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HYBRID-EXTRAGRADIENT TYPE METHODS FOR A GENERALIZED EQUILIBRIUM PROBLEM AND VARIATIONAL INEQUALITY PROBLEMS OF NONEXPANSIVE SEMIGROUPS

PHAYAP KATCHANG* AND POOM KUMAM**,[1]

*Department of Mathematics and Statistics, Faculty of Science and Agricultural Technology,
Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand
E-mail: p.katchang@hotmail.com

**Department of Mathematics, Faculty of Science, King Mongkut's University of Technology
Thonburi (KMUTT), Bangmod, Bangkok 10140, Thailand
E-mail: poom.kum@kmutt.ac.th

Abstract. We study and introduce modified mann iterative algorithms for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of variational inequalities and the set of fixed points for nonexpansive semigroups. Then, we prove strong convergence theorems in a real Hilbert space by using the hybrid-extragradient type methods in the mathematical programming under some appropriate control conditions.
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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $G : C \to H$ be a nonlinear mapping. In 2008 Takahashi and Takahashi [21] and Peng and Yao [14, 15] considered the following *generalized equilibrium problem*: Find $x \in C$ such that

$$F(x, y) + \langle Gx, y - x \rangle \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GEP(F, G)$. In the case of $G = 0$, then the problem (1.1) becomes the following *equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.2}$$

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[1]Corresponding author.

The set of solutions of (1.2) is denoted by *EP(F)*. If $F = 0$ for all $x, y \in C$, then the problem (1.1) becomes the following *variational inequality problem* is to find $x \in C$ such that

$$\langle Gx, y - x \rangle \geq 0 \text{ for all } y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by *VI(C,G)*. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games and others; see for instance [2, 3, 6, 11, 21]. Recently, many authors considered the problem of finding a common element of the set of solutions to the equilibrium problem (1.2) and variational inequality problem (1.3) and of the set of fixed points of nonexpansive mapping in Hilbert spaces; see, for example, [2, 16, 11, 12, 14, 15, 21] and the references therein.

Recall that $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of *fixed points of* T by $F(T)$, that is $F(T) = \{x \in C : x = Tx\}$. A family $\mathcal{T} = \{T(t) : t \geq 0\}$ of mappings of C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:
(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C, s \mapsto T(s)x$ is continuous.

We denote by $F(\mathcal{T})$ the set of all common *fixed points of* $\mathcal{T}$, that is,

$$F(\mathcal{T}) = \bigcap_{t=0}^{\infty} F(T(t)) = \{x \in C : T(t)x = x,\ 0 \leq t < \infty\}.$$

It is know that $F(\mathcal{T})$ is closed and convex.

In 1953, Mann [10] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.4}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [17]. In an infinite-dimensional Hilbert space, the Mann iteration can conclude only weak convergence [8]. Attempts to modify the Mann iteration method (1.4) so that strong convergence is guaranteed have recently been made. Generally speaking, the algorithm suggested by Takahashi and Toyoda [22] is based on two well-known types of methods, namely, on the projection-type methods for solving variational inequality problems and so-called hybrid or outer-approximation methods for solving fixed point problems. The idea of "hybrid" or "outer-approximation" types of methods was originally introduced by Haugazeau in 1968; see [1] for more details.

In 2002, Suzuki [19] was the first one to introduced the following implicit iteration process in Hilbert spaces:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)(x_n),\ n \geq 1, \tag{1.5}$$

for the nonexpansive semigroup. In 2007, Xu [24] established a Banach space version of the sequence (1.5) of Suzuki [19]. In [4], Chen and He considered the viscosity approximation process for a nonexpansive semigroup and proved another strong convergence theorem for a nonexpansive semigroup in Banach spaces, which is defined by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \ \forall n \in \mathbb{N}, \tag{1.6}$$

where, $f : C \to C$ be a fixed contractive mapping. Korpelevich [9] introduced the following so-called extragradient method also:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \tag{1.7}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{k}), C$ is a closed convex subset of $\mathbb{R}^n$ and A is a monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.7), converge to the same point $z \in VI(C, A)$.

In 2008, Saejung [18] proved the strong convergence theorems for nonexpansive semigroups without Bochner integrals in Hilbert spaces. The sequence $\{x_n\}$ defined by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)x_n, \\ C_{n+1} = \{z \in C_n \mid \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0, \end{cases} \tag{1.8}$$

and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T(t_n)x_n, \\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ \forall n \geq 0, \end{cases} \tag{1.9}$$

where $\{t_n\}$ is a real sequence, $\{\alpha_n\} \subset [0, 1)$ and $\{T(t) : t \geq 0\}$ is a nonexpansive semigroup on C.

In the same year, Takahashi and Takahashi [21] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. The sequence $\{x_n\}$ defined by: $u, x_1 \in C$ and

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)u_n] \end{cases} \tag{1.10}$$

for all $n \geq 0$. Where F is a bifunction from $C \times C$ into $\mathbb{R}$, $A : C \to H$ are an inverse-strongly monotone mapping and S is a nonexpansive mapping of C into itself. They proved some strong convergence theorems under suitable conditions.

In this paper, we prove the strong convergence theorems of modified mann iterative algorithms for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of two variational inequalities and the set of solutions of nonexpansive semigroups in a Hilbert space under some appropriate control conditions by using the new hybrid-extragradient methods in the mathematical programming. The results presented in this paper extend and improve the corresponding ones announced by Saejung [18], Takahashi and Takahashi [21] and many others.

2. Preliminaries

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. Then

$$\|x-y\|^2=\|x\|^2-\|y\|^2-2\langle x-y,y\rangle \tag{2.1}$$

and

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x,y\in H$ and $\lambda\in\mathbb{R}$.

A space X is said to satisfy Opial's condition [13], if for each sequence $\{x_n\}$ in X which converges weakly to a point $x\in X$, we have

$$\liminf_{n\to\infty}\|x_n-x\|<\liminf_{n\to\infty}\|x_n-y\|,\ \forall y\in X,\ y\neq x.$$

Recall that, for every point $x\in H$, there exists a unique nearest point in C, denoted by P_Cx, such that

$$\|x-P_Cx\|\leq\|x-y\|\quad\text{for all } y\in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y,P_Cx-P_Cy\rangle\geq\|P_Cx-P_Cy\|^2 \tag{2.3}$$

for every $x,y\in H$. Moreover, P_Cx is characterized by the following properties: $P_Cx\in C$ and

$$\langle x-P_Cx,y-P_Cx\rangle\leq 0, \tag{2.4}$$

$$\|x-y\|^2\geq\|x-P_Cx\|^2+\|y-P_Cx\|^2 \tag{2.5}$$

for all $x\in H, y\in C$.

Hilbert space H satisfies the *Kadec-Klee property* [7, 20], that is, for any sequence $\{x_n\}$ with $x_n\rightharpoonup x$ and $\|x_n\|\to\|x\|$ together imply $\|x_n-x\|\to 0$.

For solving the equilibrium problem, let us give the following assumptions for the bifunction $F:C\times C\to\mathbb{R}$ satisfies the following condition:

(A1) $F(x,x)=0$ for all $x\in C$;
(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;
(A3) for each $x,y,z\in C$, $\lim_{t\to 0}F(tz+(1-t)x,y)\leq F(x,y)$;
(A4) for each $x\in C, y\mapsto F(x,y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

Lemma 2.1. (Blum and Oettli [3]) *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C\times C$ into $\mathbb{R}$ satisfies (A1)-(A4). Let $r>0$ and $z\in H$. Then, there exists $x\in C$ such that*

$$F(x,y)+\frac{1}{r}\langle y-x,x-z\rangle\geq 0,\ \forall y\in C. \tag{2.6}$$

Lemma 2.2. (Combettes and Hirstoaga [5]) *Let* C *be a nonempty closed convex subset of* H. *Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfies (A1)-(A4). For $r>0$ and $z\in H$, define a mapping $T_r:H\to C$ as follows:*

$$T_r(z)=\{x\in C:F(x,y)+\frac{1}{r}\langle y-x,x-z\rangle\geq 0,\quad\forall y\in C\}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,* $\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle$;
(3) $F(T_r) = EP(F)$;
(4) *$EP(F)$ is closed and convex.*

Remark 2.3. Replacing z with $z - rGz \in H$ in (2.6), then there exists $x \in C$, such that $F(x, y) + \langle Gz, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \ge 0,\ \forall y \in C$.

3. Main results

In this section, we prove strong convergence theorems for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of two variational inequalities and the set of fixed points for a nonexpansive semigroup in a real Hilbert space.

3.1. **The hybrid method.**

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \ge 0\}$ be a nonexpansive semigroup on C, let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $G, A, B : C \to H$ be three α, β, λ-inverse-strongly monotone mappings, respectively. Suppose that $\Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap VI(C, A) \cap VI(C, B) \cap GEP(F, G) \ne \emptyset$. Let $\{\alpha_n\} \subset [0, a) \subset [0, 1)$, $\{\beta_n\} \subset [b, b'] \subset (0, 2\beta)$, $\{\lambda_n\} \subset [l, l'] \subset (0, 2\lambda)$, $\{r_n\} \subset [r, r'] \subset (0, 2\alpha)$ and $\{t_n\} \subset [0, \infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n(t_{n+1} - t_n) = 0$. For $x_0 \in H$, let the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $u_n \in C$ and*

$$\begin{cases} F(u_n, y) + \langle Gx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0,\ \forall y \in C, \\ v_n = P_C(u_n - \lambda_n Bu_n), \\ z_n = P_C(v_n - \beta_n Av_n), \\ y_n = \alpha_n u_n + (1 - \alpha_n)S(t_n)z_n, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0,\ \forall n \ge 0. \end{cases} \tag{3.1}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. It is obvious that C_n and Q_n are closed and convex for all $n \ge 0$. Thus that $C_n \cap Q_n$ is closed and convex for all $n \ge 0$. Let $x^* \in \Omega$ and $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2 then, $x^* = T_{r_n}(x^* - r_n Gx^*) = P_C(x^* - \beta_n Ax^*) = P_C(x^* - \lambda_n Bx^*)$ and $u_n = T_{r_n}(x_n - r_n Gx_n) \in C$. Note that $I - r_n G$ is nonexpansive for all $n \ge 0$, for all $u, v \in C$ and $\{r_n\} \subset (0, 2\alpha)$, we have

$$\begin{aligned} \|(I - r_n G)u - (I - r_n G)v\|^2 &= \|(u - v) - r_n(Gu - Gv)\|^2 \\ &= \|u - v\|^2 - 2r_n\langle u - v, Gu - Gv\rangle + r_n^2\|Gu - Gv\|^2 \\ &\le \|u - v\|^2 + r_n(r_n - 2\alpha)\|Gu - Gv\|^2 \le \|u - v\|^2. \end{aligned} \tag{3.2}$$

By the same method, we obtain that

$$\|(I-\beta_n A)u-(I-\beta_n A)v\| \leq \|u-v\|$$

and

$$\|(I-\lambda_n B)u-(I-\lambda_n B)v\| \leq \|u-v\|.$$

We note that

$$\begin{aligned}
\|u_n-x^*\| &= \|T_{r_n}(x_n-r_nGx_n)-T_{r_n}(x^*-r_nGx^*)\| \\
&\leq \|(x_n-r_nGx_n)-(x^*-r_nGx^*)\| \\
&\leq \|x_n-x^*\| \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
\|v_n-x^*\| &= \|P_C(u_n-\lambda_nBu_n)-P_C(x^*-\lambda_nBx^*)\| \\
&\leq \|(u_n-\lambda_nBu_n)-(x^*-\lambda_nBx^*)\| \\
&\leq \|u_n-x^*\| \\
&\leq \|x_n-x^*\| \qquad (3.4)
\end{aligned}$$

hence

$$\begin{aligned}
\|z_n-x^*\| &= \|P_C(v_n-\beta_nAv_n)-P_C(x^*-\beta_nAx^*)\| \\
&\leq \|(v_n-\beta_nAv_n)-(x^*-\beta_nAx^*)\| \\
&\leq \|v_n-x^*\| \\
&\leq \|x_n-x^*\|. \qquad (3.5)
\end{aligned}$$

It follows by (3.3), we obtain

$$\begin{aligned}
\|y_n-x^*\| &= \|\alpha_nu_n+(1-\alpha_n)S(t_n)z_n-x^*\| \\
&\leq \alpha_n\|u_n-x^*\|+(1-\alpha_n)\|S(t_n)z_n-x^*\| \\
&\leq \alpha_n\|u_n-x^*\|+(1-\alpha_n)\|z_n-x^*\| \\
&\leq \alpha_n\|u_n-x^*\|+(1-\alpha_n)\|u_n-x^*\| \\
&= \|u_n-x^*\| \\
&\leq \|x_n-x^*\|. \qquad (3.6)
\end{aligned}$$

Therefore, $\Omega \subset C_n$ for all $n \geq 0$.

By induction, we show that $\Omega \subset C_n \cap Q_n$ for all $n \geq 0$. Form $x_1 = P_C x_0$, we have

$$\langle x_1-y, x_0-x_1\rangle \geq 0 \ \text{ for all } \ y \in C,$$

and hence $Q_1 = C$. So, we have $\Omega \subset Q_1$. Then, $\Omega \subset C_1 \cap Q_1$. Suppose that $\Omega \subset C_k \cap Q_k$ for some $k \geq 0$. From $x_{k+1} = P_{C_k\cap Q_k}x_0$, we have

$$\langle x_{k+1}-y, x_0-x_{k+1}\rangle \geq 0 \ \text{ for all } \ y \in C_k \cap Q_k.$$

Since $\Omega \subset C_k \cap Q_k$, we have

$$\langle x_{k+1}-u, x_0-x_{k+1}\rangle \geq 0 \ \text{ for all } \ u \in \Omega,$$

and hence $\Omega \subset Q_{k+1}$. Since $\Omega \subset C_n$ for all $n \geq 0$, we have $\Omega \subset C_{k+1} \cap Q_{k+1}$. So, we have that $\Omega \subset C_n \cap Q_n$ for all $n \geq 0$. Then, $\{x_n\}$ is well-defined.

Let $z_0 = P_\Omega x_0$. From $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $z_0 \in \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\| \tag{3.7}$$

for all $n \geq 0$. Therefore, $\{x_n\}$ is bounded. So, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded.

Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n} x_0$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, for all $n \geq 0$. It follows that $\{x_n\}$ in nondecreasing and from $\{x_n\}$ bounded. So there exists the limit of $\|x_n - x_0\|$.

Since $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, we have $\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0$ and hence

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle \\
&\quad + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n \rangle + 2\langle x_n - x_0, x_n - x_{n+1} \rangle \\
&\quad + \|x_0 - x_{n+1}\|^2 \\
&\leq \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n \rangle + \|x_0 - x_{n+1}\|^2 \\
&\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty} \|x_0 - x_n\|$ exists, implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.8}$$

Since $x_{n+1} \in C_n$, we have

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}$$

By (3.2) and (3.6), we obtain

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Gx_n - Gx^*\|^2,$$

therefore,

$$\begin{aligned}
r(2\alpha - r')\|Gx_n - Gx^*\|^2 &\leq r_n(2\alpha - r_n)\|Gx_n - Gx^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\
&\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|.
\end{aligned}$$

It follows from (3.9) and since $\{x_n\}$ and $\{y_n\}$ are bounded that

$$\lim_{n\to\infty} \|Gx_n - Gx^*\| = 0. \tag{3.10}$$

By the same method, we have

$$\lim_{n\to\infty} \|Av_n - Ax^*\| = 0, \tag{3.11}$$

and

$$\lim_{n\to\infty} \|Bu_n - Bx^*\| = 0. \tag{3.12}$$

For $x^* \in \Omega$, from Lemma 2.2, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n G x_n) - T_{r_n}(x^* - r_n G x^*)\|^2 \\
&\leq \langle T_{r_n}(x_n - r_n G x_n) - T_{r_n}(x^* - r_n G x^*), \\
&\quad (x_n - r_n G x_n) - (x^* - r_n G x^*)\rangle \\
&= \frac{1}{2}\{\|u_n - x^*\|^2 + \|(x_n - r_n G x_n) - (x^* - r_n G x^*)\|^2 \\
&\quad -\|(x_n - r_n G x_n) - (x^* - r_n G x^*) - (u_n - x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|(x_n - u_n) \\
&\quad -r_n(G x_n - G x^*)\|^2\} \\
&= \frac{1}{2}\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \\
&\quad +2r_n\langle G x_n - G x^*, x_n - u_n\rangle - r_n^2\|G x_n - G x^*\|^2\},
\end{aligned}$$

hence,

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle G x_n - G x^*, x_n - u_n\rangle.$$

By (3.6), it follows that

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle G x_n - G x^*, x_n - u_n\rangle,$$

therefore,

$$\begin{aligned}
\|x_n - u_n\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2r_n\langle G x_n - G x^*, x_n - u_n\rangle \\
&\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\| \\
&\quad +2r_n\|G x_n - G x^*\|\|x_n - u_n\|.
\end{aligned}$$

From (3.9) and (3.10), we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.13}$$

For $x^* \in \Omega$, from (2.3) and (3.4), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(v_n - \beta_n A v_n) - P_C(x^* - \beta_n A x^*)\|^2 \\
&\leq \langle (v_n - \beta_n A v_n) - (x^* - \beta_n A x^*), z_n - x^*\rangle \\
&= \frac{1}{2}\{\|z_n - x^*\|^2 + \|(v_n - \beta_n A v_n) - (x^* - \beta_n A x^*)\|^2 \\
&\quad -\|(v_n - \beta_n A v_n) - (x^* - \beta_n A x^*) - (z_n - x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|v_n - x^*\|^2 - \|(v_n - z_n) - \beta_n(A v_n - A x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|z_n - x^*\|^2 + \|x_n - x^*\|^2 - \|v_n - z_n\|^2 \\
&\quad +2\beta_n\langle A v_n - A x^*, v_n - z_n\rangle - r_n^2\|A v_n - A x^*\|^2\},
\end{aligned}$$

hence,

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|v_n - z_n\|^2 + 2\beta_n\langle A v_n - A x^*, v_n - z_n\rangle.$$

By (3.5), it follows that

$$\|u_n - x^*\|^2 \leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|v_n - z_n\|^2 + 2\beta_n \langle Av_n - Ax^*, v_n - z_n\rangle,$$

therefore,

$$\begin{aligned}\|v_n - z_n\|^2 &\leq \|x_n - x^*\|^2 - \|u_n - x^*\|^2 + 2\beta_n \langle Av_n - Ax^*, v_n - z_n\rangle \\ &\leq (\|x_n - x^*\| + \|u_n - x^*\|)\|x_n - u_n\| \\ &\quad +2\beta_n \|Av_n - Ax^*\|\|v_n - z_n\|.\end{aligned}$$

From (3.11) and (3.13), we obtain

$$\lim_{n\to\infty} \|v_n - z_n\| = 0. \tag{3.14}$$

By the same way, using (3.12) and (3.14) we get

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.15}$$

Since $y_n - x_n = \alpha_n u_n + (1-\alpha_n)S(t_n)z_n - x_n = \alpha_n(u_n - x_n) + (1-\alpha_n)(S(t_n)z_n - x_n)$, it follows that

$$\|x_n - S(t_n)z_n\| = \frac{\alpha_n}{1-\alpha_n}\|u_n - x_n\| + \frac{1}{1-\alpha_n}\|x_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Since $S(t_n)$ is a nonexpansive mapping, we have

$$\begin{aligned}\|x_n - S(t_n)x_n\| &\leq \|x_n - S(t_n)z_n\| + \|S(t_n)z_n - S(t_n)x_n\| \\ &\leq \|x_n - S(t_n)z_n\| + \|z_n - x_n\| \\ &\leq \|x_n - S(t_n)z_n\| + \|z_n - v_n\| + \|v_n - u_n\| + \|u_n - x_n\|.\end{aligned}$$

From (3.13), (3.14), (3.15) and (3.16), we obtain

$$\lim_{n\to\infty} \|x_n - S(t_n)x_n\| = 0. \tag{3.17}$$

Since $\{x_n\}$ is bounded, we choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and assume that $x_{n_i} \rightharpoonup x'$. Let us show that $x' \in \Omega$. First, we show that $x' \in \cap_{t=0}^{\infty} F(S(t))$. Suppose that $x' \notin \cap_{t=0}^{\infty} F(S(t))$, that is $x' \neq S(t)x'$. From Opial's condition and (3.17), we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i} - x'\| &< \liminf_{i\to\infty}\|x_{n_i} - S(t)x'\| \\ &\leq \liminf_{i\to\infty}(\|x_{n_i} - S(t)x_{n_i}\| + \|S(t)x_{n_i} - S(t)x'\|) \\ &\leq \liminf_{i\to\infty}\|x_{n_i} - x'\|.\end{aligned}$$

This is a contradiction. Thus, we obtain $x' \in \cap_{t=0}^{\infty} F(S(t))$.

Next, let us show $x' \in GEP(F, G)$. Since $u_n = T_{r_n}(x_n - r_n G x_n)$ and

$$F(u_n, y) + \langle Gx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

From (A2), we also have

$$\langle Gx_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C,$$

and hence

$$\langle Gx_n, y - u_n\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{3.18}$$

From (3.13), we get $u_{n_i} \rightharpoonup x'$. For t with $0 < t \leq 1$ and $y \in C$, put $y_t = ty + (1-t)x'$. Since $y \in C$ and $x' \in C$, we have $y_t \in C$. So, from (3.18), we have and hence

$$\begin{aligned}\langle Gy_t, y_t - u_{n_i}\rangle &\geq \langle Gy_t, y_t - u_{n_i}\rangle - \langle Gx_{n_i}, y_t - u_{n_i}\rangle - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\ &\quad + F(y_t, u_{n_i}) \\ &= \langle Gy_t - Gu_{n_i}, y_t - u_{n_i}\rangle + \langle Gu_{n_i} - Gx_{n_i}, y_t - u_{n_i}\rangle \\ &\quad - \langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(y_t, u_{n_i}).\end{aligned}$$

From $\|u_{n_i} - x_{n_i}\| \to 0$, we obtain $\|Gu_{n_i} - Gx_{n_i}\| \to 0$. By the α-inverse-strongly monotonicity of G, we know that $\langle Gy_t - Gu_{n_i}, y_t - u_{n_i}\rangle \geq 0$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, it follows by (A4) that

$$F(y_t, x') \leq \lim_{i\to\infty} F(y_t, u_{n_i}) \leq \lim_{i\to\infty} \langle Gy_t, y_t - u_{n_i}\rangle = \langle Gy_t, y_t - x'\rangle.$$

So, from (A1) and (A4), we have
$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, x') \leq tF(y_t, y) + (1-t)\langle Gy_t, y_t - x'\rangle \leq tF(y_t, y) + (1-t)t\langle Gy_t, y - x'\rangle$, and hence

$$F(y_t, y) + (1-t)\langle Gy_t, y - x'\rangle \geq 0.$$

Letting $t \to 0$, we have for each $y \in C$, $F(x', y) + \langle Gx', y - x'\rangle \geq 0$. This implies that $x' \in GEP(F, G)$.

Next, let us show that $x' \in VI(C, B)$. Let

$$Uy = \begin{cases} By + N_C y, & y \in C, \\ \emptyset, & y \notin C. \end{cases}$$

Then U is maximal monotone. Let $(y, w) \in G(U)$. Since $w - By \in N_C y$ and $v_n \in C$, we have $\langle y - v_n, w - By\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n Bu_n)$, we have $\langle y - v_n, v_n - (u_n - \lambda_n Bu_n)\rangle \geq 0$, that is, $\langle y - v_n, \frac{v_n - u_n}{\lambda_n} + Bu_n\rangle \geq 0$.

Therefore, we have

$$\begin{aligned}\langle y - v_{n_i}, w\rangle &\geq \langle y - v_{n_i}, By\rangle \\ &\geq \langle y - v_{n_i}, By\rangle - \langle y - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i}\rangle \\ &= \langle y - v_{n_i}, By - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} - Bu_{n_i}\rangle \\ &= \langle y - v_{n_i}, By - Bv_{n_i}\rangle + \langle y - v_{n_i}, Bv_{n_i} - Bu_{n_i}\rangle \\ &\quad - \langle y - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle y - v_{n_i}, Bv_{n_i}\rangle - \langle y - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i}\rangle \\ &\geq \|y - v_{n_i}\|\|Bv_{n_i} - Bu_{n_i}\| - \|y - v_{n_i}\|\|\frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\|. \qquad (3.19)\end{aligned}$$

Notice that $\|v_{n_i} - u_{n_i}\| \to 0$ as $i \to \infty$ and B is Lipschitz continuous, hence from (3.19), we obtain $\langle y - x', w\rangle \geq 0$ as $i \to \infty$. Since U is maximal monotone, we have $x' \in U^{-1}0$, and hence $x' \in VI(C,B)$. In the same manner as the proof of $x' \in VI(C,B)$, we obtain $x' \in VI(C,A)$. Therefore $x' \in \Omega$.

Finally, we will show that $x_n \to P_\Omega x_0$. Since $x' \in \Omega$, we have

$$\|P_\Omega x_0 - x_0\| \le \|x' - x_0\| \le \liminf_{i\to\infty} \|x_{n_i} - x_0\| \le \limsup_{i\to\infty} \|x_{n_i} - x_0\| \le \|P_\Omega x_0 - x_0\|.$$

Thus, we obtain that $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x' - x_0\| = \|P_\Omega x_0 - x_0\|$. Using the Kadec-Klee property of H, we obtain that $\lim_{i\to\infty} x_{n_i} = x' = P_\Omega x_0$. Hence the whole sequence must converge to $x' = P_\Omega x_0$. This completes the proof. □

Corollary 3.2. [18, Theorem 2.2] *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \ge 0\}$ be a nonexpansive semigroup on C and let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A4)$. Suppose that $\Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap EP(F) \ne \emptyset$. Let $\{\alpha_n\} \subset [0,a) \subset [0,1)$, $\{r_n\} \subset [r,r'] \subset (0,2\alpha)$ and $\{t_n\} \subset [0,\infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For $x_0 \in H$, let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are generated by $u_n \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C, \\ y_n = \alpha_n u_n + (1-\alpha_n)S(t_n)u_n, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ \forall n \ge 0. \end{cases} \tag{3.20}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. If $G, A, B \equiv 0$, in Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \ge 0\}$ be a nonexpansive semigroup on C. Suppose that $\Omega := \cap_{t=0}^{\infty} F(S(t)) \ne \emptyset$. Let $\{\alpha_n\} \subset [0,a) \subset [0,1)$ and $\{t_n\} \subset [0,\infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For $x_0 \in H$, let the sequences $\{x_n\}$ and $\{y_n\}$ are generated by*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)S(t_n)x_n, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ \forall n \ge 0. \end{cases} \tag{3.21}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. If $F(x,y) \equiv 0$ for all $x, y \in C$ and $G, A, B \equiv 0$, by Theorem 3.1 we obtain the desired result. □

Corollary 3.4. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \ge 0\}$ be a nonexpansive semigroup on C and let $G, A, B : C \to H$ be three α, β, λ-inverse-strongly monotone mappings, respectively. Suppose that $\Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap VI(C,A) \cap VI(C,B) \cap VI(C,G) \ne \emptyset$. Let $\{\alpha_n\} \subset [0,a) \subset [0,1)$, $\{\beta_n\} \subset [b,b'] \subset (0,2\beta)$, $\{\lambda_n\} \subset [l,l'] \subset (0,2\lambda)$, $\{r_n\} \subset [r,r'] \subset (0,2\alpha)$ and $\{t_n\} \subset [0,\infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For*

$x_0 \in H$, let the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by $u_n \in C$ and

$$\begin{cases} u_n = P_C(x_n - r_n G x_n), \\ v_n = P_C(u_n - \lambda_n B u_n), \\ z_n = P_C(v_n - \beta_n A v_n), \\ y_n = \alpha_n u_n + (1-\alpha_n) S(t_n) z_n, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ \forall n \ge 0. \end{cases} \tag{3.22}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. If $F \equiv 0$, then $u_n = P_C(x_n - r_n G x_n)$ for all $n \ge 0$, by Theorem 3.1, we obtain the desired result. □

3.2. **The shrinking projection method.**

Theorem 3.5. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \ge 0\}$ be a nonexpansive semigroup on C, let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $G, A, B : C \to H$ be three α, β, λ-inverse-strongly monotone mappings, respectively. Suppose that $\Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap VI(C,A) \cap VI(C,B) \cap GEP(F,G) \ne \emptyset$. Let $\{\alpha_n\} \subset [0,a) \subset [0,1)$, $\{\beta_n\} \subset [b,b'] \subset (0,2\beta)$, $\{\lambda_n\} \subset [l,l'] \subset (0,2\lambda)$, $\{r_n\} \subset [r,r'] \subset (0,2\alpha)$ and $\{t_n\} \subset [0,\infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$, let the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by $u_n \in C$ and*

$$\begin{cases} F(u_n, y) + \langle G x_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \ \forall y \in C, \\ v_n = P_C(u_n - \lambda_n B u_n), \\ z_n = P_C(v_n - \beta_n A v_n), \\ y_n = \alpha_n u_n + (1-\alpha_n) S(t_n) z_n, \\ C_{n+1} = \{z \in C_n \mid \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \ge 0. \end{cases} \tag{3.23}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. Since for any $x^* \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.2. Then, we have $x^* = T_{r_n}(x^* - r_n G x^*)$ and $u_n = T_{r_n}(x_n - r_n G x_n) \in C$ for all $n \ge 0$. We already have (3.3), (3.4), (3.5) and (3.6). Thus, we get $x^* \in C_{n+1}$. This implies that $\Omega \subset C_n$ for all $n \ge 0$. By using the same argument in the proof of [23, Theorem 3.3 pp. 281–282] , we obtain that $\{x_n\}$ bounded and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. As in the proofs of Theorem 3.1, we already have (3.17). Since $\{x_n\}$ is bounded, we can choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and assume that $x_{n_i} \rightharpoonup x'$. In the same time, as in the proof of Theorem 3.1, we also have $x' \in \Omega$.

Finally, we have to show that $x_n \to P_\Omega x_0$. Since $x' \in \Omega$, we have

$$\|P_\Omega x_0 - x_0\| \le \|x' - x_0\| \le \liminf_{i\to\infty} \|x_{n_i} - x_0\| \le \limsup_{i\to\infty} \|x_{n_i} - x_0\| \le \|P_\Omega x_0 - x_0\|.$$

Thus, we obtain that $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x' - x_0\| = \|P_\Omega x_0 - x_0\|$. Using the Kadec-Klee property of H, we obtain that $\lim_{i\to\infty} x_{n_i} = x' = P_\Omega x_0$. Hence the whole sequence must converge to $x' = P_\Omega x_0$. This completes the proof. □

Corollary 3.6. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \geq 0\}$ be a nonexpansive semigroup on C and let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying $(A1) - (A4)$. Suppose that $\Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, a) \subset [0, 1)$, $\{r_n\} \subset [r, r'] \subset (0, 2\alpha)$ and $\{t_n\} \subset [0, \infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$, let the sequences $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are generated by $u_n \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) u_n, \\ C_{n+1} = \{z \in C_n \mid \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0. \end{cases} \tag{3.24}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. If $G, A, B \equiv 0$, by Theorem 3.1 we obtain the desired result. □

Corollary 3.7. [18, Theorem 2.1] *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \geq 0\}$ be a nonexpansive semigroup on C. Suppose that $\Omega := \cap_{t=0}^{\infty} F(S(t)) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, a) \subset [0, 1)$ and $\{t_n\} \subset [0, \infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$, let the sequences $\{x_n\}$ and $\{y_n\}$ are generated by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) S(t_n) x_n, \\ C_{n+1} = \{z \in C_n \mid \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0. \end{cases} \tag{3.25}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. If $F(x, y) \equiv 0$ for all $x, y \in C$ and $G, A, B \equiv 0$, by Theorem 3.1 we obtain the desired result. □

Corollary 3.8. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\{S(t) : t \geq 0\}$ be a nonexpansive semigroup on C and let $G, A, B : C \to H$ be three α, β, λ-inverse-strongly monotone mappings, respectively. Suppose that $\Omega := (\cap_{t=0}^{\infty} F(S(t))) \cap VI(C, A) \cap VI(C, B) \cap VI(C, G) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, a) \subset [0, 1)$, $\{\beta_n\} \subset [b, b'] \subset (0, 2\beta)$, $\{\lambda_n\} \subset [l, l'] \subset (0, 2\lambda)$, $\{r_n\} \subset [r, r'] \subset (0, 2\alpha)$ and $\{t_n\} \subset [0, \infty)$ satisfying $\liminf_n t_n = 0, \limsup t_n > 0$, and $\lim_n (t_{n+1} - t_n) = 0$. For $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$, let the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{z_n\}$ are generated by*

$$\begin{cases} u_n = P_C(x_n - r_n G x_n), \\ v_n = P_C(u_n - \lambda_n B u_n), \\ z_n = P_C(v_n - \beta_n A v_n), \\ y_n = \alpha_n u_n + (1 - \alpha_n) S(t_n) z_n, \\ C_{n+1} = \{z \in C_n \mid \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \forall n \geq 0. \end{cases} \tag{3.26}$$

Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Proof. If $F \equiv 0$, then $u_n = P_C(x_n - r_n G x_n)$ for all $n \geq 0$, by Theorem 3.1, we obtain the desired result. □

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