Fixed Point Theory, 13(2012), No. 1, 99-105
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

EXISTENCE OF BEST PROXIMITY POINTS OF P-CYCLIC CONTRACTIONS

S. KARPAGAM AND SUSHAMA AGRAWAL

Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai, India
E-mails: karapagam.saravanan@gmail.com sushamamdu@gmail.com

Abstract. We consider a self map T on union of p subsets, $A_1, A_2, ..., A_p$, $(p \geq 2)$ of a metric space, which is a contraction under the condition $T(A_i) \subseteq A_{i+1}$, $1 \leq i \leq p$, $(A_{p+1} = A_1)$. We give sufficient conditions for the existence of a unique best proximity point of T, that is, a point $\xi \in A_i$, such that $d(\xi, T\xi) = dist(A_i, A_{i+1})$ and approximation of this point by a Picard type iterative method.
Key Words and Phrases: Best proximity point, uniformly convex Banach space, contraction.
2010 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

Kirk, Srinivasan and Veeramani in [3], introduced the notion of contractions under cyclical conditions. They defined a self map T, on union of nonempty subsets A and B of a metric space X, such that,

(1) $T(A) \subseteq B$ *and* $T(B) \subseteq A$
(2) For some $k \in (0,1), d(Tx, Ty) \leq kd(x, y),\ x \in A,\ y \in B$.

Further, they extended this notion to p sets, $p \geq 2$, and obtained the following result.

Theorem 1.1. *Let $A_1, A_2, ..., A_p$ be non empty closed subsets of a complete metric space X. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ satisfy the following conditions:*

(1) $T(A_i) \subseteq A_{i+1},\ 1 \leq i \leq p,$ *where* $A_{p+1} = A_1$
(2) *For some* $k \in (0,1),\ d(Tx, Ty) \leq kd(x, y),\ x \in A_i,\ y \in A_{i+1};$

then there exists a unique fixed point of T.

Actually, condition (2) imply the sets to intersect and T restricted to the intersection is a Banach contraction. Hence there exists a unique fixed point of T in the intersection. When the sets do not intersect, Eldred and Veeramani in [1], weakened the contraction condition for two sets and obtained the following result of best proximity point.

Theorem 1.2. *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space. Let $T : A \cup B \to A \cup B$ be such that*

(1) *$T(A) \subseteq B$ and $T(B) \subseteq A$*
(2) *For some $k \in (0,1)$, $\|Tx - Ty\| \leq k\|x - y\| + (1-k)dist(A,B)$, $x \in A$, $y \in B$;*

then there exists a unique best proximity point $x \in A$ (that is with $\|x - Tx\| = dist(A,B)$). Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

In this paper, as an extension of cyclic contraction (for two sets), we define a map p-cyclic contraction (Definition 3.1) on the union of p sets ($p \geq 2$). The p-cyclic contraction differs from the cyclic contraction, in the sense that, for $1 \leq i \leq p$, the image of A_i is contained in A_{i+1} and the image of A_{i+1} is contained in A_{i+2} and not in A_i. The image of A_p is contained in A_1. It is interesting to note that the distances between the adjacent sets are equal under p-cyclic contraction (Lemma 3.2). This fact plays an important role in obtaining a best proximity point. It is remarkable to note that the obtained best proximity point is also a periodic point with period p. In addition, if $z \in A_i$ is a best proximity point, then $T^j z$ is a best proximity point in A_{i+j}, for j = 1,2,...,(p-1).

2. Preliminaries

It is well known that if X_0 is a convex subset of a strictly convex normed linear space X, and $x \in X$, then a best approximation of x from X_0, if it exists, is unique.

We use the following lemmas proved in [1].

Lemma 2.1. *Let A be a nonempty closed and convex subset, and B be a nonempty, closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(1) $\|z_n - y_n\| \longrightarrow dist(A,B)$,
(2) *For every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$, $\|x_m - y_n\| \leq dist(A,B) + \epsilon$;*

then for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n \geq N_1$,

$$\|x_m - z_n\| \leq \epsilon.$$

Lemma 2.2. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space, let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(1) $\|x_n - y_n\| \longrightarrow dist(A,B)$
(2) $\|z_n - y_n\| \longrightarrow dist(A,B)$;

then $\|x_n - z_n\| \longrightarrow 0$.

3. Main results

Definition 3.1. *Let $A_1, A_2, ...A_p$ be nonempty subsets of a metric space X, let $T : \bigcup_{i=1}^{p} A_i \longrightarrow \bigcup_{i=1}^{p} A_i$; T is called p-cyclic contraction, if it satisfies the following condition:*

(1) $T(A_i) \subseteq A_{i+1}$, $1 \leq i \leq p$, *where* $A_{p+i} = A_i$

(2) *For some* $k,\ 0 < k < 1$,
$d(Tx,Ty) \leq kd(x,y) + (1-k)dist(A_i, A_{i+1}),\ x \in A_i,\ y \in A_{i+1},\ 1 \leq i \leq p.$

A point $x \in A_i$ *is said to be a best proximity point, if* $d(x,Tx) = dist(A_i, A_{i+1})$.

The following lemma shows that the distances between the adjacent sets are equal under p-cyclic contraction.

Lemma 3.2. *Let* $A_1, A_2, ..., A_p$ *be nonempty closed subsets of a metric space* X, *let* $T : \bigcup_{i=1}^p A_i \longrightarrow \bigcup_{i=1}^p A_i$ *be a p-cyclic contraction; then*

$$dist(A_i, A_{i+1}) = dist(A_{i+1}, A_{i+2}),$$

for all i, i = *1,2,..,p, where* $A_{p+i} = A_i$.

Proof. Let $x \in A_i$ and $y \in A_{i+1}$; then,

$$\begin{aligned} dist(A_{i+1}, A_{i+2}) &\leq d(Tx,Ty) \\ &\leq kd(x,y) + (1-k)dist(A_i, A_{i+1}) \\ &\leq kd(x,y) + (1-k)d(x,y) \\ &= d(x,y). \end{aligned}$$

This implies that $dist(A_{i+1}, A_{i+2}) \leq dist(A_i, A_{i+1})$ for all i = 1,2,.., p. Hence

$$dist(A_p, A_1) \leq dist(A_{p-1}, A_p) \leq ... \leq dist(A_1, A_2) \leq dist(A_p, A_1).$$

Therefore, $dist(A_i, A_{i+1}) = dist(A_{i+1}, A_{i+2})$ for all i, i = 1,2,..,p, where $A_{p+i} = A_i$.

Lemma 3.3. *Let* $A_1, A_2, ..., A_p$ *be nonempty closed subsets of a metric space* X, *let* $T : \bigcup_{i=1}^p A_i \longrightarrow \bigcup_{i=1}^p A_i$ *be a* $p-$ *cyclic contraction; then for every* $x,\ y \in A_i,\ for\ 1 \leq i \leq p$,

(1) $d(T^{pn}x, T^{pn+1}y) \longrightarrow dist(A_i, A_{i+1})$ *as* $n \to \infty$
(2) $d(T^{pn\pm p}x, T^{pn+1}y) \longrightarrow dist(A_i, A_{i+1})$ *as* $n \to \infty$.

Proof. To prove (1), Lemma 3.2 is repeatedly used.

$$\begin{aligned} dist(A_i, A_{i+1}) &\leq d(T^{pn}x, T^{pn+1}y) \\ &\leq kd(T^{pn-1}x, T^{pn}y) + (1-k)dist(A_i, A_{i+1}) \\ &\leq k^2 d(T^{pn-2}x, T^{pn-1}y) + k(1-k)dist(A_{i-1}, A_i) \\ &+ (1-k)dist(A_i, A_{i+1}) \\ &= k^2 d(T^{pn-2}x, T^{pn-1}y) + (1-k^2)dist(A_i, A_{i+1}), \\ &\leq \dots \\ &\leq k^{pn} d(x,Ty) + (1-k^{pn})dist(A_i, A_{i+1}) \\ &\to dist(A_i, A_{i+1}) \ as\ n \to \infty. \end{aligned}$$

Similarly (2) can also be proved.

Remark 3.4. *If* X *is a uniformly convex Banach space and if each* A_i *is convex, then by Lemma* 3.3, *for* $x \in A_i$, $\|T^{pn}x - T^{pn+1}x\| \longrightarrow dist(A_i, A_{i+1})$ *as* $n \to \infty$ *and* $\|T^{pn\pm p}x - T^{pn+1}x\| \longrightarrow dist(A_i, A_{i+1})$, *as* $n \to \infty$. *Then by Lemma* 2.2, $\|T^{pn}x - T^{pn\pm p}x\| \longrightarrow 0$. *Similarly,* $\|T^{pn+1}x - T^{pn+2}x\| \longrightarrow dist(A_i, A_{i+1})$ *as* $n \to \infty$

and $\|T^{pn\pm p+1}x - T^{pn+2}x\| \longrightarrow dist(A_i, A_{i+1})$, *as* $n \to \infty$. *Then by Lemma 2.2,* $\|T^{pn+1}x - T^{pn\pm p+1}x\| \longrightarrow 0$.

Theorem 3.5. *Let* $A_1, A_2, ..., A_p$ *be nonempty closed subsets of a metric space, let* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *be a* $p-$*cyclic contraction; if for some* i, $x \in A_i$, *is such that the sequence* $\{T^{pn}x\}$ *in* A_i *contains a convergent subsequence* $\{T^{pn_j}x\}$, *converging to* $\xi \in A_i$, *then* ξ *is a best proximity point of* T *in* A_i.

Proof. Consider $d(T^{pn_j-1}, \xi) \leq d(T^{pn_j-1}x, T^{pn_j}x) + d(T^{pn_j}x, \xi)$ which tends to $dist(A_i, A_{i-1})$ as $j \longrightarrow to\ \infty$. Now

$$\begin{aligned} dist(A_i, A_{i+1}) &\leq d(\xi, T\xi) \\ &= \lim_{j\to\infty} d(T^{pn_j}x, T\xi) \\ &\leq \lim_{j\to\infty} kd(T^{pn_j-1}x, \xi) + (1-k)dist(A_i, A_{i+1}) \\ &= kdist(A_i, A_{i+1}) + (1-k)dist(A_i, A_{i+1}) \\ &= dist(A_i, A_{i+1}). \end{aligned}$$

Therefore, $d(\xi, T\xi) = dist(A_i, A_{i+1})$.

Theorem 3.6. *Let* $A_1, A_2, ..., A_p$ *be nonempty, closed and convex subsets of a uniformly convex Banach space. Let* $T : \bigcup_{i=1}^{p} A_i \longrightarrow \bigcup_{i=1}^{p} A_i$ *be a* $p-$*cyclic contraction. Then there exists a* $z_i \in A_i$ $(1 \leq i \leq p)$, *such that, if* x *is any point of* A_i, *the sequence* $\{T^{pn}x\}$ *converges to* z_i *and* z_i *is a best proximity point of* T *in* A_i. *Moreover,* $T^j z_i = z_{i+j}$ *is a best proximity point in* A_{i+j}, *for* $j = 1$ *to* $(p-1)$ *and* z_i *is the unique periodic point of* T *with period* p.

Proof. If $dist(A_i, A_{i+1}) = 0$ for some i, then $dist(A_i, A_{i+1}) = 0$ for all i. Then by Theorem 1.1, T has a unique fixed point. Hence we assume that $dist(A_i, A_{i+1}) > 0$, for all i. Let $x \in A_i$. Then $T^{pn}x \in A_i$ and $T^{pn+1}x \in A_{i+1}$, for all n. By Lemma 3.3, $\|T^{pn}x - T^{pn+1}x\| \longrightarrow dist(A_i, A_{i+1})$. If, for given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$, such that for $m > n > n_0$,

$$\|T^{pm}x - T^{pn+1}x\| \leq dist(A_i, A_{i+1}) + \epsilon, \tag{3.1}$$

then by Lemma 2.1, for given $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$, such that, for $m > n > n_1$, $\|T^{pm}x - T^{pn}x\| \leq \epsilon$. Therefore, $\{T^{pn}x\}$ is a Cauchy sequence and converges to some $z \in A_i$. By Theorem 3.5, z is a best proximity point in A_i. Therefore, assume the contrary of (3.1). Then, there exists an $\epsilon_o > 0$ such that, for every $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$ such that,

$$\|T^{pm_k}x - T^{pn_k+1}x\| \geq dist(A_i, A_{i+1}) + \epsilon_0. \tag{3.2}$$

Let m_k be the smallest integer greater than n_k, to satisfy the above inequality. Now,

$$\begin{aligned} dist(A_i, A_{i+1}) + \epsilon_0 &\leq \|T^{pm_k}x - T^{pn_k+1}x\| \\ &\leq \|T^{pm_k}x - T^{pm_k-p}x\| + \|T^{pm_k-p}x - T^{pn_k+1}x\|. \end{aligned}$$

By Remark (3.4), $\|T^{pm_k}x - T^{pm_k - p}x\| \longrightarrow 0 \ as \ k \longrightarrow \infty$. Therefore,

$$dist(A_i, A_{i+1}) + \epsilon_0 \leq \lim_{k\to\infty} \|T^{pm_k}x - T^{pn_k+1}x\| \leq dist(A_i, A_{i+1}) + \epsilon_0.$$

So, $\lim_{k\to\infty} \|T^{pm_k}x - T^{pn_k+1}x\| = dist(A_i, A_{i+1}) + \epsilon_0$. Now,

$$\|T^{pm_k}x - T^{pn_k+1}x\| \leq \{\|T^{pm_k}x - T^{pm_k+p}x\| + \|T^{pm_k+p}x - T^{pn_k+p+1}x\| + \|T^{pn_k+p+1}x - T^{pn_k+1}x\|\}.$$

By Remark (3.4), $\|T^{pm_k}x - T^{pm_k+p}x\| \longrightarrow 0 \ as \ k \longrightarrow \infty$ and

$$\|T^{pn_k+p+1}x - T^{pn_k+1}x\| \longrightarrow 0 \ as \ k \longrightarrow \infty.$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty} \|T^{pm_k}x - T^{pn_k+1}x\| &\leq \lim_{k\to\infty} \|T^{pm_k+p}x - T^{pn_k+p+1}x\| \\
&\leq \lim_{k\to\infty} k^p\|T^{pm_k}x - T^{pn_k+1}x\| + (1-k^p)dist(A_i, A_{i+1}).
\end{aligned}$$

That is, $dist(A_i, A_{i+1}) + \epsilon_0 \leq k^p(dist(A_i, A_{i+1}) + \epsilon_0) + (1-k^p)dist(A_i, A_{i+1})$.

That is, $\epsilon_0 \leq k^p\epsilon_0$, which is a contradiction. Hence $\{T^{pn}x\}$ is a Cauchy sequence and converges to some $z \in A_i$, such that $\|z - Tz\| = dist(A_i, A_{i+1})$. Now let $y \in A_i$ be such that, $y \neq x$ and $\{T^{pn}y\}$ converges to $z' \in A_i$. By Theorem 3.5, z' is a best proximity point. That is, $\|z' - Tz'\| = dist(A_i, A_{i+1})$. To prove $z' = z$, consider,

$$\begin{aligned}
\|z' - T^{p+1}z'\| &= \lim_{n\to\infty} \|T^{pn}y - T^{p+1}z'\| \\
&\leq \lim_{n\to\infty} k^p\|T^{pn-p}y - Tz'\| + (1-k^p)dist(A_i, A_{i+1}) \\
&= k^p\|z' - Tz'\| + (1-k^p)dist(A_i, A_{i+1}) \\
&= k^p dist(A_i, A_{i+1}) + (1-k^p)dist(A_i, A_{i+1}) \\
&= dist(A_i, A_{i+1}).
\end{aligned}$$

Therefore, $dist(A_i, A_{i+1}) \leq \|z' - T^{p+1}z'\| \leq dist(A_i, A_{i+1})$. Hence

$$\|z' - T^{p+1}z'\| = dist(A_i, A_{i+1}).$$

Since A_{i+1} is a convex set and X is a uniformly convex Banach space,

$$Tz' = T^{p+1}z'. \tag{3.3}$$

Now,

$$\begin{aligned}
\|z - Tz'\| &= \lim_{n\to\infty} \|T^{pn}x - T^{p+1}z'\| \\
&\leq \lim_{n\to\infty} k^p\|T^{pn-p}x - Tz'\| + (1-k^p)dist(A_i, A_{i+1}) \\
&= k^p\|z - Tz'\| + (1-k^p)dist(A_i, A_{i+1}) \\
&\leq k^p\|z - Tz'\| + dist(A_i, A_{i+1}) - k^p\|z - Tz'\|.
\end{aligned}$$

Therefore, $\|z - Tz'\| \leq dist(A_i, A_{i+1})$. Hence $\|z - Tz'\| = dist(A_i, A_{i+1})$. Since A_i is a convex set, $z' = z$.

Now, we observe that, since

$$\|T^{p-1}z - T^p z\| \leq \|T^{p-2}z - T^{p-1}z\| \leq \|T^{p-3}z - T^{p-2}z\| \leq \cdots \leq \\ \leq \|z - Tz\| = dist(A_i, A_{i+1}),$$

$T^j z$ is a best proximity point in A_{i+j}, for $j = 0$ *to* $(p-1)$.

Next, to prove that z is a periodic point of T with period p, we see that by similar argument of (3.3), $T^{p+1}z = Tz$. Now,

$$\begin{aligned} \|T^p z - Tz\| &= \|T^p z - T^{p+1}z\| \\ &\leq k^p\|z - Tz\| + (1 - k^p)dist(A_i, A_{i+1}) \\ &= dist(A_i, A_{i+1}). \end{aligned}$$

Since A_i is a convex set, we have $T^p z = z$. Hence $T^{pm}z = z$ and $T^{pm+1}z = Tz$ for all $m \in \mathbb{N}$.

Now suppose there exists a $\xi \in A_i$ such that $T^p\xi = \xi$, then $\{T^{pn}\xi\}$ converges to ξ. Since $z_i \in A_i$ is the unique element in A_i such that for any $x \in A_i$, $\{T^{pn}x\}$ converges to z_i, we have $\xi = z_i$. Since $T^p z_i = z_i$ and $\xi = z_i$ implies z_i is the unique periodic point of T in A_i.

Now, by what we have proved, there exists a unique $z_{i+1} \in A_{i+1}$, such that for any $y \in A_{i+1}$, the sequence $\{T^{pn}y\}$ converges to z_{i+1}, which is a best proximity point of T in A_{i+1}. Now z_i is a best proximity point in A_i. $Tz_i \in A_{i+1}$ implies $\{T^{pn}(Tz_i)\}$ converges to z_{i+1}. Moreover, $T^{p+1}z_i = Tz_i$. Therefore $T^{pn+1}z_i = Tz_i$. That is $\{T^{pn}(Tz_i)\}$ converges to Tz_i. This implies $z_{i+1} = Tz_i$. Similarly, $z_{i+j} = T^j z_i$ for $j = 1, 2..., (p-1)$.

The following example illustrates Theorem 3.6.

Example 3.7. *Let $X = \mathbb{R}^2$ be the Euclidean plane equipped with the usual Euclidean metric. Let the subsets A_i, $i = 1$ to 4 be as follows:*

$$A_1 = \{(0, 1 + x) : 0 \leq x \leq 1\},\ A_2 = \{(1 + x, 0) : 0 \leq x \leq 1\},$$

$$A_3 = \{(0, -(1 + x)) : 0 \leq x \leq 1\} \text{ and } A_4 = \{(-(1 + x), 0) : 0 \leq x \leq 1\}.$$

Note that $dist(A_i, A_{i+1}) = \sqrt{2}$, for $i = 1$ to 4, where $A_{4+i} = A_i$.
Define $T : \bigcup_{i=1}^4 A_i \to \bigcup_{i=1}^4 A_i$ as follows:

$$T(0, 1 + x) = (1 + \frac{x}{10}, 0)$$

$$T(1 + x, 0) = (0, -(1 + \frac{x}{10}))$$

$$T(0, -(1 + x)) = (-(1 + \frac{x}{10}), 0)$$

$$T(-(1 + x), 0) = (0, (1 + \frac{x}{10})),\ \text{where } x,\ y \in [0, 1].$$

Clearly, $T(A_i) \subseteq A_{i+1}$, for $i = 1$ to 4.

Now, let

$$z_1 = (0, 1 + y) \in A_1,\ z_2 = (1 + x, 0) \in A_2,$$

$$z_3 = (0, -(1 + y)) \in A_3, z_4 = (-(1 + x), 0) \in A_4,\ \text{where } x, y \in [0, 1].$$

For each $i = 1$ to 4, we note that

$$d(z_i, z_{i+1}) = \sqrt{(1+x)^2 + (1+y)^2}$$

$$d(Tz_i, Tz_{i+1}) = \sqrt{\left(1+\frac{x}{10}\right)^2 + \left(1+\frac{y}{10}\right)^2}$$

$$\begin{aligned}
[d(Tz_i, Tz_{i+1})]^2 &\le \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \frac{x}{2} + \frac{y}{2} + 1 + \frac{1}{\sqrt{2}}\sqrt{(1+x)^2 + (1+y)^2} \\
&= \left(\frac{1}{4} + \frac{x^2}{4} + \frac{2x}{4}\right) + \left(\frac{1}{4} + \frac{y^2}{4} + \frac{2y}{4}\right) \\
&+ \left(\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}}\sqrt{(1+x)^2 + (1+y)^2}\right) \\
&= \frac{1}{4}[(1+x)^2 + (1+y)^2] + \left(\frac{1}{\sqrt{2}}\right)^2 \\
&+ \left(2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)\sqrt{(1+x)^2 + (1+y)^2}\right) \\
&= \left(\frac{1}{2}\sqrt{(1+x)^2 + (1+y)^2} + \frac{1}{\sqrt{2}}\right)^2 \\
&= \left(\frac{1}{2}\sqrt{(1+x)^2 + (1+y)^2} + \left(1 - \frac{1}{2}\right)\sqrt{2}\right)^2.
\end{aligned}$$

Hence, for $k = \frac{1}{2}$ the following condition is satisfied

$$d(Tz_i, Tz_{i+1}) \le kd(z_i, z_{i+1}) + (1-k)\sqrt{2},$$

for all $z_i \in A_i$ and $z_{i+1} \in A_{i+1}$. Therefore, T is a p-cyclic contraction.
Let $x = (0, 1+y) \in A_1$ where $y \in [0,1]$. Then $\{T^{4n}x\} = \{(0, 1 + \frac{y}{10^{4n}})\}$.
Clearly, $\{T^{4n}x\} \to (0,1)$ as $n \to \infty$, which is a best proximity point of T in A_1.
Also, $T(0,1) = (1,0)$. So, (1,0) is a best proximity point in A_2. $T^2(0,1) = (0,-1)$ and $T^3(0,1) = (-1,0)$ are unique best proximity points in A_3 and A_4 respectively.

Acknowledgement. Both the authors are thankful to Dr. A. Anthony Eldred, St. Joseph College, Trichy 620 001 (India) for the valuable discussions the authors had with him which were useful for the article.

References

[1] A.A. Eldred, P. Veeramani, *Existence and convergence of best proximity Points*, J. Math. Anal. Appl., **323**(2006), 1001-1006.
[2] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley and Sons, Inc, 2001.
[3] W.A. Kirk, P.S. Srinivasan, P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, **4**(2003), 79-89.

Received: December 14, 2008; Accepted: September 29, 2010.