F) < \epsilon/2 \text{ for all integers } i \geq k \tag{2.3}$$

and

$$x_{i+1} \in T(x_i) \text{ for all integers } i \geq 0. \tag{2.4}$$

Assume that $q \geq k$ is an integer and that we have already constructed a sequence $\{x_i\}_{i=0}^{q} \subset X$ such that

$$x_0 = x, \ x_{i+1} \in T(x_i) \text{ for all integers } i = 0, \ldots, q-1,$$

and

$$\rho(x_i, F) < \epsilon/2, \ i = k, \ldots, q. \tag{2.5}$$

(Clearly, for $q = k$ this assumption holds.)

By (2.5), there is

$$y_q \in F \tag{2.6}$$

such that

$$\rho(x_q, y_q) < \epsilon/2. \tag{2.7}$$

By (1.1) and (2.7), we have

$$H(T(x_q), T(y_q)) \leq \rho(x_q, y_q) < \epsilon/2. \tag{2.8}$$

By (1.5) and (2.6), there is

$$y_{q+1} \in T(y_q) \cap F. \tag{2.9}$$

In view of (2.8) and (2.9),

$$\rho(y_{q+1}, T(x_q)) < \epsilon/2$$

and therefore there is

$$x_{q+1} \in T(x_q) \tag{2.10}$$

such that

$$\rho(y_{q+1}, x_{q+1}) < \epsilon/2. \tag{2.11}$$

By (2.9) and (2.11),

$$\epsilon/2 > \rho(y_{q+1}, x_{q+1}) \geq \rho(x_{q+1}, F)$$

and the assumption we have made regarding q also holds for $q + 1$. Therefore we have constructed by induction a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying (2.3) and (2.4). Proposition 1.3 is proved.

3. An auxiliary result

The following result has been obtained in [13, Lemma 2.1]. Its setting is that of Theorem 1.1 (see (1.3) and (1.4)).

Lemma 3.1 *Let $q \geq 0$ be an integer. Let the sequence $\{x_i\}_{i=q}^{\infty} \subset X$ satisfy*

$$x_{i+1} \in T(x_i)$$

for each integer $i \geq q$. Then there is a sequence $\{y_i\}_{i=q}^{\infty} \subset X$ such that

$$y_q = x_q, \ y_{i+1} \in T_i(y_i) \text{ for all integers } i \geq q,$$

and for all integers $j \geq q + 1$,

$$\rho(y_j, x_j) \leq \sum_{i=q}^{j-1} 2\epsilon_i.$$

4. Proof of Theorem 1.5

By (1.3), there is a natural number k_0 such that

$$\sum_{i=k_0}^{\infty} \epsilon_i < \epsilon/8. \tag{4.1}$$

There is also a sequence $\{x_i\}_{i=0}^{k_0} \subset X$ such that

$$x_0 = x, \ x_{i+1} \in T_i(x_i), \ i = 0, \dots, k_0 - 1. \tag{4.2}$$

By (1.5), property (P) and Proposition 1.3, there is a sequence $\{z_i\}_{i=k_0}^{\infty} \subset X$ such that

$$z_{k_0} = x_{k_0}, \ z_{i+1} \in T(z_i), \ i = k_0, k_0 + 1, \dots, \tag{4.3}$$

and

$$\rho(z_i, F) < \epsilon/8 \text{ for all suficiently large natural numbers } i. \tag{4.4}$$

By Lemma 3.1, (4.3) and (4.1), there is a sequence $\{x_i\}_{i=k_0}^{\infty} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \geq k_0, \tag{4.5}$$

and such that for all integers $j \geq k_0 + 1$,

$$\rho(z_j, x_j) \leq \sum_{i=k_0}^{j-1} 2\epsilon_i < \epsilon/4. \tag{4.6}$$

By (4.4), there is a natural number $k_1 > k_0$ such that for all integers $j \geq k_1$,

$$\rho(z_j, F) \leq \epsilon/8. \tag{4.7}$$

By (4.6) and (4.7), for all integers $j \geq k_1$ we have

$$\rho(x_j, F) \leq \rho(x_j, z_j) + \rho(z_j, F) < \epsilon/4 + \epsilon/8 < \epsilon.$$

Theorem 1.5 is proved.

5. Proof of Theorem 1.6

Let $x \in X$. Set

$$S_0 = 0 \text{ and } x_0 = x. \tag{5.1}$$

Assume that $q \geq 0$ is an integer and that we have already defined a strictly increasing sequence of nonnegative integers $S_i,\ i = 0, \ldots, q$, and a sequence $\{x_i\}_{i=0}^{S_q} \subset X$ such that (5.1) holds,

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \text{ satisfying } 0 \leq i < S_q, \tag{5.2}$$

and that for all integers j satisfying $1 \leq j \leq q$,

$$\rho(x_{S_j}, F) \leq 1/j. \tag{5.3}$$

(Note that for $q = 0$ this assumption holds). By (1.3), there is a natural number $R_1 > S_q + 4$ such that

$$\sum_{i=R_1}^{\infty} \epsilon_i < (4(q+1))^{-1}. \tag{5.4}$$

There is a sequence $\{x_i\}_{i=S_q}^{R_1} \subset X$ such that

$$x_{i+1} \in T_i(x_i),\ i = S_q, \ldots, R_1 - 1. \tag{5.5}$$

By property (P), there exist a natural number $S_{q+1} > R_1 + 4$ and a sequence $\{z_i\}_{i=R_1}^{S_{q+1}} \subset X$ such that

$$z_{R_1} = x_{R_1},\ z_{i+1} \in T(z_i) \text{ for all integers } i = R_1, \ldots, S_{q+1} - 1, \tag{5.6}$$

and

$$\rho(z_{S_{q+1}}, F) < (2(q+1))^{-1}. \tag{5.7}$$

By (5.6), (5.7), Lemma 3.1 and (5.4), there is a sequence $\{x_i\}_{i=R_1}^{S_{q+1}} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i = R_1, \ldots, S_{q+1} - 1, \tag{5.8}$$

and for all integers $j = R_1 + 1, \ldots, S_{q+1}$,

$$\rho(x_j, z_j) \leq \sum_{i=R_1}^{j-1} 2\epsilon_i < 2\sum_{i=R_1}^{\infty} \epsilon_i < (2(q+1))^{-1}.$$

When combined with (5.7), this inequality implies that

$$\rho(x_{S_{q+1}}, F) \le \rho(x_{S_{q+1}}, z_{S_{q+1}}) + \rho(z_{S_{q+1}}, F) < (q+1)^{-1}.$$

Thus the assumption we made for q also holds for $q+1$. Therefore we have constructed by induction a strictly increasing sequence of nonnegative integers $\{S_q\}_{q=0}^{\infty}$ and a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$x_{i+1} \in T_i(x_i) \text{ for all integers } i \ge 0$$

and

$$\rho(x_{S_q}, F) \le q^{-1} \text{ for all integers } q \ge 1.$$

This completes the proof of Theorem 1.6.

Acknowledgments. This research was supported by the Israel Science Foundation (Grant No. 647/07), the Fund for the Promotion of Research at the Technion and by the Technion President's Research Fund.

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Received: May 11, 2011; Accepted: May 31, 2011.