

EXISTENCE, MULTIPLICITY OF POSITIVE SOLUTIONS FOR FOUR-POINT BOUNDARY VALUE PROBLEM WITH DEPENDENCE ON THE FIRST ORDER DERIVATIVE

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Abstract. The paper consider positive solutions for second-order four-point boundary value problem

$$\begin{aligned}u''(t) + f(t, u, u') &= 0, t \in (0, 1) \\ u(0) = \alpha u(\eta), u(1) &= \beta u(\xi)\end{aligned}$$

where the first order derivative is involved in the nonlinear term explicitly. By using Krasnoselskii fixed point theorem and triple fixed point theorem, we show the existence, multiplicity of positive solutions for the problem. Some examples are given to illustrate the main results of the article.

Key Words and Phrases: Four point boundary value problem, positive solution, cone, fixed point.
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1. INTRODUCTION

In this paper we are interested in the existence of positive solutions of the following four-point boundary value problem

$$u''(t) + f(t, u, u') = 0, t \in (0, 1) \tag{1.1}$$

$$u(0) = \alpha u(\eta), u(1) = \beta u(\xi) \tag{1.2}$$

under following conditions

C_1) $f : [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \rightarrow [0, +\infty)$ is continuous.

C_2) $\eta, \xi \in (0, 1)$ satisfying $0 < \alpha < \frac{1}{1-\eta}, 0 < \beta < \frac{1}{\xi}, (1-\beta\xi)(1-\alpha) + \alpha\eta(1-\beta) > 0$.

Problem (1.1), (1.2) arise from the discussion of problem (1.1) under the following boundary condition

$$u(0) = \sum_{i=1}^m \alpha_i u(\eta_i), u(1) = \sum_{j=1}^n \beta_j u(\xi_j), \tag{1.3}$$

where $0 < \eta_1 < \eta_2 < \dots < \eta_m < 1, 0 < \xi_1 < \xi_2 < \dots < \xi_n < 1$ and all the α_i, β_j have the same sign. In fact, existence of solutions for boundary value problem (BVP for

short) (1.1), (1.3) can be obtained via the existence subject to the boundary condition (1.2), where $\alpha = \sum_{i=1}^m \alpha_i, \beta = \sum_{j=1}^n \beta_j, \eta \in (\eta_1, \eta_m), \xi \in (\xi_1, \xi_n)$.

The multi-point boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have received a great deal of attentions. To identify a few, we refer the reader to [1-3, 5, 8] and references along this line.

As for four-point boundary value problem, Liu [8] obtained existence of one or two positive solutions for problem

$$\begin{cases} u''(t) + h(t)f(u) = 0, t \in (0, 1) \\ u(0) = \alpha u(\eta), u(1) = \beta u(\xi) \end{cases}$$

under assumptions that

$C_3) f \in C([0, +\infty), [0, +\infty))$;

$C_4) h \in C([0, 1], [0, +\infty))$ and there exists $t_0 \in [0, 1]$ such that $h(t_0) > 0$;

$C_5) \eta \leq \xi, 0 < \alpha < \frac{1}{1-\eta}, 0 < \beta < \frac{1}{\xi}, \Lambda := (1 - \beta\xi)(1 - \alpha) + \alpha\eta(1 - \beta) > 0$.

However all the above works about positive solutions were done under the assumption that f depends on u only and the variable t and first order derivative u' is not involved explicitly. On the other hand, to my best knowledge, there are very few works considering positive solutions of multi-point boundary value problems by Krasnoselskii fixed point theorem which the nonlinear term involved first order u' explicitly. Thus the following question is naturally asked: whether can we obtain a similar conclusion or not, if f depends on both t, u and u' by using Krasnoselskii fixed point theorem. By extending the general definitions of f_0, f_∞ (see [1],[8]), we give a positive answer to above question.

We note that if the nonlinear term $f(t, u, u') = a(t)f(u, u')$, recently paper [9] obtained existence results for a class of boundary value problem with one-dimensional p-Laplacian operator. In [9] the authors claimed that

$$\delta \|u\| \leq [|u|^2 + |u'|^2]^{\frac{1}{2}} \leq \|u\|, t \in [\delta, 1 - \delta], \delta \in (0, \frac{1}{2}), \quad (*)$$

where the norm is defined

$$\|u(t)\| := \max_{0 \leq t \leq 1} [(u(t))^2 + (u'(t))^2]^{\frac{1}{2}},$$

and $u(t)$ is concave on $[0, 1]$.

In fact, (*) can't be obtained because we can't get the relationship between $\min_{\delta \leq t \leq 1-\delta} |u'(t)|$ and $\max_{0 \leq t \leq 1} |u'(t)|$ though the relationship between $\min_{\delta \leq t \leq 1-\delta} |u(t)|$ and $\max_{0 \leq t \leq 1} |u(t)|$ is get by $\min_{\delta \leq t \leq 1-\delta} |u(t)| > \delta \max_{0 \leq t \leq 1} |u(t)|$ through the concavity of $u(t)$. For the results of [9] considerably depended on (*), we think that the existence results of [9] should be revised.

Furthermore, all known results about positive solutions of problem (1.1), (1.2) ([8], [13]) are established under the case $\eta \leq \xi$. We see in section 2 that the case $\eta \geq \xi$

may cause some difficulties. In the paper we overcome it and extend existence results to arbitrary $\eta, \xi \in (0, 1)$. In this sense we extend the main results of [8] and [13].

2. PRELIMINARIES

Let $E := C^1[0, 1]$ be the Banach space endowed with the norm

$$\|u\| := \max_{0 \leq t \leq 1} |u|.$$

Suppose for any $H > 0, |u'| \leq H$, following limits exist. We denote

$$\begin{aligned} \bar{f}_0 &= \lim_{u \rightarrow 0} \max_{0 \leq t \leq 1} \frac{f(t, u, v)}{u}, & \underline{f}_0 &= \lim_{u \rightarrow 0} \min_{0 \leq t \leq 1} \frac{f(t, u, v)}{|u| + |v|}, \\ \bar{f}_\infty &= \lim_{u \rightarrow \infty} \max_{0 \leq t \leq 1} \frac{f(t, u, v)}{u}, & \underline{f}_\infty &= \lim_{u \rightarrow \infty} \min_{0 \leq t \leq 1} \frac{f(t, u, v)}{|u| + |v|}. \end{aligned}$$

These definitions of $\bar{f}_0, \underline{f}_0, \bar{f}_\infty, \underline{f}_\infty$ are of great importance in the proof of our main results. To our best knowledge, this is the first paper in discussing the positive solutions of boundary value problems by Krasnoselskii fixed point theorem which t, u, u' are all involved in the nonlinear term.

The main tools of this paper are the following well-known Krasnoselskii's fixed point theorem and the fixed point index theorem.

Lemma 2.1. ([7]) *Let E be a Banach space and $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and let*

$$A : K \cap (\Omega_2 \setminus \bar{\Omega}_1) \rightarrow K$$

be a completely continuous operator such that

$$\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_1, \text{ and } \|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2;$$

or

$$\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1, \text{ and } \|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2.$$

Then A has a fixed point in $K \cap (\Omega_2 \setminus \bar{\Omega}_1)$.

Lemma 2.2. ([7]) *Let E be a Banach space, $K \subset E$ be a cone. For $\rho > 0$, define $K_\rho = \{u \in K : \|u\| \leq \rho\}$ and assume that $A : K_\rho \rightarrow K$ is a completely continuous operator such that $Au \neq u$ for $x \in \partial K_\rho = \{u \in K : \|u\| = \rho\}$. Then*

$$\|Au\| \leq \|u\|, u \in \partial K_\rho, \text{ then } i(A, K_\rho, K) = 1.$$

$$\|Au\| \geq \|u\|, u \in \partial K_\rho, \text{ then } i(A, K_\rho, K) = 0.$$

Lemma 2.3. *Suppose $y(t) \geq 0$ for $t \in [0, 1]$. Then the problem*

$$u''(t) + y(t) = 0, t \in (0, 1) \tag{2.1}$$

$$u(0) = \alpha u(\eta), u(1) = \beta u(\xi) \tag{2.2}$$

has no positive solution under the following cases:

- (1) $\alpha > \frac{1}{1 - \eta}$, or $\beta > \frac{1}{\xi}$;
- (2) $\Lambda = \alpha\eta(1 - \beta) + (1 - \alpha)(1 - \beta\xi) < 0$.

Proof. For (1), see Liu [8]. Now we consider the case $\Lambda < 0$.

Suppose that (2.1), (2.2) has a positive solution $u(t)$, from the fact that $u''(t) = -y(t) \leq 0$, we know that $u(t)$ is concave on $[0,1]$. If $\eta = \xi, \Lambda = 1 - \beta\xi$, see [1]. If $\eta \neq \xi$, we distinguish two cases:

(1) $\eta < \xi$. From the concavity of $u(t)$, we have

$$\frac{u(\eta) - u(0)}{\eta} \geq \frac{u(\xi) - u(0)}{\xi} \quad \text{and} \quad \frac{u(1) - u(\xi)}{1 - \xi} \leq \frac{u(1) - u(\eta)}{1 - \eta} \quad (2.3)$$

rearranging (2.3) we have

$$\eta u(\xi) \leq [\alpha\eta + (1 - \alpha)\xi]u(\eta) \quad \text{and} \quad (1 - \xi)u(\eta) \leq [1 - \beta\xi + (\beta - 1)\eta]u(\xi),$$

thus

$$\eta(1 - \xi) \leq [\alpha\eta + (1 - \alpha)\xi][1 - \beta\xi + (\beta - 1)\eta], \quad (2.4)$$

which means $(\xi - \eta)\Lambda \geq 0$. A contradiction with $\eta < \xi, \Lambda < 0$. (2) $\eta > \xi$. Similarly with above we can get $(\xi - \eta)\Lambda \leq 0$, also contradict to $\eta > \xi, \Lambda < 0$. This completes the proof. \square

Lemma 2.4. *Let $\Lambda \neq 0$. Then for $y(t) \in C[0,1]$, the problem (2.1), (2.2) has a unique solution*

$$u(t) = \int_0^1 G(t,s)y(s)ds, \quad (2.5)$$

where $G(t,s)$ is the Green's function of problem

$$\begin{cases} -u''(t) = 0, & t \in (0,1) \\ u(0) = \alpha u(\eta), u(1) = \beta u(\xi) \end{cases}$$

which is given as follows:

For $\eta \leq \xi$

$$G(t,s) = \frac{1}{\Lambda} \begin{cases} (\alpha\eta + t - \alpha t)[(1 - \beta\xi) + (\beta - 1)s] + \alpha(s - \eta)[(1 - \beta\xi) + (\beta - 1)t] & 0 \leq s \leq \eta, t \leq s \\ s[(1 - \beta\xi) + (\beta - 1)t] & 0 \leq s \leq \eta, t \geq s \\ (\alpha\eta + t - \alpha t)[(1 - \beta\xi) + (\beta - 1)s] & \eta \leq s \leq \xi, t \leq s \\ (\alpha\eta + s - \alpha s)[(1 - \beta\xi) + (\beta - 1)t] & \eta \leq s \leq \xi, t \geq s \\ (1 - s)[\alpha\eta + (1 - \alpha)t] & \xi \leq s \leq 1, t \leq s \\ (\alpha\eta + s - \alpha s)[(1 - \beta\xi) + (\beta - 1)t] + \beta(\xi - s)[\alpha\eta + (1 - \alpha)t] & \xi \leq s \leq 1, t \geq s \end{cases}$$

For $\eta \geq \xi$

$$G(t,s) = \frac{1}{\Lambda} \begin{cases} (\alpha\eta + t - \alpha t)[(1 - \beta\xi) + (\beta - 1)s] + \alpha(s - \eta)[(1 - \beta\xi) + (\beta - 1)t] & 0 \leq s \leq \xi, t \leq s \\ s[(1 - \beta\xi) + (\beta - 1)t] & 0 \leq s \leq \xi, t \geq s \\ (1 - s)[\alpha\eta + (1 - \alpha)t] + \alpha(s - \eta)[(1 - \beta\xi) + (\beta - 1)t] & \xi \leq s \leq \eta, t \leq s \\ s[(1 - \beta\xi) + (\beta - 1)t] + \beta(\xi - s)[\alpha\eta + (1 - \alpha)t] & \xi \leq s \leq \eta, t \geq s \\ (1 - s)[\alpha\eta + (1 - \alpha)t] & \eta \leq s \leq 1, t \leq s \\ (\alpha\eta + s - \alpha s)[(1 - \beta\xi) + (\beta - 1)t] + \beta(\xi - s)[\alpha\eta + (1 - \alpha)t] & \eta \leq s \leq 1, t \geq s \end{cases}$$

Proof. Considering the definition and property of Green's function together with boundary condition (2.2), we can get the expression of the Green's function by computation. For $\eta > \xi$, the conclusion is new.

Denote

$$G_1(t,s) = \frac{1}{\Lambda} \begin{cases} (\alpha\eta + t - \alpha t)[(1 - \beta\xi) + (\beta - 1)s] & t \leq s \\ (\alpha\eta + s - \alpha s)[(1 - \beta\xi) + (\beta - 1)t] & t \geq s \end{cases}$$

$$P(t) = \frac{\alpha}{\Lambda} \int_0^\eta [(1-\beta\xi) + (\beta-1)t](s-\eta)y(s)ds + \frac{\beta}{\Lambda} \int_\xi^1 [\alpha\eta + (1-\alpha)t](\xi-s)y(s)ds. \quad (2.6)$$

Then (2.5) can be given by

$$u(t) = \int_0^1 G_1(t,s)y(s)ds + P(t), 0 \leq t \leq 1. \quad (2.7)$$

□

Remark 2.1. Considering the Green's function is not symmetric, we can give the expression of the solution by using a symmetric kernel function and a linear function. We claim that (2.7) is satisfied under arbitrary $\eta, \xi \in (0, 1)$. In fact, if $\eta \leq \xi$, $P(t)$ is given by (2.6). If $\eta > \xi$, by computation we get

$$\begin{aligned} P(t) &= \frac{\alpha}{\Lambda} \int_0^\xi [(1-\beta\xi) + (\beta-1)t](s-\eta)y(s)ds \\ &+ \frac{1}{\Lambda} \int_\xi^\eta [\alpha(1-\beta\xi + \beta t - t)(s-\eta) + \beta(\alpha\eta + t - \alpha t)(\xi-s)]y(s)ds \\ &+ \frac{\beta}{\Lambda} \int_\eta^1 [\alpha\eta + (1-\alpha)t](\xi-s)y(s)ds \\ &= \frac{\alpha}{\Lambda} \int_0^\eta [(1-\beta\xi) + (\beta-1)t](s-\eta)y(s)ds + \frac{\beta}{\Lambda} \int_\xi^1 [\alpha\eta + (1-\alpha)t](\xi-s)y(s)ds. \end{aligned}$$

Further considering condition (C_2) it's easy to see

$$\alpha\eta + (1-\alpha)t \geq 0, (1-\beta\xi) + (\beta-1)t \geq 0, t \in [0, 1],$$

then we can easily prove that $G(t, s) \geq 0$, if $\eta \leq \xi$. But we also can see $G(t, s)$ can change sign on $[0, 1] \times [0, 1]$ if $\eta > \xi$ even if (C_2) is satisfied. In this case we prove that the solution is positive mainly by (2.7).

The following lemma takes a great role in the proof of our main results.

Lemma 2.5. Suppose $y(t) \geq 0$ is such that there exists $t_0 \in [0, 1]$ with $y(t_0) > 0$ and the condition (C_2) is satisfied. Then, we have:

- (1) $u(t) > 0, t \in [0, 1]$;
- (2) $u(t) \geq \gamma \max_{0 \leq t \leq 1} u(t), t \in [0, 1]$, where

$$\gamma = \min\left\{\frac{\beta(1-\xi)}{1-\beta\xi}, \frac{\beta\xi}{\beta\xi + 1 - \beta}, \frac{\alpha\eta}{\alpha\eta + 1 - \alpha}, \frac{\alpha(1-\eta)}{1-\alpha\eta}\right\} \leq 1.$$

Proof. (1) For $\eta \leq \xi$, form (2.7) we have

$$\begin{aligned} u(\eta) &= \frac{1}{\Lambda} \int_0^\eta [(1-\beta\xi) + (\beta-1)\eta][\alpha\eta + (1-\alpha)s]y(s)ds \\ &+ \frac{\eta}{\Lambda} \int_\eta^1 [(1-\beta\xi) + (\beta-1)s]y(s)ds \\ &+ \frac{\alpha}{\Lambda} \int_0^\eta (s-\eta)[(1-\beta\xi) + (\beta-1)\eta]y(s)ds + \frac{\beta\eta}{\Lambda} \int_\xi^1 (\xi-s)y(s)ds \end{aligned}$$

$$\begin{aligned}
&= \frac{1 - \beta\xi + (\beta - 1)\eta}{\Lambda} \int_0^\eta sy(s)ds + \frac{\eta}{\Lambda} \int_\eta^\xi [(1 - \beta\xi) + (\beta - 1)s]y(s)ds \\
&\quad + \frac{\eta}{\Lambda} \int_\xi^1 (1 - s)y(s)ds > 0.
\end{aligned}$$

Similarly, for $\eta \geq \xi$ we have

$$\begin{aligned}
u(\eta) &= \frac{1 - \beta\xi + (\beta - 1)\eta}{\Lambda} \int_0^\xi sy(s)ds + \frac{1}{\Lambda} \int_\xi^\eta [s(1 - \eta) + \beta\xi(\eta - s)]y(s)ds \\
&\quad + \frac{\eta}{\Lambda} \int_\eta^1 (1 - s)y(s)ds > 0.
\end{aligned}$$

So $u(0) = \alpha u(\eta) > 0$.

Then we show $u(1) > 0$. Otherwise suppose $u(1) \leq 0$, then $u(\xi) \leq 0$. Considering the concavity of $u(t)$, then

$$\frac{u(\xi) - u(0)}{\xi - 0} \geq \frac{u(1) - u(0)}{1 - 0},$$

we get $u(1) \leq \frac{1}{\xi}u(\xi) + \frac{\xi - 1}{\xi}u(0) < \frac{1}{\xi}u(\xi)$, which means $\beta\xi u(\xi) < u(\xi)$, thus $\beta\xi > 1$.

A contradiction to $\beta < \frac{1}{\xi}$. So $u(1) > 0$. Considering $u(0) > 0, u(1) > 0$ together with the concavity of $u(t)$ we get the conclusion of (1).

(2) Let $\max_{0 \leq t \leq 1} u(t) = u(t_1), \min_{0 \leq t \leq 1} u(t) = u(t_2)$. Obviously $t_2 = 0$, or $t_2 = 1$. We distinguish two cases:

(i) $t_2 = 1$. Here $0 < \beta \leq 1, u(0) \geq u(1)$. For $t_1 \leq \xi < 1$, we see

$$\frac{u(t_1) - u(1)}{t_1 - 1} \geq \frac{u(\xi) - u(1)}{\xi - 1}, \text{ so } u(1) \geq \frac{\beta(1 - \xi)}{1 - \beta\xi}u(t_1).$$

For $0 < \xi \leq t_1$, similarly with above we can get

$$u(t_1) \leq \frac{t_1 u(\xi) - (t_1 - \xi)u(0)}{\xi} \leq \frac{t_1 u(\xi) - (t_1 - \xi)u(1)}{\xi} \leq \frac{t_1 - \beta(t_1 - \xi)}{\beta\xi}u(1),$$

then $u(1) \geq \frac{\beta\xi}{\beta\xi + 1 - \beta}u(t_1)$

(ii) $t_2 = 0$. Here $0 < \alpha \leq 1, u(0) \leq u(1)$. For $\eta \in (0, t_1]$ we have

$$\frac{u(t_1) - u(0)}{t_1} \leq \frac{u(\eta) - u(0)}{\eta}, \text{ then } u(0) \geq \frac{\alpha\eta}{\alpha\eta + 1 - \alpha}u(t_1),$$

and for $\eta \in [t_1, 1)$ we get

$$u(t_1) \leq \frac{(1 - t_1)u(\eta) - (\eta - t_1)u(1)}{1 - \eta} \leq \frac{1 - t_1 - \alpha(\eta - t_1)}{\alpha(1 - \eta)}u(0) \leq \frac{1 - \alpha\eta}{\alpha(1 - \eta)}u(0).$$

Let $\gamma = \min\left\{\frac{\beta(1 - \xi)}{1 - \beta\xi}, \frac{\beta\xi}{\beta\xi + 1 - \beta}, \frac{\alpha\eta}{\alpha\eta + 1 - \alpha}, \frac{\alpha(1 - \eta)}{1 - \alpha\eta}\right\}$, we see

$$u(t) \geq \gamma \max_{0 \leq t \leq 1} u(t), t \in [0, 1]. \quad \square$$

3. EXISTENCE RESULTS OF AT LEAST ONE POSITIVE SOLUTION

Problem (1.1), (1.2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 G(t, s)f(s, u(s), u'(s))ds := Au(t), 0 \leq t \leq 1. \tag{3.1}$$

Define

$$K = \{u \geq 0 \in E : u(0) = \alpha u(\eta), u(1) = \beta u(\xi), u(t) \text{ is concave on } [0, 1]\},$$

it is obviously that K is a cone in E . Moreover by Lemma 2.4, 2.5, $A(K) \subset K$.

From Lemma 2.5 we know that

$$\int_0^1 G(t, s)ds > 0, t \in [0, 1].$$

For the convenience of the reader, we denote

$$M_1 = \max_{0 \leq t \leq 1} \int_0^1 G(t, s)ds, m = \min_{0 \leq t \leq 1} \int_0^1 G(t, s)ds$$

$$M_2 = \max\{|\int_0^1 \frac{\partial G(t, s)}{\partial t}|_{t=0}ds|, |\int_0^1 \frac{\partial G(t, s)}{\partial t}|_{t=1}ds|\}.$$

Theorem 3.1. *Assume that $(C_1), (C_2)$ hold. Then problem (1.1), (1.2) has at least one positive solution if*

(1) $\bar{f}_0 = 0, \underline{f}_\infty = \infty;$

or

(2) $\underline{f}_0 = \infty, \bar{f}_\infty = 0.$

Proof. We consider first the case (1).

By $\bar{f}_0 = 0$, we choose $\varepsilon_1 > 0$ satisfying $\varepsilon_1 M_1 \leq 1$, there exists $H_1 > 0$ such that

$$f(t, u, u') \leq \varepsilon_1 u, \text{ for } u \leq H_1.$$

Define $\Omega_1 = \{u \in E : \|u\| < H_1\}$, then for $u \in K \cap \partial\Omega_1$ we have

$$\begin{aligned} \|Au\| &= \max_{0 \leq t \leq 1} |Au| \\ &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s)f(s, u(s), u'(s))ds \\ &\leq \max_{0 \leq t \leq 1} \varepsilon_1 M_1 u \leq \|u\|. \end{aligned}$$

On the other hand, by $\underline{f}_\infty = \infty$, choose ε_2 satisfying $\varepsilon_2 m \gamma \geq 1$, there exists $\bar{H}_2 > 0$ such that

$$f(t, u, u') \geq \varepsilon_2(|u| + |u'|) \text{ for } u \geq \bar{H}_2.$$

Let $H_2 = \max\{2H_1, \frac{1}{\gamma}\bar{H}_2\}$, $\Omega_2 = \{u \in E : \|u\| < H_2\}$. Then for $u \in K \cap \partial\Omega_2$, we see

$u \geq \gamma \frac{1}{\gamma}\bar{H}_2 = \bar{H}_2$, then

$$\|Au\| = \max_{0 \leq t \leq 1} |Au|$$

$$\begin{aligned}
&= \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s), u'(s)) ds \\
&\geq \varepsilon_2 m(u + |u'|) \geq \varepsilon_2 m \gamma \|u\| \geq \|u\|.
\end{aligned}$$

Then by Lemma 2.1, A has a fixed point u , which implies problem (1.1), (1.2) has at least one positive solution.

Next the case $\underline{f}_0 = \infty, \bar{f}_\infty = 0$ is considered. For $\underline{f}_0 = \infty$, there exists $H_3 > 0$ such that

$$f(t, u, u') \geq \varepsilon_3(|u| + |u'|) \text{ for } 0 < u \leq H_3,$$

where $\varepsilon_3 > 0$ satisfying $\varepsilon_3 m \gamma \geq 1$.

Denote $\Omega_3 = \{u \in E : \|u\| < H_3\}$, then for $u \in K \cap \partial\Omega_3$ we have

$$\begin{aligned}
\|Au\| &= \max_{0 \leq t \leq 1} |Au| \\
&= \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s), u'(s)) ds \\
&\geq \varepsilon_3 m(u + |u'|) \geq \varepsilon_3 m \gamma \|u\| \geq \|u\|.
\end{aligned}$$

Considering $\bar{f}_\infty = 0$ for $|u'| \leq H$, there exists $\bar{H}_4 > 0$ such that

$$f(t, u, u') \leq \varepsilon_4 u \text{ for } u \geq \bar{H}_4,$$

where $\varepsilon_4 > 0$ satisfying $\varepsilon_4 M_1 \leq 1$.

We distinguish two cases :

Case (i). Suppose that f is bounded, then there exists $N > 0$ satisfying $f(t, u(t), u'(t)) < N$. Taking $H_4 = \max\{2H_3, NM_1\}$, for $u \in K \cap \partial\Omega_4, \Omega_4 = \{u \in E : \|u\| < H_4\}$, we have

$$\begin{aligned}
\|Au\| &= \max_{0 \leq t \leq 1} |Au| \\
&= \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s), u'(s)) ds \\
&\leq NM_1 \leq H_4 \leq \|u\|.
\end{aligned}$$

Case (ii). Suppose that f is unbounded, then there certainly exists $H_4 > \max\{2H_3, \bar{H}_4\}$ such that

$$f(t, u, u') \leq f(t, H_4, u'), 0 \leq u \leq H_4.$$

For $u \in K \cap \partial\Omega_4, \Omega_4 = \{u \in E : \|u\| < H_4\}$, we have

$$\begin{aligned}
\|Au\| &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s), u'(s)) ds \\
&\leq M_1 f(t, H_4, u') \leq \varepsilon_4 M_1 H_4 \leq \|u\|.
\end{aligned}$$

So, in either case, we can choose $H_4 > 0$ such that $\|Au\| \leq \|u\|$, for $u \in K \cap \partial\Omega_4, \Omega_4 = \{u \in E : \|u\| < H_4\}$. Then by Lemma 2.1 problem (1.1), (1.2) has at least one positive solution. We completed the proof of Theorem 3.1. \square

Following we consider existence results under the case $\bar{f}_0, \underline{f}_0, \bar{f}_\infty, \underline{f}_\infty \notin \{0, \infty\}$.

Theorem 3.2. Assume that $(C_1), (C_2)$ hold. Then problem (1.1), (1.2) has at least one positive solution in one of the following cases:

(3) $\bar{f}_0 = a \in (0, +\infty), \underline{f}_\infty = b \in (0, +\infty)$ and $aM_1 < 1, bm\gamma > 1$;

- (4) $\underline{f}_0 = a \in (0, +\infty), \bar{f}_\infty = b \in (0, +\infty)$ and $bM_1 < 1, am\gamma > 1$;
- (5) $\bar{f}_0 = a \in (0, +\infty), \underline{f}_\infty = \infty$ and $aM_1 < 1$;
- (6) $\underline{f}_0 = a, \bar{f}_\infty = 0$ and $am\gamma > 1$;
- (7) $\bar{f}_0 = 0, \underline{f}_\infty = b \in (0, +\infty)$ and $bm\gamma > 1$;
- (8) $\underline{f}_0 = \infty, \bar{f}_\infty = b$ and $bM_1 < 1$.

Proof. We consider the case (3) first.

Since $\bar{f}_0 = a \in (0, +\infty)$, for $\varepsilon_5 > 0$ satisfying $(a + \varepsilon_5)M_1 \leq 1$, there exists $H_5 > 0$, such that

$$f(t, u, v) \leq (a + \varepsilon_5)u \text{ for } u \leq H_5.$$

Define $\Omega_5 = \{u \in E : \|u\| < H_5\}$, then for $x \in K \cap \partial\Omega_5$, we see

$$\begin{aligned} \|Au\| &= \max_{0 \leq t \leq 1} |Au| \\ &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s)f(s, u(s), u'(s))ds \\ &\leq \max_{0 \leq t \leq 1} (a + \varepsilon_5)M_1 u \leq \|u\|. \end{aligned}$$

Considering $\underline{f}_\infty = b$, there exists $\bar{H}_6 \geq 0$ such that

$$f(t, u, u') \geq (b - \varepsilon_6)(|u| + |u'|), \text{ for } u \geq \bar{H}_6,$$

where ε_6 satisfy $(b - \varepsilon_6)m\gamma \geq 1$.

Let $H_6 = \max\{2H_5, \frac{1}{\gamma}\bar{H}_6\}$, $\Omega_6 = \{u \in E : \|u\| < H_6\}$, then for $u \in K \cap \partial\Omega_6$, we get $u \geq \bar{H}_6$, then

$$\begin{aligned} \|Au\| &= \max_{0 \leq t \leq 1} |Au| \\ &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s)f(s, u(s), u'(s))ds \\ &\geq (b - \varepsilon_6)m(u + |u'|) \geq (b - \varepsilon_6)m\gamma\|u\| \geq \|u\|. \end{aligned}$$

Then by Lemma 2.1, BVP (1.1), (1.2) has at least one positive solution. As for other cases ((4), (5), (6), (7), (8)), the proof can easily be obtained as in case (1), (2), (3) and are omitted here. \square

4. EXISTENCE OF DOUBLE POSITIVE SOLUTIONS

In this section we study the existence of at least two positive solutions for problem (1.1), (1.2) under

$$\bar{f}_0 = \bar{f}_\infty = 0 \text{ or } \underline{f}_0 = \underline{f}_\infty = \infty.$$

Theorem 4.1. *Assume that (C₁), (C₂) hold and following conditions are satisfied:*

(C₃) $\bar{f}_0 = \bar{f}_\infty = 0$;

(C₄) *there exist $r^* > 0$ such that $f(t, u, u') \geq \frac{r^*}{m}$ for any $H > 0, (t, u, u') \in [0, 1] \times [\gamma r^*, r^*] \times [-H, H]$.*

Then, the problem (1.1), (1.2) has at least two positive solutions u_1, u_2 satisfying

$$0 \leq \|u_1\| \leq r^* \leq \|u_2\|.$$

Proof. Considering $\bar{f}_0 = 0$, similarly with Theorem 3.1, we claim there exists $r_1 \in (0, r^*)$ such that

$$i(A, K_{r_1}, K) = 1, \text{ where } K_{r_1} = \{u \in E : \|u\| < r_1\}.$$

Since $\bar{f}_\infty = 0$, similarly with above there exists $r_2 \in (r^*, +\infty)$ such that

$$i(A, K_{r_2}, K) = 1, \text{ where } K_{r_2} = \{u \in E : \|u\| < r_2\}.$$

If $u \in K \cap \partial K_{r^*}, K_{r^*} = \{u \in E : \|u\| < r^*\}$, then

$$\|Au\| = \max_{0 \leq t \leq 1} |Au| = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s), u'(s)) ds \geq m \times \frac{r^*}{m} = r^* \geq \|u\|.$$

Hence $i(A, K_{r^*}, K) = 0$. Therefore,

$$i(A, K_{r^*} \setminus K_{r_1}, K) = -1, i(A, K_{r_2} \setminus K_{r^*}, K) = 1.$$

So, there exists at least positive solutions u_1, u_2 such that $0 < \|u_1\| < r^* < \|u_2\|$. \square

Theorem 4.2. Assume that $(C_1), (C_2)$ hold and following conditions are satisfied:

$(C_5) \underline{f}_0 = \underline{f}_\infty = \infty;$

(C_6) there exists $R^* > 0$ such that $f(t, u, u') \leq \frac{R^*}{M_1}$ for any $H > 0, (t, u, u') \in$

$[0, 1] \times [\gamma R^*, R^*] \times [-H, H],$

then problem (1.1), (1.2) has at least two positive solutions u_1, u_2 satisfying

$$0 \leq \|u_1\| \leq R^* \leq \|u_2\|.$$

Proof. The proof is similar to the proof of Theorem 4.1, so we omitted it here. \square

5. EXISTENCE OF TRIPLE POSITIVE SOLUTIONS

As for multiple positive solutions of second order BVPs of ordinary differential equations, when the nonlinear term doesn't depends on first order derivative, some results are obtained mainly by the Leggett-Williams fixed point theorem. But the presence of u' causes some considerable difficulties. To overcome this predicament, Bai. et al. [12], [13] generalized the Leggett-Williams fixed point theorem and apply their main results to obtain three positive solutions of problem (1.1) subject to (1.2). But the results are obtained under the case $\eta \leq \xi$. By using the results established in Lemmas 2.4, 2.5, we obtain here the existence results of three positive solutions for $\eta > \xi$ which can be seen as an extension of [13].

In this section, let the Banach space $E = C^1[0, 1]$ be endowed with the ordering $u \leq v$ for all $u(t) \leq v(t), t \in [0, 1]$, and the maximum norm $\|u\| = \max\{\max_{0 \leq t \leq 1} |u(t)|, \max_{0 \leq t \leq 1} |u'(t)|\}$. Define the cone $P \in E$ by

$$P = \{u \in E : u(t) \geq 0, \min_{0 \leq t \leq 1} u(t) \geq \gamma \max_{0 \leq t \leq 1} u(t), u(t) \text{ is concave on } [0, 1]\},$$

where γ is defined in Lemma 2.5.

Define functionals

$$\varphi(x) = \max_{0 \leq t \leq 1} |u(t)|, \psi(x) = \max_{0 \leq t \leq 1} |u'(t)|, \omega(x) = \min_{0 \leq t \leq 1} |u(t)|, \text{ for } u \in E.$$

Noting the definitions of φ, ψ, ω , these functionals satisfy the conditions of Lemma 2.1 in [13].

Remark 5.1. The proof of (2) in Lemma 2.5 allow us to define $\omega(x) = \min_{0 \leq t \leq 1} |u(t)|$. We see that in [13] it is defined by $\omega(x) = \min_{\eta \leq t \leq \xi} |u(t)|$. Comparison with [13], the definition is more natural.

We establish now a result which is similar to Theorem 3.1 of [13].

Theorem 5.1. *Suppose that the conditions $(C_1), (C_2)$ hold and there exist constants $h_2 \geq b/\gamma > b > h_1 > 0, L_2 \geq L_1 > 0$ such that $b/m \leq \min\{h_2/M_1, L_2/M_2\}$. If the following assumptions are satisfied*

$$A_1) f(t, u, v) < \min\{h_1/M_1, L_1/M_2\}, \text{ for } (t, u, v) \in [0, 1] \times [0, h_1] \times [-L_1, L_1];$$

$$A_2) f(t, u, v) > b/m, \text{ for } (t, u, v) \in [0, 1] \times [b, b/\gamma] \times [-L_2, L_2];$$

$$A_3) f(t, u, v) \leq \min\{h_2/M_1, L_2/M_2\}, \text{ for } (t, u, v) \in [0, 1] \times [0, h_2] \times [-L_2, L_2],$$

then the problem (1.1) and (1.2) has at least three positive solutions u_1, u_2, u_3 with

$$\max_{0 \leq t \leq 1} u_1(t) \leq r_1, \max_{0 \leq t \leq 1} |u'_1(t)| \leq L_1;$$

$$b < \min_{0 \leq t \leq 1} u_2(t) \leq \max_{0 \leq t \leq 1} u_2(t) \leq h_2, \max_{0 \leq t \leq 1} |u'_2(t)| \leq L_2;$$

$$\max_{0 \leq t \leq 1} u_3(t) \leq \frac{b}{\gamma}, \max_{0 \leq t \leq 1} |u'_3(t)| \leq L_2.$$

Proof. Lemmas 2.4, 2.5 are considered in both cases $\eta \leq \xi$ and $\eta > \xi$. Considering these results together with the proof of Theorem 3.1 in [13], it's easy to get the proof, thus here we omitted it. \square

6. EXAMPLES

In this section we given some examples to illustrate the importance of our main results.

Example 6.1. Consider the second order boundary value problem

$$u'' + e^{(t+u')}u^2 = 0, 0 \leq t \leq 1, \tag{6.1}$$

$$u(0) = \frac{2}{3}u\left(\frac{\pi}{4}\right), \quad u(1) = \frac{1}{2}u\left(\frac{\pi}{6}\right). \tag{6.2}$$

We see that

$$\alpha = \frac{2}{3}, \beta = \frac{1}{2}, \eta = \frac{\pi}{4}, \xi = \frac{\pi}{6}, f(t, u, u') = e^{(t+u')}u^2 \geq 0 \text{ and}$$

$$\alpha < \frac{1}{1-\eta}, \beta < \frac{1}{\xi}, \Lambda = \frac{7\pi}{18} > 0.$$

Conditions $(C_1), (C_2)$ are satisfied and $\bar{f}_0 = 0, \underline{f}_\infty = \infty$. Then problem (6.1), (6.2) has at least one positive solutions.

Example 6.2. Consider boundary value problem

$$u'' + \pi^4(u + \sin(uu'))^{-2}e^{u+\sin(uu')} = 0, 0 \leq t \leq 1, \tag{6.3}$$

$$u(0) = u\left(\frac{2}{3}\right), \quad u(1) = \frac{1}{2}u\left(\frac{1}{3}\right). \tag{6.4}$$

We see that

$$\alpha = 1, \beta = \frac{1}{2}, \eta = \frac{2}{3}, \xi = \frac{1}{3} \text{ and } \alpha < \frac{1}{1-\eta}, \beta < \frac{1}{\xi}, \Lambda = \frac{1}{3} > 0, \gamma = \frac{1}{4}.$$

We can check that conditions $(C_1), (C_2)$ are satisfied. Furthermore $f_0 = f_\infty = \infty$. By computation we get $m = \frac{2}{3}$. Choosing $r^* = 8$, we know that for any $H > 0$

$$f(t, u, u') \geq \frac{\pi^4 e}{9} \geq \frac{r^*}{m} = 12 \text{ for } (t, u, u') \in [0, 1] \times [2, 8] \times [-H, H].$$

Then by Theorem 4.2 we know that problem (6.3), (6.4) has at least two positive solutions u_1, u_2 satisfying $0 < \|u_1\| < 8 < \|u_2\|$.

Remark 6.1. For the above problems, we notice that the nonlinear term depends on t, u, u' explicitly and $\eta > \xi$. Thus, the previous results for positive solutions are not applicable to the problem.

Example 6.3. Consider the boundary value problem

$$u''(t) + f(t, u, u') = 0, 0 < t < 1, \tag{6.5}$$

$$u(0) = \frac{1}{2}u\left(\frac{3}{4}\right), \quad u(1) = u\left(\frac{1}{2}\right), \tag{6.6}$$

where

$$f(t, u, v) = \begin{cases} \frac{1}{6}e^t + u^3 + \frac{1}{6}\left(\frac{v}{1800}\right)^3 & 0 \leq u \leq 10 \\ \frac{1}{6}e^t + 1000 + \frac{1}{6}\left(\frac{v}{1800}\right)^3 & u > 10 \end{cases}$$

We see

$$\alpha = \frac{1}{2}, \beta = 1, \eta = \frac{3}{4}, \xi = \frac{1}{2} \text{ and } \alpha < \frac{1}{1-\eta}, \beta < \frac{1}{\xi}, \Lambda = \frac{1}{4} > 0, \gamma = \frac{1}{5}.$$

It's obviously that conditions $C_1), C_2)$ hold.

Choose $h_1 = 1, h_2 = 1800, b = 7, L_1 = 10, L_2 = 1800$. By computation we get $M_1 = \frac{9}{16}, M_2 = \frac{2}{9}, m = \frac{7}{16}$. Consequently $f(t, u, v)$ satisfy

$$f(t, u, v) \leq 3200, (t, u, v) \in [0, 1] \times [0, 1800] \times [-1800, 1800];$$

$$f(t, u, v) > 16, (t, u, v) \in [0, 1] \times [7, 35] \times [-1800, 1800];$$

$$f(t, u, v) < \frac{16}{9}, (t, u, v) \in [0, 1] \times [0, 1] \times [-10, 10].$$

Then all assumptions of Theorem 5.1 are satisfied. Thus problem (6.5)-(6.6) has three positive solutions u_1, u_2, u_3 satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} u_1(t) &\leq 1, \max_{0 \leq t \leq 1} |u'_1(t)| \leq 10; \\ 7 < \min_{0 \leq t \leq 1} u_2(t) &\leq \max_{0 \leq t \leq 1} u_2(t) \leq 1800, \max_{0 \leq t \leq 1} |u'_2(t)| \leq 1800; \\ \max_{0 \leq t \leq 1} u_3(t) &\leq 35, \max_{0 \leq t \leq 1} |u'_3(t)| \leq 1800. \end{aligned} \tag{6.7}$$

Remark 6.2. In the example we see $\eta > \xi$, hence the results of [13] and some other results on positive solution for four point BVPs are not applicable to this problem.

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