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AN EXTENSION OF BROWDER'S NON-EJECTIVE FIXED POINT THEOREM

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Abstract. We consider a continuous map $f : Q \to Q$, where Q is the Hilbert cube. It is shown that f will be guaranteed to have a fixed point that is not contained in any ejective invariant set satisfying certain conditions. This is a generalization of the non-ejective fixed point theorem.
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1. Introduction

An invariant set E with respect to a self map f of the Hilbert cube Q is said to be *ejective* if there is an open neighborhood U of E such that for all $x \in U \backslash E$, there is some $k \in \mathbb{N}$ such that $f^k(x) \in Q \backslash U$. In particular, a fixed point x is ejective if the singleton $\{x\}$ is ejective. By compactness of Q, the set $\mathcal{E}$ of ejective subsets of Q must be finite. Notice that each ejective set must also be closed. Felix Browder [1] proved

Theorem 1. *Every continuous function $f : Q \to Q$ has a fixed point which is not ejective.*

This theorem has been very useful in proving the existence of non-trivial periodic solutions to certain types of functional differential equations (see for example [3]). But it will apply only when the trivial solutions correspond to ejective fixed points. We wish to consider cases in which the trivial solutions themselves are not ejective, but may be part of some larger set which is ejective. This may occur, for example, if a point attracts other points nearby which are in a set which is unstable. Also of interest are cases such as in [5], where the set S of trivial solutions forms a line segment, and numerical computations suggest the existence of a nontrivial solution which attracts all points outside of S.

As we shall show below, every $f : Q \to Q$ will have a fixed point not in any given ejective invariant set E, provided E satisfies certain (fairly general) assumptions (which include, for example, the case in which E is a line segment).

2. A Counterexampe and a Reformulation of the Problem

It seems reasonable to expect that if each of the ejective invariant sets is contractible, then f will have a fixed point which is not contained in any of them. But this is false: take[1] Q to be $I^\omega = \prod_{i\in\mathbb{N}}[-1,1]$, and define $g : [-1,1] \to [-1,1]$ by

$$g(x) = -sgn(x)\,|x|^{1/2}.$$

Let

$$A = \{0\} \times \prod_{n=2}^{\infty}[-1,1]$$

and define $h : Q \to Q$ by

$$h((x_i)) = (g(x_1), x_2, \ldots, x_n, \ldots).$$

All of the fixed points of h are in A, which is ejective and convex.

This example, however, seems not to be typical of what one finds in applications. Note that after the first coordinate, h is the identity on Q, so in a sense it is like a self mapping of a finite dimensional set. Indeed, in constructing this example we exploited the fact that the finite-dimensional analogue of Browder's theorem is false. We can rule out these kinds of cases by restricting our attention to ejective sets which are *Z-sets.*

Recall that the topology on I^ω is the one given by the metric

$$d(x,y) = \sum_{1}^{\infty} |x_n - y_n|/2^n.$$

Definition 2. *$A \subset Q$ is called a* Z-set *provided that it is closed and has the property that for any $\epsilon > 0$, there exists a map $f : Q \to Q\backslash A$ such that $d(f(x), x) \le \epsilon$ for all $x \in Q$.*

A Z-set $A \subset Q$ is said to be *contractible in U* where U is a neighborhood of A if there is a map $H : A \times [0,1] \to U$ such that H is the identity on $A \times \{0\}$ and is constant on $A \times \{1\}$. We say that the Z-set A has *trivial shape* if for any neighborhood U of A in Q, A is contractible in U.

These definitions will provide us with the proper context in which to prove our main result.

3. Main Result

One of the key ingredients in the proof of the non-ejective fixed point theorem is the topological homogeneity of the Hilbert cube - that is, the fact that the space around any given point of Q is topologically the same as the space around every other point of Q. To extend Browder's result, we will need the following fact from infinite dimensional topology, which states that the same is true more generally when points are replaced by Z-sets with trivial shape.

[1] In [1] Browder uses the Hilbert parallelotope P. Our use of I^ω instead is merely a convenience since these two spaces are affinely homeomorphic.

Theorem 3. (The Complement Theorem [2]) *Let A be a Z-set in Q. Then $Q\backslash A$ is homeomorphic to $Q\backslash\{p\}$ if and only if A has trivial shape (where $p \in Q$ is arbitrary).*

Lemma 4. *Let A be a Z-set in Q. Then the quotient space Q/A is homeomorphic to Q if and only if A has trivial shape.*

Proof. If A and $\{p\}$ have the same shape, then Q/A and Q are both the one point compactification of the same topological space, since $Q\backslash\{p\}$ is homeomorphic to $Q\backslash A$. But for any locally compact Hausdorff space, the one point compactification is unique up to homeomorphism.

Conversely, suppose that Q/A is homeomorphic to Q. Then for $p \in Q$, by the topological homogeneity of the Hilbert cube there is a homeomorphism h such that $h([a]) = p$, where $[a]$ is the equivalence class equal to A. This induces a homeomorphism $h : (Q/A)\backslash\{[a]\} \to Q\backslash\{p\}$. Since $(Q/A)\backslash\{[a]\}$ and $Q\backslash A$ are obviously homeomorphic, the result again follows from the previous theorem. □

The second implication above will not be needed in what follows but is included here because it suggests that Z-sets with trivial shape are the most general kinds of sets for which the theorem below is true.

Theorem 5. *For a map $f : Q \to Q$, let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a family of ejective Z-sets which have trivial shape. Then f has a non-ejective fixed point which is not contained in any element of $\mathcal{E}$.*

Proof. Define the equivalence relation $\sim$ on Q by $x{\sim}y$ if x and y are contained in the same element of $\mathcal{E}$, and $x{\sim}x$ otherwise. We want to show that the quotient space $Q/{\sim}$ is homeomorphic to Q. The proof is by induction on the cardinality of $\mathcal{E}$. The base case follows from the lemma. Now suppose that for any $\mathcal{E} = \{E_1, \ldots, E_k\}$ satisfying the hypothesis of the theorem, the corresponding quotient $Q/{\sim}$ is homeomorphic to Q. Note that for any $\{E_1, \ldots, E_n\}$, the set

$$Q/{\sim_n} = (\ldots((Q/E_1)/E_2)/\ldots/E_n)$$

is homeomorphic to $Q/{\sim}$. So, if $\mathcal{E}$ has cardinality $k+1$, then since $Q/{\sim_k}$ is homeomorphic to Q (by the inductive hypothesis), $Q/{\sim}$ is homeomorphic to Q/E_{k+1}, which is homeomorphic to Q since E_{k+1} has tivial shape.

Now, suppose that all nonejective fixed points of f are in elements of $\mathcal{E}$. Define the induced mapping $f_* : Q/{\sim} \to Q/{\sim}$ by

$$f_*([x]) = [f(x)].$$

f_* is well defined because if $[x] = [y]$, then either x and y both belong to a single element of $\mathcal{E}$ (in which case $[f(x)] = [f(y)]$), or $x = y$. Let $U \subset Q/{\sim}$ be open, and let $\pi : Q \to Q/{\sim}$ be the quotient map. Then

$$\begin{aligned} f_*^{-1}(U) \quad &= \{[x] \in Q/{\sim} : [f(x)] \in U\} \\ &= \{[x] : f(x) \in \pi^{-1}(U)\} \\ &= \pi(f^{-1}(\pi^{-1}(U))) \end{aligned}$$

which is open since f is continuous. So f_* is a self mapping of the Hilbert cube (represented as $Q/{\sim}$) with all fixed points ejective, which is impossible. Thus f must have a fixed point x which is not contained in any element of $\mathcal{E}$. We can take x to be non-ejective since otherwise we can assume $\{x\} \in \mathcal{E}$. □

If $E_j \in \mathcal{E}$ is finite dimensional and convex, then $\pi_n(E_j) = [-1, 1]$ for only finitely many n, where $\pi_n : I^\omega \to [-1, 1]$ is the projection onto the nth coordinate. This is sufficient for E_j to be a Z-set [4]. Thus we have

Corollary 6. *Suppose that each element of $\mathcal{E}$ is convex and finite dimensional. Then the same conclusion holds.*

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