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A CLASS OF NEWTON-LIKE METHODS WITH CUBIC CONVERGENCE FOR NONLINEAR EQUATIONS

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Abstract. In this paper, we present a new class of Newton-like methods for solving nonlinear equations, which deduces, as particular cases, some known results. It is proved that each method in the family is cubically convergent. A general error analysis is given, and the computational efficiency in term of function evaluations is provided. Several numerical examples are given to illustrate the performance of the presented methods by comparing with some other methods.

Key Words and Phrases: Nonlinear equations, Newton method, iterative method, third order convergence, computational efficiency.
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1. Introduction

One of the most studied problems in numerical analysis is to find the solution of a nonlinear equation $f(x) = 0$, where $f : \mathbf{D} \subset \mathbb{R} \to \mathbb{R}$ is a scalar function defined on an open interval $\mathbf{D}$.

A powerful tool to solve such an equation is by means of iterative methods. The well known and frequently used one is the classical Newton method, also called the Newton-Raphson method. It is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.1}$$

which converges quadratically to simple roots. Researchers are very interested in getting higher order iterative methods. In [6], Potra and Pták proposed a third-order method, also known as two-step method, which can be rewritten as

$$x_{n+1} = x_n - \frac{f(x_n) + f\big(x_n - \frac{f(x_n)}{f'(x_n)}\big)}{f'(x_n)}. \tag{1.2}$$

This method only needs the first derivative. It is cheaper than any other third-order methods requiring the evaluation of the second derivative at least. Since the evaluations of higher-order derivatives are generally more expensive than the evaluation of the first derivative.

In recent years, numerous higher-order iterative methods with free second derivatives have been developed and analyzed for solving nonlinear equations that improve some classical methods such as Newton method.

Weerakoon and Fernando [9] used quadrature approximations to Newton-Leibniz formula

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt,$$

they gave a modified Newton-type iterative method as follows

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'\big(x_n - \frac{f(x_n)}{f'(x_n)}\big)}, \tag{1.3}$$

which is proved converging cubically. Lu and Xu [4] combined Newton method and Lagrange polynomial interpolation for $f(x)$. Consider the following interpolation

$$f(x) \approx \sum_{i=0}^{m} f(x_i) l_i(x),$$

where $x_i = x_n - \beta_i \frac{f(x_n)}{f'(x_n)}$, β_i is a set of disposable parameters, and $l_i(x)$ is the set of basis functions. They obtained a new class of methods with third order convergence, which includes

$$x_{n+1} = x_n - \frac{f(x_n) + f\big(x_n + 2\frac{f(x_n)}{f'(x_n)}\big)}{4f'(x_n)}, \tag{1.4}$$

Motivated by the recent work in this direction. In this paper we present and analyze a new family of modified Newton methods with cubical convergence, which include some known results such as those in [4], [6] and [9].

The structure of this paper is as follows: the proposed method is described in Section 2, and the general convergence analysis is also given there. In Section 3, several numerical examples are given to show the performance of the presented methods by comparing with six existed methods.

2. Main results

In this section, we describe our proposed methods for finding a simple root x^* of a nonlinear equation $f(x) = 0$, and give a detailed analysis of the convergence. The following definition is used for the convergence of our methods.

Definition 2.1.[7] A sequence $\{x_n\}$ generated by a numerical method is said to converge to x^* with order $p \geq 1$ if there exists $C > 0$ such that

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|^p} \leq C, \quad n \geq n_0, \text{ where } n_0 \geq 0 \text{ is a suitable integer.}$$

In such a case, the method is said to be of order p. Notice that if $p = 1$, in order for x_n to converge to x^*, it is necessary that $C < 1$, If $p = 2$ or 3, the convergence is said to be quadratic or cubic, respectively.

Let $e_n = x_n - x^*$ be the error at the nth iteration, the relation

$$e_{n+1} = ce_n^p + O(e_n^{p+1})$$

is called the error equation, where $c \neq 0$. It is clear that p is the order of this method.

Our new family of parametric Newton-like methods is given by

$$x_{n+1} = x_n - \frac{\alpha_1 f(x_n) + f\big(x_n + \beta_1 \frac{f(x_n)}{f'(x_n)}\big)}{\alpha_2 f'(x_n) + f'\big(x_n + \beta_2 \frac{f(x_n)}{f'(x_n)}\big)}. \tag{2.1}$$

where α_i, β_i $(i = 1, 2)$ are parameters meeting certain conditions.

Remark 2.1. (i). Obviously, the well-known Newton method (1.1) can be obtained from (2.1), if $\alpha_1 = \alpha_2 \neq -1$, $\beta_1 = \beta_2 = 0$.

Also, it is easy to get the two-step method (1.2) from (2.1), if $\alpha_1 = 1$, $\beta_1 = -1$, $\alpha_2 = \beta_2 = 0$, and Weerakoon-Fernando method (1.3) if $\alpha_1 = 1$, $\beta_1 = 0$, $\alpha_2 = 1$, $\beta_2 = -1$. For Lu-Xu method, we have $\alpha_1 = 1$, $\beta_1 = 2$, $\alpha_2 = 3$, $\beta_2 = 0$.

(ii). If we choose $\alpha_1 = \alpha_2 = \beta_1 = 0$, $\beta_2 = -\frac{1}{2}$, the following method in [1, 5] is obtained,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\big(x_n - \frac{f(x_n)}{2f'(x_n)}\big)}. \tag{2.2}$$

While, it is not difficult to see that the method in [2] can be obtained if we choose $\alpha_1 = -1$, $\alpha_2 = \beta_2 = 0$, $\beta_1 = 1$, that is

$$x_{n+1} = x_n - \frac{f\big(x_n + \frac{f(x_n)}{f'(x_n)}\big) - f(x_n)}{f'(x_n)}. \tag{2.3}$$

We now give the proposed theorem as follows:

Theorem 2.1. *Let $x^* \in \mathbb{R}$ be a simple root of a third differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$, where the open interval D is a neighborhood of x^*. If x_0 is sufficiently close to x^*, then the class of iterative methods defined by* (2.1) *is cubically convergent if and only if the disposable parameters α_i, β_i $(i = 1, 2)$ satisfy the following conditions:*

$$\begin{cases} \alpha_2 = \alpha_1 + \beta_1, \\ \beta_1^2 = 1 + \alpha_1 + \beta_1 + 2\beta_2, \\ \alpha_2 \neq -1. \end{cases}$$

The error equation is

$$e_{n+1} = \left[(\frac{\beta_1^2}{1+\alpha_2} + 1)A_2^2 + (\frac{3\beta_2^2 - \beta_1^3}{1+\alpha_2} - 1)A_3 \right] e_n^3 + O(e_n^4), \tag{2.4}$$

where $e_n = x_n - x^*$ *and* $A_k = \dfrac{f^{(k)}(x^*)}{k! f'(x^*)}, \quad k = 2, 3.$

Proof. Since f has derivatives up to the third order, by expanding $f(x_n)$ and $f'(x_n)$ at x^* in Taylor's expansion, we get

$$f(x_n) = f(e_n + x^*) = f'(x^*)\left[e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\right], \tag{2.5}$$

and

$$f'(x_n) = f'(e_n + x^*) = f'(x^*)\big[1 + 2A_2 e_n + 3A_3 e_n^2 + O(e_n^3)\big]. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\beta_i \frac{f(x_n)}{f'(x_n)} = \beta_i\big[e_n - A_2 e_n^2 + 2(A_2^2 - A_3)e_n^3 + O(e_n^4)\big], \quad i = 1, 2.$$

Set $\bar{e}_{n+1} := x_n - x^* + \beta_i \frac{f(x_n)}{f'(x_n)}$. Then

$$\bar{e}_{n+1} = (1 + \beta_i)e_n - \beta_i A_2 e_n^2 + 2\beta_i(A_2^2 - A_3)e_n^3 + O(e_n^4).$$

Hence,

$$\begin{aligned} f\big(x_n + \beta_1 \frac{f(x_n)}{f'(x_n)}\big) &= f'(x^*)(\bar{e}_{n+1} + A_2\bar{e}_{n+1}^2 + A_3\bar{e}_{n+1}^3) + O(\bar{e}_{n+1}^4) \\ &= f'(x^*)\big\{(1 + \beta_1)e_n + (\beta_1^2 + \beta_1 + 1)A_2 e_n^2 \\ &\quad + \big[(\beta_1^3 + 3\beta_1^2 + \beta_1 + 1)A_3 - 2\beta_1^2 A_2^2\big]\, e_n^3\big\} + O(e_n^4). \end{aligned}$$

$$\begin{aligned} f'\big(x_n + \beta_2 \frac{f(x_n)}{f'(x_n)}\big) &= f'(x^*)(1 + 2A_2\bar{e}_{n+1} + 3A_3\bar{e}_{n+1}^2) + O(\bar{e}_{n+1}^3) \\ &= f'(x^*)\left\{1 + 2(1 + \beta_2)A_2 e_n + (-2\beta_2 A_2^2 + 3(1 + \beta_2)^2 A_3)e_n^2\right\} \\ &\quad + O(e_n^3). \end{aligned}$$

If we denote $K_1 := \alpha_1 f(x_n) + f\big(x_n + \beta_1 \frac{f(x_n)}{f'(x_n)}\big)$ and $K_2 := \alpha_2 f'(x_n) + f'\big(x_n + \beta_2 \frac{f(x_n)}{f'(x_n)}\big)$, then we have

$$\begin{aligned} K_1 &= f'(x^*)\{(1 + \alpha_1 + \beta_1)e_n + (\beta_1^2 + \beta_1 + \alpha_1 + 1)A_2 e_n^2 \\ &\quad + \big[(\beta_1^3 + 3\beta_1^2 + \beta_1 + \alpha_1 + 1)A_3 - 2\beta_1^2 A_2^2\big]e_n^3\} + O(e_n^4), \\ K_2 &= f'(x^*)\big\{1 + \alpha_2 + 2(1 + \alpha_2 + \beta_2)A_2 e_n \\ &\quad + \big(-2\beta_2 A_2^2 + 3(\alpha_2 + (1 + \beta_2)^2)A_3\big)e_n^2\big\} + O(e_n^3). \end{aligned} \tag{2.7}$$

From (2.5), (2.6) and (2.7), after some simplifications, we get

$$\frac{K_1}{K_2} = \varphi_1 e_n + \varphi_2 A_2 e_n^2 + \varphi_3 e_n^3 + O(e_n^4),$$

where

$$\begin{aligned} \varphi_1 &= \frac{1 + \alpha_1 + \beta_1}{1 + \alpha_2}, \\ \varphi_2 &= \frac{(1 + \alpha_2)(1 + \alpha_1 + \beta_1 + \beta_1^2) - 2(1 + \alpha_1 + \beta_1)(1 + \alpha_2 + \beta_2)}{(1 + \alpha_2)^2}, \\ \varphi_3 &= \eta_1 A_2^2 + \eta_2 A_3, \end{aligned}$$

and $\eta_1 = \frac{2\beta_2\varphi_1 - 2\beta_1}{1+\alpha_2} - 2\big(1 + \frac{\beta_2}{1+\alpha_2}\varphi_2\big)$, $\eta_2 = \frac{\beta_1^3 + 3\beta_1^2}{1+\alpha_2} - \frac{3(\beta_2^2 + \beta_2)\varphi_1}{1+\alpha_2} + \varphi_1 - 3\varphi_1^2$.

Thus the error equation of the iterative class defined by (2.1) is

$$e_{n+1} = (1-\varphi_1)e_n - \varphi_2 A_2 e_n^2 - \varphi_3 e_n^3 + O(e_n^4). \tag{2.8}$$

Therefore, the methods are cubically convergent if and only if

$$\begin{cases} 1-\varphi_1 = 0 \\ \varphi_2 = 0. \end{cases} \tag{2.9}$$

Simplifying (2.9), we get its equivalent form

$$\begin{cases} \alpha_2 = \alpha_1 + \beta_1, \\ \beta_1^2 = 1 + \alpha_1 + \beta_1 + 2\beta_2, \\ \alpha_2 \neq -1. \end{cases} \tag{2.10}$$

Substituting (2.9) into (2.8), we have

$$e_{n+1} = \left[(\frac{\beta_1^2}{1+\alpha_2} + 1)A_2^2 + (\frac{3\beta_2^2 - \beta_1^3}{1+\alpha_2} - 1)A_3 \right] e_n^3 + O(e_n^4). \tag{2.11}$$

The proof is complete.

Now using the result of Theorem 2.1, we can get an iterative method with cubic convergence by choosing $\alpha_1 = \alpha_2 = -2$, $\beta_1 = 0$ and $\beta_2 = \frac{1}{2}$ in (2.1) as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n) - f'\big(x_n + \frac{f(x_n)}{2f'(x_n)}\big)}. \tag{2.12}$$

If we consider the definition of efficiency index defined in [8] by $E = \rho^{\frac{1}{w}}$, where ρ is the order of the method and w is the cost of the evaluations of the function and its derivatives per iteration, then the efficiency index of Newton's method is $E = 2^{\frac{1}{2}} \approx 1.414$, and for the new methods, we have higher computational efficiency $E = 3^{\frac{1}{3}} \approx 1.442$.

3. Numerical examples

In this section, we present the following functions to illustrate the efficiency of our method (2.12) (YXM). The method is compared with methods mentioned above, that is the classical Newton method (1.1) (NM), Potra-Pták method (1.2) (PPM), Weerakoon-Fernando method (1.3) (WFM), Lu-Xu method (1.4) (LXM), Homeier method (2.2) (HM) and Kou-Li-Wang method (2.3) (KM). All numerical computations have been carried out in a Matlab 6.1 environment based PC. The stopping criterion is taken as $|x_n - x_{n-1}| \leq 10^{-13}$.

We use the following functions:

$f_1(x) = e^{x^2+7x-30} - 1$,

$f_2(x) = (x-2)^{23} - 1$,

$f_3(x) = x^3 + 4x^2 - 10$,

$f_4(x) = \dfrac{1}{x} - 1$,

$f_5(x) = \sin x - x^2 + 1$,

$f_6(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$
$f_7(x) = \arctan x,$
$f_8(x) = x^2 - e^x - 3x + 2.$

For every function, we try to seek an approximation x_n of the root x^* of equation $f(x) = 0$ after n times iterations. The corresponding function absolute values $|f(x_n)|$ are also given. The number of function evaluations (NFE) is counted as the sum of the number of evaluations of the function f itself plus the number of evaluations of the first derivative f'.

Table 1: Comparison of various iterative methods

	n	NFE	x_n	$\|f(x_n)\|$
	$f_1, x_0 = 6$			
NM	54	108	3	0
PPM	40	120	3	0
WFM	37	111	3	0
LXM	Divergence			
HM	33	99	3	7.1054e-15
KM	32	94	3	0
YXM	20	60	3	0
	$f_2, x_0 = 5$			
NM	30	60	3	0
PPM	22	66	3	0
WFM	21	63	3	0
LXM	Divergence			
HM	19	57	3	0
KM	19	57	3	0
YXM	13	39	3	0
	$f_3, x_0 = 4$			
NM	8	16	1.3652	0
PPM	6	18	1.3652	0
WFM	5	15	1.3652	0
LXM	Divergence			
HM	5	15	1.3652	0
KM	5	15	1.3652	3.5527e-015
YXM	5	15	1.3652	0
	$f_4, x_0 = 2$			
NM	Failure (Divided by zero)			
PPM	Failure (Divided by zero)			
WFM	Failure (Divided by zero)			
LXM	15	45	1	0

	n	NFE	x_n	$\|f(x_n)\|$
HM	6	18	1	0
KM	2	6	1	0
YXM	5	15	1	0
	$f_5, x_0 = 2$			
NM	6	12	1.4096	1.1102e-16
PPM	4	12	1.4096	1.1102e-16
WFM	4	12	1.4096	1.1102e-16
LXM	Divergence			
HM	4	12	1.4096	1.1102e-16
KM	4	12	1.4096	1.1102e-16
YXM	4	12	1.4096	1.1102e-16
	$f_6, x_0 = -5.5$			
NM	37	74	-1.2076	2.6645e-15
PPM	27	81	-1.2076	3.5527e-15
WFM	25	75	-1.2076	2.6645e-15
LXM	Divergence			
HM	23	69	-1.2076	3.5527e-15
KM	22	66	-1.2076	3.5527e-15
YXM	14	42	-1.2076	3.5527e-15
	$f_7, x_0 = 2$			
NM	Failure (Divided by zero)			
PPM	Divergence			
WFM	Failure (Divided by zero)			
LXM	13	39	0	0
HM	6	18	0	0
KM	6	18	0	0
YXM	6	18	0	0
	$f_8, x_0 = 2$			
NM	7	14	0.2575	0
PPM	5	15	0.2575	0
WFM	5	15	0.2575	0
LXM	Divergence			
HM	5	15	0.2575	0
KM	5	15	0.2575	0
YXM	5	15	0.2575	0

One can easily see from this table that our method behaves either similarly or better than the compared methods, especially our new method seems to be superior in difficult cases where some other methods fail in convergence, as functions f_4 and f_7.

4. Conclusion

In this work, we considered developing a new family of iterative methods (2.1) with some parameters. Among this family of methods, we can obtain efficient iterative methods different from any known schemes. Some examples were tested, showing equal or better performance than some other methods of the same kind.

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