Fixed Point Theory, 12(2011), No. 1, 127-136
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ULAM STABILITY OF SOME VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper we investigate some new applications of the Gronwall lemmas to Ulam stability of some Volterra integral equations in higher dimensions. In this case we present two types of Ulam stability for Volterra integral equations: Ulam-Hyers stability and Ulam-Hyers-Rassias stability.
Key Words and Phrases: Volterra integral equations, Picard operators, abstract Gronwall lemmas, Ulam-Hyers stability, Ulam-Hyers-Rassias stability.
2010 Mathematics Subject Classification: 45G10, 45M10, 45N05, 47N10.

1. Introduction

First we begin with some notions from Picard operators theory. Throughout this paper we follow the terminology and notations in [9]. For the convenience of the reader we shall recall some of them.

Definition 1.1. ([9], [14], [15]). Let $(X, \to)$ be an L-space. By definition an operator $A : X \to X$ is a Picard operator if $F_A = \{x_A^*\}$ and $A^n(x) \to x_A^*$ as $n \to \infty$, for all $x \in X$.

The following abstract lemmas are well known ([9], [10], [14]).

Lemma 1.1. (**Abstract Gronwall Lemma** ([9], [10], [14])). *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We suppose that:*

(i) A is a Picard operators ($F_A = \{x_A^*\}$);

(ii) A is an increasing operator.

Then we have:

(a) $x \in X,\ x \leq A(x) \ \Rightarrow\ x \leq x_A^*$;

(b) $x \in X,\ x \geq A(x) \ \Rightarrow\ x \geq x_A^*$.

Lemma 1.2. (**Abstract Gronwall-Comparison Lemma** ([10])). *Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that:*

(i) A and B are Picard operators;

(ii) A is an increasing operator;

(iii) $A \leq B$.
Then

$$x \leq A(x) \;\Rightarrow\; x \leq x_A^* \leq x_B^*.$$

Remark 1.1. In order to use the Abstract Gronwall Lemma (Lemma 1.1) for to have a concrete Gronwall lemma we need to determine the fixed point x_A^*. If this is a difficult problem we choose an operator B, as in Lemma 1.2, for to have an upper bound for the solution x of the inequation $x \leq A(x)$.

Lemma 1.3. ([5], [7], [9], [10], [11]). *Let* $(X, \to)$ *be an L-space and* $A, B : X \to X$ *two operators.*

We suppose that:

(i) A *and* B *are increasing operators;*

(ii) A *and* B *are POs;*

(iii) $x = A(x) \;\Rightarrow\; x \leq B(x)$.

Then

$$x \leq A(x) \;\Rightarrow\; x \leq x_B^*.$$

For other results see D. Bainov, P. Simeonov [1], C. Crăciun, N. Lungu [2], N. Lungu [4], [5], N. Lungu, I. A. Rus [6], I. A. Rus [12], [13], [15].

2. Ulam-Hyers stability of Volterra integral equations in higher dimensions

We consider the following integral equation in higher dimensions

$$\begin{aligned} u(x_1, x_2, \ldots, x_n) = \alpha + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) u(s_1, x_2, \ldots, x_n) ds_1 + \\ + \int_0^{x_1}\int_0^{x_2} K_2(s_1, s_2, x_3, \ldots, x_n) u(s_1, s_2, x_3, \ldots, x_n) ds_1 ds_2 + \ldots + \\ + \int_0^{x_1} \ldots \int_0^{x_n} K_n(s_1, \ldots, s_n) u(s_1, \ldots, s_n) ds_1 \ldots ds_n, \end{aligned} \tag{2.1}$$

where

$$\alpha > 0, \quad a_i > 0, \quad i = \overline{1,n}, \quad D = \prod_{i=1}^{n} [0, a_i], \quad K_i \in C(D), \quad i = \overline{1,n}$$

and $M_{K_i} > 0$ is such that

$$|K_i(x)| \leq M_{K_i}, \; \forall \; x \in D, \quad i = \overline{1,n}.$$

Let $A : C(D) \to C(D)$ be the operator defined by

$$A(u)(x_1, \ldots, x_n) := \text{second part of (2.1)}.$$

We have

Theorem 2.1. ([7]) *We suppose that* $\alpha > 0$, $K_i \in C(D, \mathbb{R}_+)$, $i = \overline{1,n}$. *Then*

(a) $u^*(x_1, x_2, \ldots, x_n) > 0, \; \forall \; (x_1, x_2, \ldots, x_n) \in D$.

(b) If $K_i(x_1, x_2, \ldots, x_n)$ *is increasing with respect to* $x_{i+1}, \ldots, x_n$*, then* u^* *is increasing.*

Where u^* *is the unique solution of the equation (2.1).*

Remark 2.1. First, we observe that under the conditions of Theorem 2.1 the operator A is increasing and we have

$$u^*(s_1,s_2,x_3,\dots,x_n)\le u^*(s_1,x_2,\dots,x_n),\dots,u^*(s_1,\dots,s_n) \le u^*(s_1,x_2,\dots,x_n) \tag{2.2}$$

for all $s,x\in D$, $s\le x$, $s=(s_1,\dots,s_n)$, $x=(x_1,\dots,x_n)$.

Hence,

$$u^*(x)=A(u^*)(x)\le\alpha+\int_0^{x_1}K_1(s_1,x_2,\dots,x_n)u^*(s_1,x_2,\dots,x_n)ds_1+ +\int_0^{x_1}\int_0^{x_2}K_2(s_1,s_2,x_3,\dots,x_n)u^*(s_1,x_2,\dots,x_n)ds_1ds_2+\dots+ +\int_0^{x_1}\dots\int_0^{x_n}K_n(s_1,\dots,s_n)u^*(s_1,x_2,\dots,x_n)ds_1\dots ds_n. \tag{2.3}$$

Consider now the operator $B:C(D)\to C(D)$ defined by

$$B(u)(x):=\text{last part of (2.3) in which we put } u \text{ instead of } u^*.$$

The operator B is PO on $(C(D),\xrightarrow{unif.})$ and is increasing. Let u_B^* be the unique fixed point of B. Thus we have $u^*\le u_B^*$.

Theorem 2.2. *We suppose that:*

(i) $K_i\in C(D,\mathbb{R}_+)$, $i=\overline{1,n}$ and let $M_{K_i}>0$ be such that $|K_i(x)|\le M_{K_i}$, $\forall\ x\in D$, $i=\overline{1,n}$.

Then:

(a) The equation (2.1) has in $C(D)$ a unique solution u^*;

(b) For each $\varepsilon>0$, if $u\in C(D)$ is a solution of the inequation

$$\Big|u(x_1,x_2,\dots,x_n)-\alpha-\int_0^{x_1}K_1(s_1,x_2,\dots,x_n)u(s_1,x_2,\dots,x_n)ds_1- -\int_0^{x_1}\int_0^{x_2}K_2(s_1,s_2,x_3,\dots,x_n)u(s_1,s_2,x_3,\dots,x_n)ds_1ds_2-\dots- -\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n}K_n(s_1,s_2,\dots,s_n)u(s_1,s_2,\dots,s_n)ds_1ds_2\dots ds_n\Big|\le\varepsilon,\ \forall\ x\in D, \tag{2.4}$$

then

$$|u(x)-u^*(x)|\le C_k\cdot\varepsilon,\ \forall\ x\in D,$$

where

$$C_k=\exp(M_{K_1}a_1+M_{K_2}a_1a_2+\dots+M_{K_n}a_1a_2\dots a_n),$$

i.e., the equation (2.1) is Ulam-Hyers stable.

Proof. (a) It is a well known result (see [7]).

(b) We have:

$$|u(x_1,x_2,\dots,x_n)-u^*(x_1,x_2,\dots,x_n)|\le\Big|u(x_1,x_2,\dots,x_n)-\alpha- -\int_0^{x_1}K_1(s_1,x_2,\dots,x_n)u(s_1,x_2,\dots,x_n)ds_1-$$

$$-\int_0^{x_1}\int_0^{x_2} K_2(s_1,s_2,x_3,\dots,x_n)u(s_1,s_2,x_3,\dots,x_n)ds_1ds_2-\dots-$$
$$-\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n} K_n(s_1,s_2,\dots,s_n)u(s_1,s_2,\dots,s_n)ds_1ds_2\dots ds_n\Big|+$$
$$+\int_0^{x_1}|K_1(s_1,x_2,\dots,x_n)||u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1+$$
$$+\int_0^{x_1}\int_0^{x_2}|K_2(s_1,s_2,x_3,\dots,x_n)|\cdot|u(s_1,s_2,x_3,\dots,x_n)-u^*(s_1,s_2,x_3,\dots,x_n)|ds_1ds_2+\dots+$$
$$+\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n}|K_n(s_1,s_2,\dots,s_n)|\cdot|u(s_1,s_2,\dots,s_n)-u^*(s_1,s_2,\dots,s_n)|ds_1ds_2\dots ds_n.$$

From Lemmas 1.2 and 1.3 and from Remark 1.1 it follows that

$$|u(x_1,x_2,\dots,x_n)-u^*(x_1,x_2,\dots,x_n)|\le$$
$$\le\Big|u(x_1,x_2,\dots,x_n)-\alpha-\int_0^{x_1}K_1(s_1,x_2,\dots,x_n)u(s_1,x_2,\dots,x_n)ds_1-$$
$$-\int_0^{x_1}\int_0^{x_2} K_2(s_1,s_2,x_3,\dots,x_n)u(s_1,s_2,x_3,\dots,x_n)ds_1ds_2-\dots-$$
$$-\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n} K_n(s_1,s_2,\dots,s_n)u(s_1,s_2,\dots,s_n)ds_1ds_2\dots ds_n\Big|+$$
$$+\int_0^{x_1}|K_1(s_1,x_2,\dots,x_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1+$$
$$+\int_0^{x_1}\int_0^{x_2}|K_2(s_1,s_2,x_3,\dots,x_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1ds_2+\dots+$$
$$+\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n}|K_n(s_1,s_2,\dots,s_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1ds_2\dots ds_n$$

and we have

$$|u(x_1,x_2,\dots,x_n)-u^*(x_1,x_2,\dots,x_n)|\le \tag{2.5}$$
$$\le\varepsilon+\int_0^{x_1}|K_1(s_1,x_2,\dots,x_n)|\cdot|u(s_1,x_2,\dots,s_n)-u^*(s_1,x_2,\dots,x_n)|ds_1+$$
$$+\int_0^{x_1}\int_0^{x_2}|K_2(s_1,s_2,x_3,\dots,x_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1ds_2+\dots+$$
$$+\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n}|K_n(s_1,s_2,\dots,s_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1ds_2\dots ds_n.$$

From the Abstract Gronwall-Comparison Lemma (Lemma 1.2) and Gronwall lemma, we have

$$|u(x_1,x_2,\dots,x_n)-u^*(x_1,x_2,\dots,x_n)|\le\varepsilon\exp\Bigg(\int_0^{x_1}K_1(s_1,x_2,\dots,x_n)ds_1+$$
$$+\int_0^{x_1}\int_0^{x_2}K_2(s_1,s_2,x_3,\dots,x_n)ds_1ds_2+\dots+$$
$$+\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n}K_n(s_1,s_2,\dots,s_n)ds_1ds_2\dots ds_n\Bigg)\le$$

$$\leq \varepsilon \exp(M_{K_1}a_1 + M_{K_2}a_1a_2 + \ldots + M_{K_n}a_1a_2\ldots a_n).$$

Hence we have

$$|u(x_1, x_2, \ldots, x_n) - u^*(x_1, x_2, \ldots, x_n)| \leq C_k \cdot \varepsilon$$

where

$$C_k = \exp(M_{K_1}a_1 + M_{K_2}a_1a_2 + \ldots + M_{K_n}a_1a_2\ldots a_n),$$

and i.e., the equation (2.1) has the Ulam-Hyers stability.

Remark 2.2. Theorem 2.2 remains true if (2.1) is replaced by

$$u(x_1, x_2, \ldots, x_n) = \alpha + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) f_1(u(s_1, x_2, \ldots, x_n)) ds_1 + \ldots +$$

$$+ \int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n} K_n(s_1, s_2, \ldots, s_n) f_n(u(s_1, s_2, \ldots, s_n)) ds_1 ds_2 \ldots ds_n,$$

where $f_i(u) \leq u$, f_i increasing and Lipschitz.

3. Ulam-Hyers-Rassias stability of Volterra integral equations in higher dimensions

In what follows we consider the equation (2.1). We have the following theorem:

Theorem 3.1. *We suppose that:*

(i) $K_i \in C(D, \mathbb{R}_+)$, $i = \overline{1,n}$ *and there exists* $M_{K_i} > 0$ *such that*

$$|K_i(x)| \leq M_{K_i}, \ \forall\, x \in D, \quad i = \overline{1,n} \quad \text{and} \quad \varphi \in C(D, \mathbb{R}_+);$$

(ii) φ *is an increasing function.*

Then:

(a) The equation (2.1) has in $C(D)$ *a unique solution* u^**;*

(b) If $u \in C(D)$ *is such that*

$$\Big| u(x_1, x_2, \ldots, x_n) - \alpha - \int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) u(s_1, x_2, \ldots, x_n) ds_1 -$$

$$- \int_0^{x_1}\int_0^{x_2} K_2(s_1, s_2, x_3, \ldots, x_n) u(s_1, s_2, x_3, \ldots, x_n) ds_1 ds_2 - \ldots - \tag{3.1}$$

$$- \int_0^{x_1}\int_0^{x_2}\ldots\int_0^{x_n} K_n(s_1, \ldots, s_n) u(s_1, \ldots, s_n) ds_1 ds_2 \ldots ds_n \Big| \leq \varphi(x_1, x_2, \ldots, x_n),$$

$\forall\, x \in D$, *then*

$$|u(x_1, x_2, \ldots, x_n) - u^*(x_1, x_2, \ldots, x_n)| \leq C_k \cdot \varphi(x_1, x_2, \ldots, x_n) \tag{3.2}$$

where

$$C_k = \exp(M_{K_1}a_1 + M_{K_2}a_1a_2 + \ldots + M_{K_n}a_1a_2\ldots a_n),$$

$u^*(x_1, x_2, \ldots, x_n)$ *is the unique fixed point of the equation (2.1), i.e., the equation (2.1) has the Ulam-Hyers-Rassias stability.*

Proof. (a) It is a well known result (see [7]).

(b) By analogous method as in Theorem 2.1, we have

$$|u(x_1, x_2, \ldots, x_n) - u^*(x_1, x_2, \ldots, x_n)| \leq \varphi(x_1, x_2, \ldots, x_n) + \tag{3.3}$$

$$+\int_0^{x_1} |K_1(s_1,x_2,\dots,x_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1+$$
$$+\int_0^{x_1}\int_0^{x_2} |K_2(s_1,s_2,x_3,\dots,x_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1ds_2+\dots+$$
$$+\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n} |K_n(s_1,s_2,\dots,s_n)|\cdot|u(s_1,x_2,\dots,x_n)-u^*(s_1,x_2,\dots,x_n)|ds_1ds_2\dots ds_n.$$

From (3.3), Lemma 1.3, Theorem 2.1 and Gronwall lemma we have

$$|u(x_1,x_2,\dots,x_n)-u^*(x_1,x_2,\dots,x_n)|\le$$
$$\le \varphi(x_1,x_2,\dots,x_n)\exp\left(\int_0^{x_1} K_1(s_1,x_2,\dots,x_n)ds_1+\right.$$
$$+\int_0^{x_1}\int_0^{x_2} K_2(s_1,s_2,x_3,\dots,x_n)ds_1ds_2+\dots+$$
$$\left.+\int_0^{x_1}\int_0^{x_2}\dots\int_0^{x_n} K_n(s_1,s_2,\dots,s_n)ds_1ds_2\dots ds_n\right)\le$$
$$\le \varphi(x_1,x_2,\dots,x_n)\exp(M_{K_1}a_1+M_{K_2}a_1a_2+\dots+M_{K_n}a_1a_2\dots a_n).$$

Hence we have

$$|u(x_1,x_2,\dots,x_n)-u^*(x_1,x_2,\dots,x_n)|\le C_k\cdot\varphi(x_1,x_2,\dots,x_n) \tag{3.4}$$

where

$$C_k=\exp(M_{K_1}a_1+M_{K_2}a_1a_2+\dots+M_{K_n}a_1a_2\dots a_n). \tag{3.5}$$

Then, the equation (2.1) is Ulam-Hyers-Rassias stable.

4. Ulam-Hyers stability of Volterra integral equation

In what follows we consider the integral equation

$$u(x)=h(x)+\int_0^x f(x,s,u(s),g(u(s)))ds, \quad x\in[0,a) \tag{4.1}$$

and $(\mathbb{B},|\cdot|)$ a (real or complex) Banach space, $f\in C([0,a)\times[0,a)\times\mathbb{B}^2,\mathbb{B})$, $h:[0,a)\to\mathbb{B}$, $g\in C([0,a)\times C([0,a)))$, $a\in(0,\infty]$.

First we consider the equation (4.1). Following I. A. Rus [12], [13], [14] and I. A. Rus, N. Lungu [11], we have a stability result of Ulam-Hyers type for the equation (4.1).

Theorem 4.1. *If we have:*

(i) $f\in C([0,a]\times[0,a]\times\mathbb{B}^2,\mathbb{B})$, $h\in C([0,a],\mathbb{B})$, $g\in C([0,a]\times C([0,a]))$;

(ii) there exists $L_1,L_2>0$ *such that*

$$|f(x,s,u,v)-f(x,s,\overline{u},\overline{v})|\le L_1|u-\overline{u}|+L_2|v-\overline{v}|,\ \forall\ s,x\in[0,a],\quad u,v,\overline{u},\overline{v}\in\mathbb{B};$$

(iii) there exists $L_3>0$ *such that*

$$|g(u)-g(v)|\le L_3|u-v|,\ \textit{for all}\ u,v\in\mathbb{B},$$

then

(a) The equation (4.1) has in $C([0,a],\mathbb{B})$ *a unique solution* u^*;

(b) For each $\varepsilon > 0$, if $u \in C([0,a],\mathbb{B})$ is a solution of the inequation

$$\left|u(x) - h(x) - \int_0^x f(x,s,u(s),g(u(s)))ds\right| \le \varepsilon,\ \forall\ x \in [0,a], \tag{4.2}$$

then

$$|u(x) - u^*(x)| \le C_f \cdot \varepsilon,\ \forall\ x \in [0,a], \tag{4.3}$$

where

$$C_f = \exp(L_1 + L_2L_3)a, \tag{4.4}$$

hence, the equation (4.1) is Ulam-Hyers stable.

Proof. (a) It is a known result ([15]).

(b) We have

$$|u(x) - u^*(x)| \le \left|u(x) - h(x) - \int_0^x f(x,s,u(s),g(u(s)))ds\right| +$$

$$+ \int_0^x |f(x,s,u(s),g(u(s))) - f(x,s,u^*(s),g(u^*(s)))|ds \le$$

$$\le \varepsilon + (L_1 + L_2L_3)\int_0^x |u(s) - u^*(s)|ds.$$

From the Gronwall lemma, we have

$$|u(x) - u^*(x)| \le \varepsilon \exp(L_1 + L_2L_3)a = C_f \cdot \varepsilon, \tag{4.5}$$

where

$$C_f = \exp(L_1 + L_2L_3)a.$$

Hence, the equation (4.1) is Ulam-Hyers stable.

In the following theorem we have a stability result of Ulam-Hyers-Rassias type ([11], [12]), for the equation (4.1).

Theorem 4.2. *We suppose that*

(i) $f \in C([0,a] \times [0,a] \times \mathbb{B}^2,\mathbb{B})$, $h \in C([0,a],\mathbb{B})$, $g \in C([0,a] \times C([0,a]))$ and $\varphi \in C([0,a],\mathbb{R}_+)$;

(ii) there exists $L_1, L_2 > 0$ such that

$$|f(x,s,u,v) - f(x,s,\overline{u},\overline{v})| \le L_1|u - \overline{u}| + L_2|v - \overline{v}|,\ \forall\ s,x \in [0,a], \quad u,v,\overline{u},\overline{v} \in \mathbb{B};$$

(iii) there exists $L_3 > 0$ such that

$$|g(u) - g(v)| \le L_3|u - v|,\ \text{for all } u,v \in \mathbb{B};$$

(iv) the function φ is an increasing function.

Then:

(a) The equation (4.1) has in $C([0,a],\mathbb{B})$ a unique solution u^*;

(b) If $u \in C([0,a],\mathbb{B})$ is such that

$$\left|u(x) - h(x) - \int_0^x f(x,s,u(s),g(u(s)))ds\right| \le \varphi(x),\ \forall\ x \in [0,a],$$

then

$$|u(x) - u^*(x)| \le \varphi(x) \cdot C_f, \tag{4.6}$$

where

$$C_f = \exp((L_1 + L_2L_3)a).$$

Proof. Is analogous as in Theorem 4.1.

Remark 4.1. The Theorems 4.1 and 4.2 remains true if (4.1) is replaced by

$$u(x,y) = h(x,y) + \int_0^x \int_0^y f(x,y,s,t,u(s,t),g(u(s,t)))dsdt, \quad x,s \in [0,a],\ y,t \in [0,a].$$

5. Stability of functional Volterra integral equation

Let $(\mathbb{B}, |\cdot|)$ a (real or complex) Banach space, $\varphi \in C([0,a) \times [0,a), \mathbb{R}_+)$, $K \in C([0,a]^4 \times \mathbb{B}^2, \mathbb{B})$, $g \in C([0,a]^2 \times \mathbb{B}, \mathbb{B})$. We consider the following functional Volterra integral equation

$$u(x,y) = g(x,y,h(u)(x,y)) + \int_0^x \int_0^y K(x,y,s,t,u(s,t),f(u(s,t)))dsdt. \tag{5.1}$$

Theorem 5.1. *If we have:*

(i) $K \in C([0,a]^4 \times \mathbb{B}^2, \mathbb{B})$, $g \in C([0,a]^2 \times \mathbb{B}, \mathbb{B})$, $\varphi \in C([0,a]^2, \mathbb{R}_+)$ *increasing;*

(ii) there exists $l_{K_1}, l_{K_2} > 0$ *such that*

$$|K(x,y,s,t,u,v) - K(x,y,s,t,\overline{u},\overline{v})| \leq l_{K_1}|u-\overline{u}| + l_{K_2}|v-\overline{v}|,$$

for all $x,y,s,t \in [0,a]$, $u,v,\overline{u},\overline{v} \in \mathbb{B}$*;*

(iii) there exists $l_g > 0$ *such that*

$$|g(x,y,e_1) - g(x,y,e_2)| \leq l_g|e_1 - e_2|;$$

(iv) there exists $l_h > 0$ *such that*

$$|h(u) - h(v)| \leq l_h|u-v|;$$

(v) there exists $l_f > 0$ *such that*

$$|f(u) - f(v)| \leq l_f|u-v|;$$

(vi) $l_g \cdot l_h < 1$.

Then:

(a) The equation (5.1) has in $C([0,a] \times [0,a], \mathbb{B})$ *a unique solution* u^**;*

(b) If $u \in C([0,a] \times [0,a], \mathbb{B})$ *is such that*

$$\left| u(x,y) - g(x,y,h(u)(x,y)) - \int_0^x \int_0^y K(x,y,s,t,u(s,t),f(u(s,t)))dsdt \right| \leq \varphi(x,y) \tag{5.2}$$

for all $x,y,s,t \in [0,a]$*, then*

$$|u(x,y) - u^*(x,y)| \leq C_{K,g,h,f} \cdot \varphi(x,y),\ \forall\ x,y \in [0,a]$$

where

$$C_{K,g,h,f} = \frac{1}{1 - l_g l_h} \exp\left(\frac{l_{K_1} + l_{K_2} l_f}{1 - l_g l_h}\right) a^2, \tag{5.3}$$

i.e., the equation (5.1) is the Ulam-Hyers-Rassias stable.

Proof. (a) It is a known result (see [6]).

(b) We have

$$|u(x,y) - u^*(x,y)| \leq \Big| u(x,y) - g(x,y,h(u)(x,y)) -$$

$$-\int_0^x \int_0^y K(x,y,s,t,u(s,t),f(u(s,t)))dsdt\Big|+$$
$$+|g(x,y,h(u)(x,y)) - g(x,y,h(u^*)(x,y))|+$$
$$+\int_0^x \int_0^y |K(x,y,s,t,u(s,t),f(u(s,t))) - K(x,y,s,t,u^*(s,t)f(u^*(s,t)))|dsdt \leq$$
$$\leq \varphi(x,y) + l_g|h(u) - h(u^*)|+$$
$$+\int_0^x \int_0^y (l_{K_1}|u(s,t) - u^*(s,t)| + l_{K_2}|f(u(s,t)) - f(u^*(s,t))|)dsdt \leq$$
$$\leq \varphi(x,y) + l_g l_h|u(x,y) - u^*(x,y)|+$$
$$+\int_0^x \int_0^y (l_{K_1}|u(s,t) - u^*(s,t)| + l_{K_2} l_f|u(s,t) - u^*(s,t)|)dsdt.$$

Hence, we have

$$|u(x,y) - u^*(x,y)| \leq \varphi(x,y) + l_g l_h|u(x,y) - u^*(x,y)|+$$
$$+\int_0^x \int_0^y (l_{K_1} + l_{K_2} l_f)|u(s,t) - u^*(s,t)|dsdt.$$

Then,

$$(1 - l_g l_h)|u(x,y) - u^*(x,y)| \leq \varphi(x,y) + (l_{K_1} + l_{K_2} l_f)\int_0^x \int_0^y |u(s,t) - u^*(s,t)|dsdt,$$

and we have

$$|u(x,y) - u^*(x,y)| \leq \frac{1}{1 - l_g l_h}\varphi(x,y) + \frac{l_{K_1} + l_{K_2} l_f}{1 - l_g l_h}\int_0^x \int_0^y |u(s,t) - u^*(s,t)|dsdt.$$

From Gronwall-lemma ([3]) it follows that

$$|u(x,y) - u^*(x,y)| \leq \frac{1}{1 - l_g l_h}\exp\left(\frac{l_{K_1} + l_{K_2} l_f}{1 - l_g l_h}a^2\right)\varphi(x,y),$$

and

$$|u(x,y) - u^*(x,y)| \leq C_{K,g,h,f} \cdot \varphi(x,y), \tag{5.4}$$

where $C_{K,g,h,f}$ is given by (5.3) and the equation (5.1) is the Ulam-Hyers-Rassias stable.

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Received: May 6, 2010; Accepted: July 21, 2010.