Fixed Point Theory, 12(2011), No. 1, 113-126
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

WEAK AND STRONG MEAN CONVERGENCE THEOREMS FOR SUPER HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first introduce a class of nonlinear mappings called extended hybrid in a Hilbert space containing the class of generalized hybrid mappings. The class is different from the class of super hybrid mappings which was defined by Kocourek, Takahashi and Yao [12]. We prove a fixed point theorem for generalized hybrid nonself-mapping in a Hilbert space. Next, we prove a nonlinear ergodic theorem of Baillon's type for super hybrid mappings in a Hilbert space. Finally, we deal with two strong convergence theorems of Halpern's type for these nonlinear mappings in a Hilbert space.
Key Words and Phrases: Hilbert space, nonexpansive mapping, nonspreading mapping, hybrid mapping, fixed point, mean convergence.
2010 Mathematics Subject Classification: 47H10, 47H05.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Then a mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. From Baillon [2] we know the following first nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

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converges weakly to an element $z \in F(T)$.

The following strong convergence theorem of Halpern's type [7] was proved by Wittmann [26]; see also [19].

Theorem 1.2. *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n = 1, 2, ...,$$

where $\{\alpha_n\} \subset [0, 1]$ satisfies $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then $\{x_n\}$ converges strongy to a fixed point of T.

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. Let C be a nonempty subset of H. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [4] and Goebel and Kirk [6]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [3] and [5]. Recently, Kohsaka and Takahashi [14], and Takahashi [21] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to H$ is called *nonspreading* [14] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called *hybrid* [21] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [13] and Iemoto and Takahashi [10]. Very recently, Kocourek, Takahashi and Yao [12] introduced a broad class of mappings $T : C \to H$ called generalized hybrid such that for some $\alpha, \beta \in \mathbb{R}$,

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Further, they defined a more braod class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. Such a class is called a class of super hybrid mappings. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point.

In this paper, we first introduce a class of nonlinear mappings called extended hybrid in a Hilbert space containing the class of generalized hybrid mappings. The class is different from the class of super hybrid mappings which was defined by Kocourek, Takahashi and Yao [12]. We prove a fixed point theorem for generalized hybrid nonself-mapping in a Hilbert space. Next, we prove a nonlinear ergodic theorem of Baillon's type for super hybrid mappings in a Hilbert space. Finally, we deal with two strong convergence theorems of Halpern's type for these nonlinear mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle\cdot,\cdot\ \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [20], we know the following basic equality: For $x, y, u, v \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Further, we know that for $x, y, u, v \in H$

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

From (2.2), we have also the following equality.

$$\begin{aligned}\|x-y+u-v\|^2 &= \|x-y\|^2 + \|u-v\|^2 + 2\langle x-y, u-v\rangle \\ &= \|x-y\|^2 + \|u-v\|^2 + \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \end{aligned} \tag{2.3}$$

Let C be a nonempty closed convex subset of H and let T be a mapping from C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x-Ty\| \le \|x-y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [11]. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y\in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \ge 0$$

for all $x \in H$ and $u \in C$. Further, we know that

$$\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle \tag{2.4}$$

for all $x, y \in H$; see [20] for more details. The following lemma was proved by Takahashi and Toyoda [23].

Lemma 2.1. *Let D be a nonempty closed convex subset of a real Hilbert space H. Let P be the metric projection of H onto D and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

Let C be a nonempty subset of H. Then, a nonself-mapping $T : C \to H$ is called *generalized hybrid* [12] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.5}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta)$-*generalized hybrid* mapping. We observe that the mapping above covers several well-known mappings. For example, an $(\alpha,\ \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|x - y\|^2 + (1-\beta)\|x - y\|^2$$

and hence $\|x - Ty\| \leq \|x - y\|$. This means that an (α, β)-generalized hybrid mapping with a fixed point is quasi-nonexpansive.

Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to H$ is called *super hybrid* [12, 25] if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \leq \\ \big(\beta + (\beta - \alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\gamma\big)\|x - y\|^2 \\ + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2 \end{aligned} \tag{2.6}$$

for all $x, y \in C$. We call such a mapping an (α, β, γ)-*super hybrid* mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is (α, β)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. Let us consider a super hybrid mapping S with $\alpha = 1$, $\beta = 0$ and $\gamma = 1$. Then, we have

$$\|Sx - Sy\|^2 + \|x - Sy\|^2 \leq -\|Sx - y\|^2 + 3\|x - y\|^2 + \|x - Sx\|^2 + \|y - Sy\|^2$$

for all $x, y \in C$. This is equivalent to

$$\|Sx - Sy\|^2 + 2\langle x - y, Sx - Sy\rangle \leq 3\|x - y\|^2$$

for all $x, y \in C$. In the case of $H = \mathbb{R}$, consider $Sx = 2\cos\ x - x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, we have

$$\begin{aligned} &|Sx - Sy|^2 + 2\langle x - y, Sx - Sy\rangle \\ &\qquad = |2\cos\ x - x - (2\cos\ y - y)|^2 + 2\langle x - y, 2\cos\ x - x - (2\cos\ y - y)\rangle \\ &\qquad = 4(\cos\ x - \cos\ y)^2 - 2\langle x - y, 2\cos\ x - 2\cos\ y\rangle + (x - y)^2 \\ &\qquad\quad - (x - y)^2 + 2\langle x - y, 2\cos\ x - 2\cos\ y\rangle \\ &\qquad \leq 4(x - y)^2 - (x - y)^2 \\ &\qquad = 3(x - y)^2 \end{aligned}$$

and hence S is super hybrid. However, S is not quasi-nonexpansive. Further, we have that

$$Tx = \frac{1}{2}(2\cos\ x - x) + \frac{1}{2}x = \cos\ x$$

and hence T is a nonexpansive mapping with a fixed point. The following theorem was proved in [25] and [12].

Theorem 2.2. *Let C be a nonempty subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let S and T be mappings of C into H such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, S is (α, β, γ)-super hybrid if and only if T is (α, β)-generalized hybrid. In this case, $F(S) = F(T)$. In particular, let C be a nonempty closed and convex subset of H and let α, β and γ be real numbers with $\gamma \geq 0$. If a mapping $S : C \to C$ is (α, β, γ)-super hybrid, then the mapping $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$ is an (α, β)-generalized hybrid mapping of C into itself.*

Kocourek, Takahashi and Yao [12] also proved the following fixed point theorem for super hybrid mappings in a Hilbert space.

Theorem 2.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an (α, β, γ)-super hybrid mapping. Then, S has a fixed point in C. In particular, if $S : C \to C$ be an (α, β)-generalized hybrid mapping, then S has a fixed point in C.*

To prove one of our main results, we need the following lemma [1]:

Lemma 2.4. *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

3. Fixed Point Theorem for Non-Self Mappings

In this section, we prove a fixed point theorem for generalized hybrid nonself-mappings in a Hilbert space. Before proving it, we need the following lemma.

Lemma 3.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let α and β be in $\mathbb{R}$. Then, a nonself-mapping $T : C \to H$ is (α, β)-generalized hybrid if and only if it satisfies that*

$$\begin{aligned}\|Tx - Ty\|^2 &\leq (\alpha - \beta)\|x - y\|^2 \\ &\quad + 2(\alpha - 1)\langle x - Tx, y - Ty\rangle - (\alpha - \beta - 1)\|y - Tx\|^2\end{aligned}$$

for all $x, y \in C$.

Proof. We have that for all $x, y \in C$,

$$\begin{aligned}&\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 \\ &\qquad + 2(\alpha - 1)\langle x - Tx, y - Ty\rangle - (\alpha - \beta - 1)\|y - Tx\|^2 \\ &\iff \|Tx - Ty\|^2 \leq (1 - \beta)\|x - y\|^2 + (\alpha - 1)\|x - y\|^2 \\ &\qquad + (\alpha - 1)(\|x - Ty\|^2 + \|y - Tx\|^2 - \|x - y\|^2 - \|Tx - Ty\|^2) \\ &\qquad + \beta\|y - Tx\|^2 - (\alpha - 1)\|y - Tx\|^2 \\ &\iff \alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \\ &\qquad \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2.\end{aligned}$$

□

Using Lemma 3.1, we have the following result.

Lemma 3.2. *Let H be a Hilbert space and let C be a nonempty bounded subset of H. If a nonself-mapping $T : C \to H$ is generalized hybrid, then TC is bounded.*

Proof. Since $T : C \to H$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.1}$$

for all $x, y \in C$. We have from Lemma 3.1 that

$$\begin{aligned}\|Tx - Ty\|^2 &\le (\alpha-\beta)\|x-y\|^2 \\ &\quad + 2(\alpha-1)\langle x - Tx, y - Ty\rangle - (\alpha-\beta-1)\|y - Tx\|^2\end{aligned}$$

for all $x, y \in C$. Fix $z \in C$. Then, we have that for any $y \in C$,

$$\begin{aligned}\|Tz - Ty\|^2 &\le (\alpha-\beta)\|z-y\|^2 \\ &\quad + 2(\alpha-1)\langle z - Tz, y - Ty\rangle - (\alpha-\beta-1)\|y - Tz\|^2 \\ &\le |\alpha-\beta|\|z-y\|^2 \\ &\quad + 2|\alpha-1|\|z - Tz\|\|y - Ty\| + |\alpha-\beta-1|\|y - Tz\|^2 \\ &= |\alpha-\beta|\|z-y\|^2 \\ &\quad + |\alpha-1|\|z - Tz\|(\|y - Tz\| + \|Tz - Ty\|) + |\alpha-\beta-1|\|y - Tz\|^2.\end{aligned}$$

So, $\{\|Tz - Ty\| : y \in C\}$ is bounded and hence TC is bounded. □

Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers. Then, $U : C \to H$ is called an (α, β, γ)-extended hybrid mapping if

$$\begin{aligned}\alpha(1+\gamma)\|Ux - Uy\|^2 &+ (1-\alpha(1+\gamma))\|x - Uy\|^2 \\ &\le (\beta+\alpha\gamma)\|Ux - y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\ &\quad - (\alpha-\beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2\end{aligned}$$

for all $x \in C$.

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \ne -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1+\gamma > 0$, $T : C \to H$ is an (α, β)-generalized hybrid mapping if and only if $U : C \to H$ is an (α, β, γ)- extended hybrid mapping.*

Proof. Since $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, we have $T = (1+\gamma)U - \gamma I$. So, we have from Theorem 3.1 that for any $x, y \in C$,

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \\ &\le \beta\|Tx - y\|^2 + (1-\beta)\|x-y\|^2 \\ &\iff \|Tx - Ty\|^2 \le (\alpha-\beta)\|x-y\|^2 \\ &+2(\alpha-1)\langle x - Tx, y - Ty\rangle - (\alpha-\beta-1)\|y - Tx\|^2 \\ &\iff \|(1+\gamma)Ux - \gamma x - (1+\gamma)Uy + \gamma y\|^2 \le (\alpha-\beta)\|x-y\|^2 \\ &+2(\alpha-1)\langle(1+\gamma)(x - Ux), (1+\gamma)(y - Uy)\rangle \\ &-(\alpha-\beta-1)\|y - (1+\gamma)Ux + \gamma x\|^2\end{aligned}$$

$$\begin{aligned}
\iff \|(1+\gamma)(Ux-Uy)-\gamma(x-y)\|^2 &\le (\alpha-\beta)\|x-y\|^2 \\
&+2(\alpha-1)(1+\gamma)^2\langle x-Ux, y-Uy\rangle \\
&-(\alpha-\beta-1)\|y-Ux+\gamma(x-Ux)\|^2 \\
\iff \alpha(1+\gamma)^2\|Ux-Uy\|^2 &+ (1+\gamma)(1-\alpha(1+\gamma))\|x-Uy\|^2 \\
\le (1+\gamma)(\beta+\alpha\gamma)\|Ux-y\|^2 &+ (1+\gamma)(1-\beta-\alpha\gamma)\|x-y\|^2 \\
-(1+\gamma)\gamma(\alpha-\beta)\|x-Sx\|^2 &- \gamma(1+\gamma)\|y-Uy\|^2 \\
\iff \alpha(1+\gamma)\|Ux-Uy\|^2 &+ (1-\alpha(1+\gamma))\|x-Uy\|^2 \\
\le (\beta+\alpha\gamma)\|Ux-y\|^2 &+ (1-\beta-\alpha\gamma)\|x-y\|^2 \\
-(\alpha-\beta)\gamma\|x-Ux\|^2 &- \gamma\|y-Uy\|^2.
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α and β be real numbers. Let T be an (α, β)-generalized hybrid mapping with $\alpha - \beta \ge 0$ of C into H. Suppose that there exists $m > 1$ such that for any $x \in C$, $Tx = x + t(y - x)$ for some $y \in C$ and t with $1 \le t \le m$. Then, T has a fixed point in C.*

Proof. By the assumption, we have that for any $x \in C$, there are $y \in C$ and t with $1 \le t \le m$ such that $Tx = x + t(y - x)$. We have $Tx = ty + (1 - t)x$ and hence $y = \frac{1}{t}Tx + \frac{t-1}{t}x$. Define $Ux \in C$ as follows:

$$Ux = (1 - \frac{t}{m})x + \frac{t}{m}(\frac{1}{t}Tx + \frac{t-1}{t}x).$$

So, we have $Ux = \frac{1}{m}Tx + \frac{m-1}{m}x$. Taking $\gamma > 0$ with $m = 1 + \gamma$, we have

$$Ux = \frac{1}{1+\gamma}Tx + \frac{\gamma}{1+\gamma}x. \tag{3.2}$$

Thus, we can define a mapping U of C into itself satisfying (3.2). Since $T : C \to H$ is an (α, β)-generalized hybrid mapping with $\alpha - \beta \ge 0$, from Theorem 3.3 U is an (α, β, γ)-extended hybrid mapping of C into itself, i.e.,

$$\begin{aligned}
\alpha(1+\gamma)\|Ux-Uy\|^2 &+ (1-\alpha(1+\gamma))\|x-Uy\|^2 \\
&\le (\beta+\alpha\gamma)\|Ux-y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2 \\
&\quad - (\alpha-\beta)\gamma\|x-Ux\|^2 - \gamma\|y-Uy\|^2
\end{aligned}$$

for all $x \in C$. From $\alpha - \beta \ge 0$ and $\gamma > 0$, we have

$$\begin{aligned}
\alpha(1+\gamma)\|Ux-Uy\|^2 &+ (1-\alpha(1+\gamma))\|x-Uy\|^2 \\
&\le (\beta+\alpha\gamma)\|Ux-y\|^2 + (1-(\beta+\alpha\gamma))\|x-y\|^2
\end{aligned}$$

for all $x \in C$. This implies that U is an $(\alpha(1+\gamma), \beta+\alpha\gamma)$-generalized hybrid mapping of C into itself. So, we have a fixed point from Theorem 2.3. This completes the proof. □

Let us give an example of mappings $T : C \to H$ such that for any $x \in C$, there are $y \in C$ and t with $1 \le t \le m$ such that $Tx = x + t(y - x)$. In the case of $H = \mathbb{R}$, consider a mapping $T : [0, \frac{\pi}{2}] \to \mathbb{R}$:

$$Tx = (1 + 2x)\cos x - 2x^2, \quad \forall x \in [0, \frac{\pi}{2}].$$

Then, we have

$$Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, \frac{\pi}{2}].$$

For any $x \in [0, \frac{\pi}{2}]$, take $t = 1 + 2x$, $y = \cos x$ and $m = 1 + \pi$. Then, we have $Tx = t(y - x) + x$, $y = \cos x \in [0, \frac{\pi}{2}]$ and $1 \le t = 1 + 2x \le 1 + \pi$.

4. Nonlinear Ergodic Theorem

In this section, using the technique developed by Takahashi [17], we prove a nonlinear ergodic theorem of Baillon's type [2] for super hybrid mappings in a Hilbert space. Before proving it, we need the following lemma.

Lemma 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping from C into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$. Then, $\lim_{n\to\infty} \|S_n x - TS_n x\| = 0$. In particular, if C is bounded, then*

$$\lim_{n\to\infty} \sup_{x\in C} \|S_n x - TS_n x\| = 0.$$

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. For any $y \in C$ and $k \in \mathbb{N}$, we have that

$$\begin{aligned}
0 &\le \beta\|T^{k+1}x - y\|^2 + (1 - \beta)\|T^k x - y\|^2 \\
&\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1 - \alpha)\|T^k x - Ty\|^2 \\
&= \beta\big\{\|T^{k+1}x - Ty\|^2 + 2\left\langle T^{k+1}x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\
&\quad + (1 - \beta)\big\{\|T^k x - Ty\|^2 + 2\left\langle T^k x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\
&\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1 - \alpha)\|T^k x - Ty\|^2 \\
&= \|Ty - y\|^2 + 2\left\langle \beta T^{k+1}x + (1 - \beta)T^k x - Ty, Ty - y\right\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\} \\
&= \|Ty - y\|^2 + 2\left\langle T^k x - Ty + \beta(T^{k+1}x - T^k x), Ty - y\right\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\}.
\end{aligned}$$

Summing up these inequalities with respect to $k = 1, 2, \dots, n$, we have

$$\begin{aligned}
0 &\le n\|Ty - y\|^2 + 2\langle \sum_{k=1}^{n} T^k x - nTy, Ty - y\rangle + 2\beta\langle T^{n+1}x - Tx, Ty - y\rangle \\
&\quad + (\beta - \alpha)(\|T^{n+1}x - Ty\|^2 - \|Tx - Ty\|^2).
\end{aligned}$$

Deviding this inequality by n, we have

$$0 \leq \|Ty-y\|^2 + 2\langle S_nx - Ty, Ty-y\rangle + \frac{1}{n}2\beta\langle T^{n+1}x - Tx, Ty-y\rangle \\ + \frac{1}{n}(\beta-\alpha)(\|T^{n+1}x-Ty\|^2 - \|Tx-Ty\|^2).$$

where $S_nx = \frac{1}{n}\sum_{k=1}^n T^kx$. Replacing y by S_nx, we obtain

$$0 \leq \|TS_nx - S_nx\|^2 \\ + 2\langle S_nx - TS_nx, TS_nx - S_nx\rangle + \frac{1}{n}2\beta\langle T^{n+1}x - Tx, TS_nx - S_nx\rangle \\ + \frac{1}{n}(\beta-\alpha)(\|T^{n+1}x - TS_nx\|^2 - \|Tx - TS_nx\|^2)$$

and hence

$$\|TS_nx - S_nx\|^2 \leq \frac{1}{n}2\beta\langle T^{n+1}x - Tx, TS_nx - S_nx\rangle \\ + \frac{1}{n}(\beta-\alpha)(\|T^{n+1}x - TS_nx\|^2 - \|Tx - TS_nx\|^2).$$

By the assumption, $\{T^nx\}$ is bounded. So, $\{S_nx\}$ is also bounded. By Lemma 3.2, $\{TS_nx\}$ is bounded. So, we have $\limsup_{n\to\infty}\|S_nx - TS_nx\| \leq 0$ and hence $\lim_{n\to\infty}\|S_nx - TS_nx\| = 0$. In particular, if C is bounded, then we have

$$\limsup_{n\to\infty}\sup_{x\in C}\|S_nx - TS_nx\| \leq 0$$

and hence $\lim_{n\to\infty}\sup_{x\in C}\|S_nx - TS_nx\| = 0$. This completes the proof. □

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let α, β and γ be real numbers with $\gamma \geq 0$ and let $S : C \to C$ be an (α, β, γ)-super hybrid mapping with $F(S) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_nx = \frac{1}{n}\sum_{k=1}^n\left(\frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I\right)^kx$$

converges weakly to $z \in F(S)$, where $z = \lim_{n\to\infty} PT^nx$ and $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$.

Proof. Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. From Theorem 2.2, we have that $T : C \to C$ is an (α, β)-generalized hybrid mapping, i.e.,

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2 \tag{4.1}$$

for all $x, y \in C$. Since T is a generalized hybrid mapping and $F(T) = F(S) \neq \emptyset$, T is quasi-nonexpansive. So, $F(T)$ is closed and convex. Let $x \in C$ and $u \in F(T)$. Then, we have $\|T^{n+1}x - u\| \leq \|T^nx - u\|$. Putting $D = F(T)$ in Lemma 2.1, we have that $\lim_{n\to\infty} PT^nx$ converges strongly. Put $z = \lim_{n\to\infty} PT^nx$. Let us show $S_nx \rightharpoonup z$. Since $\{T^nx\}$ is bounded, so is $\{S_nx\}$. Let $\{S_{n_i}x\}$ be a subsequence of $\{S_nx\}$ such that $S_{n_i}x \rightharpoonup v$. By Lemma 4.1, we know $\lim_{n\to\infty}\|S_nx - TS_nx\| = 0$. If $v \neq Tv$, we have from Opial's theorem and Lemma 3.1 that

$$\begin{aligned}
&\liminf_{i\to\infty}\|S_{n_i}x - v\|^2 \\
&\quad< \liminf_{i\to\infty}\|S_{n_i}x - Tv\|^2 \\
&\quad= \liminf_{i\to\infty}(\|S_{n_i}x - TS_{n_i}x\|^2 + \|TS_{n_i}x - Tv\|^2 \\
&\qquad + 2\langle S_{n_i}x - TS_{n_i}x, TS_{n_i}x - Tv\rangle) \\
&\quad= \liminf_{i\to\infty}\|TS_{n_i}x - Tv\|^2 \\
&\quad\leq \liminf_{i\to\infty}((\alpha-\beta)\|S_{n_i}x - v\|^2 + 2(\alpha-1)\langle S_{n_i}x - TS_{n_i}x, v - Tv\rangle \\
&\qquad - (\alpha-\beta-1)\|v - TS_{n_i}x\|^2 \\
&\quad\leq \liminf_{i\to\infty}((\alpha-\beta)\|S_{n_i}x - v\|^2 - (\alpha-\beta-1)\|v - TS_{n_i}x\|^2) \\
&\quad\leq \liminf_{i\to\infty}((\alpha-\beta)\|S_{n_i}x - v\|^2 - (\alpha-\beta-1)\|v - S_{n_i}x + S_{n_i}x - TS_{n_i}x\|^2) \\
&\quad\leq \liminf_{i\to\infty}((\alpha-\beta)\|S_{n_i}x - v\|^2 - (\alpha-\beta-1)\|v - S_{n_i}x\|^2) \\
&\quad= \liminf_{i\to\infty}\|S_{n_i}x - v\|^2,
\end{aligned}$$

which is a contradiction. Therefore, we have $v \in F(T)$. To show $S_nx \rightharpoonup z$, it is sufficient to prove $z = v$. From $v \in F(T)$, we have

$$\begin{aligned}
\langle v - z, T^kx - PT^kx\rangle &= \langle v - PT^kx, T^kx - PT^kx\rangle + \langle PT^kx - z, T^kx - PT^kx\rangle \\
&\leq \langle PT^kx - z, T^kx - PT^kx\rangle \\
&\leq \|PT^kx - z\|\|T^kx - PT^kx\| \\
&\leq \|PT^kx - z\|L
\end{aligned}$$

for all $k \in \mathbb{N}$, where $L = \sup\{\|T^kx - PT^kx\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to n_i and dividing by n_i, we have

$$\left\langle v - z, S_{n_i}x - \frac{1}{n_i}\sum_{k=1}^{n_i} PT^kx \right\rangle \leq \frac{1}{n_i}\sum_{k=1}^{n_i}\|PT^kx - z\|L.$$

Since $S_{n_i}x \rightharpoonup v$ as $i \to \infty$ and $PT^nx \to z$ as $n \to \infty$, we have $\langle v - z, v - z\rangle \leq 0$. This implies $z = v$. Therefore, $\{S_nx\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n\to\infty} PT^nx$. So, we get the desired result. □

5. Strong Convergence Theorems

In this section, we first prove a strong convergence theorem of Halpern's type [7] for super hybrid nonself-mappings in a Hilbert space.

Theorem 5.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let γ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be a mapping such that*

$$\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq (1 + 2\gamma)\|x - y\|^2$$

for all $x, y \in C$. Let $\{\alpha_n\} \subset [0,1]$ be a sequence of real numbers such that $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Suppose $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_C\Big\{\frac{1}{1+\gamma} S x_n + \frac{\gamma}{1+\gamma} x_n\Big\}, \quad n = 1, 2, \dots.$$

If $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element v of $F(S)$, where $v = P_{F(S)}u$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Proof. We have that for any $x, y \in C$,

$$\begin{aligned}
&\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \le (1+2\gamma)\|x-y\|^2 \\
&\quad\Longleftrightarrow \|Sx - Sy\|^2 + \gamma(\|x - Sy\|^2 + \|Sx - y\|^2 - \|Sx - x\|^2 - \|y - Sy\|^2) \\
&\qquad\quad \le (1+2\gamma)\|x-y\|^2 \\
&\quad\Longleftrightarrow \|Sx - Sy\|^2 + \gamma\|x - Sy\|^2 \\
&\qquad\quad \le -\gamma\|Sx - y\|^2 + (1+2\gamma)\|x-y\|^2 + \gamma\|Sx - x\|^2 + \gamma\|y - Sy\|^2.
\end{aligned}$$

So, S is a (1, 0, γ)-super hybrid mapping of C into H. Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, we have from Theorem 2.2 that T is a (1, 0)-generalized hybrid mapping of C into H, i.e., T is a nonexpansive mapping of C into H. Furthermore, we have $F(S) = F(T)$. From Wittmann's theorem [26], we obtain $x_n \to P_{F(P_C T)}u$; see also Takahashi [19]. Let us show $F(P_C T) = F(T) = F(S)$. We know $F(T) = F(S)$. It is obvious that $F(T) \subset F(P_C T)$. We show $F(P_C T) \subset F(T)$. If $P_C T v = v$, we have from the property of P_C (2.4) that for $u \in F(T)$,

$$\begin{aligned}
2\|v - u\|^2 &= 2\|P_C T v - u\|^2 \\
&\le 2\langle Tv - u, P_C T v - u\rangle \\
&= \|Tv - u\|^2 + \|P_C T v - u\|^2 - \|Tv - P_C T v\|^2
\end{aligned}$$

and hence

$$2\|v-u\|^2 \le \|v-u\|^2 + \|v-u\|^2 - \|Tv - v\|^2.$$

So, we have $0 \le -\|Tv - v\|^2$ and hence $Tv = v$. This completes the proof. □

Remark 5.2. *We do not know whether a strong convergence theorem of Halpern's type for generalized hybrid mappings holds or not.*

Next, using an idea of mean convergence, we prove a strong convergence theorem of Halpern's type for super hybrid mappings in a Hilbert space.

Theorem 5.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let α, β and γ be real numbers with $\gamma \ge 0$. Let $S : C \to C$ be a (α, β, γ)-super hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(S)$. Suppose $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\Big(\frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I\Big)^k x_n \end{cases}$$

for all $n = 1, 2, ...,$ *where* $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to* Pu.

Proof. For a (α, β, γ)-super hybrid mapping $S : C \to C$, define

$$T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I.$$

Then, from Theorem 2.2 $T : C \to C$ is an (α, β)-generalized hybrid mapping such that $F(T) = F(S)$. Since $F(T)$ is nonempty, we take $q \in F(T)$. Put $r = \|u - q\|$. We define

$$D = \{y \in H : \|y - q\| \le r\} \cap C.$$

Then D is a nonempty bounded closed convex subset of C. D is T-invariant and contains u. Thus we assume that C is boundeda without loss of generality. T is quasi-nonexpansive. So, we have that for all $q \in F(T)$ and $n = 1, 2, 3, ...,$

$$\begin{aligned} \|z_n - q\| &= \left\| \frac{1}{n} \sum_{k=1}^{n} T^k x_n - q \right\| \le \frac{1}{n} \sum_{k=1}^{n} \|T^k x_n - q\| \\ &\le \frac{1}{n} \sum_{k=1}^{n} \|x_n - q\| = \|x_n - q\|. \end{aligned} \tag{5.1}$$

Let us show $\limsup_{n\to\infty} \langle u - Pu, z_n - Pu\rangle \le 0$. Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ with $z_{n_i} \rightharpoonup v$. We may assume without loss of generality

$$\limsup_{n\to\infty} \langle u - Pu, z_n - Pu\rangle = \lim_{i\to\infty} \langle u - Pu, z_{n_i} - Pu\rangle.$$

By Lemma 4.1, we have $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. Using this equality and Opial's theorem, we have $v \in F(T)$. In fact, if $v \ne Tv$, we have

$$\begin{aligned} &\liminf_{i\to\infty} \|z_{n_i} - v\|^2 \\ &\quad < \liminf_{i\to\infty} \|z_{n_i} - Tv\|^2 \\ &\quad = \liminf_{i\to\infty} (\|z_{n_i} - Tz_{n_i}\|^2 + \|Tz_{n_i} - Tv\|^2 + 2\langle z_{n_i} - Tz_{n_i}, Tz_{n_i} - Tv\rangle) \\ &\quad = \liminf_{i\to\infty} \|Tz_{n_i} - Tv\|^2 \\ &\quad \le \liminf_{i\to\infty} ((\alpha - \beta)\|z_{n_i} - v\|^2 + 2(\alpha - 1)\langle z_{n_i} - Tz_{n_i}, v - Tv\rangle \\ &\qquad - (\alpha - \beta - 1)\|v - Tz_{n_i}\|^2) \\ &\quad \le \liminf_{i\to\infty} ((\alpha - \beta)\|z_{n_i} - v\|^2 - (\alpha - \beta - 1)\|v - Tz_{n_i}\|^2) \\ &\quad \le \liminf_{i\to\infty} ((\alpha - \beta)\|z_{n_i} - v\|^2 - (\alpha - \beta - 1)\|v - z_{n_i} + z_{n_i} - Tz_{n_i}\|^2) \\ &\quad \le \liminf_{i\to\infty} ((\alpha - \beta)\|z_{n_i} - v\|^2 - (\alpha - \beta - 1)\|v - z_{n_i}\|^2) \\ &\quad = \liminf_{i\to\infty} \|z_{n_i} - v\|^2, \end{aligned}$$

which is a contradiction. Therefore, we have $v \in F(T)$. Since P is the metric projection of H onto $F(T)$, we have

$$\lim_{i\to\infty} \langle u - Pu, z_{n_i} - Pu\rangle = \langle u - Pu, v - Pu\rangle \le 0.$$

This implies

$$\limsup_{n\to\infty} \langle u - Pu, z_n - Pu\rangle \le 0. \tag{5.2}$$

Since $x_{n+1} - Pu = (1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)$, from (5.1) we have

$$\begin{aligned}\|x_{n+1} - Pu\|^2 &= \|(1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2 \\ &\le (1-\alpha_n)^2\|z_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle.\end{aligned}$$

Putting $s_n = \|x_n - Pu\|^2$, $\beta_n = 0$ and $\gamma_n = 2\langle u - Pu, x_{n+1} - Pu\rangle$ in Lemma 2.4, from $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (5.2) we have

$$\lim_{n\to\infty} \|x_n - Pu\| = 0.$$

□

Acknowledgements. The second author and the third author are partially supported by Grant-in-Aid for Scientific Research No. 19540167 from Japan Society for the Promotion of Science and by the Grant NSC 99-2115-M-110-004-MY3, respectively.

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Received: March 22, 2010; Accepted: September 21, 2010.