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THE DARBOUX PROBLEM FOR THIRD ORDER HYPERBOLIC INCLUSIONS VIA CONTINUOUS SELECTIONS FOR CONTINUOUS MULTIFUNCTIONS.

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Abstract. In this paper we present, via a continuous selection theorem, an existence result for the Darboux problem corresponding to a third order hyperbolic inclusion of the form $u_{xyz} \in F(x,y,z,u)$.
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1. Introduction

In this paper we consider the following Darboux problem for a third order hyperbolic inclusion:

$$\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z} \in F(x,y,z,u), \quad (x,y,z) \in D = [0,a]\times[0,b]\times[0,c],\ u \in B \subset \mathbb{R}^n, \tag{1.1}$$

with the initial values

$$\begin{cases} u(x,y,0) = \varphi(x,y), & (x,y) \in D_1 = [0,a]\times[0,b], \\ u(0,y,z) = \psi(y,z), & (y,z) \in D_2 = [0,b]\times[0,c], \\ u(x,0,z) = \chi(x,z), & (x,z) \in D_3 = [0,a]\times[0,c], \end{cases} \tag{1.2}$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense functions (see [4, 565-570]) and they satisfy the conditions

$$\begin{cases} u(x,0,0) = \varphi(x,0) = \chi(x,0) = v^1(x), & x \in [0,a], \\ u(0,y,0) = \varphi(0,y) = \psi(y,0) = v^2(y), & y \in [0,b], \\ u(0,0,z) = \psi(0,z) = \chi(0,z) = v^3(z), & z \in [0,c], \\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0, \end{cases} \tag{1.3}$$

$F : D \times B \to \operatorname{comp} A$ is a continuous multifunction whose values are non-empty compact not necessarily convex subsets of $\mathbb{R}^n$, A is the closed ball centered at the origin of $\mathbb{R}^n$ with radius M and B is the closed ball centered at the origin of $\mathbb{R}^n$ with radius $r = M_1 + M\,abc$.

Using a continuous selection result and Schauder's fixed point theorem we will prove an existence theorem of a solution for the above mentioned Darboux problem.

2. Preliminaries

We recall first some definitions and theorems from [7]-[26]. If X and Y are two non-empty sets, then a multifunction $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

Definition 2.1. Let X and Y be two non-empty sets and $\Phi : X \to 2^Y$ be a multifunction. An element $x \in X$ with the property that $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.2. A singlevalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

For some continuity concepts for multifunctions see, for example, [15].

Definition 2.3. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* (*weakly measurable*), if $\Phi^-(B) \in \mathcal{F}$ for every closed (open) subset $B \subseteq Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Definition 2.4. ([14]) The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is absolutely *continuous in Carathéodory's sense* [4, 565-570] if and only if $u(x,y,z)$ is continuous on D, absolutely continuous in each variable (for any pair of the other two variables) and similarly for $u_x(x,y,z)$, $u_y(x,y,z)$, $u_z(x,y,z)$, $u_{xy}(x,y,z)$, $u_{yz}(x,y,z)$, $u_{xz}(x,y,z)$, and u_{xyz} is Lebesgue-integrable on D.

Theorem 2.5. *The function $u : D \to \mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$, is absolutely continuous in Carathéodory's sense on D if and only if there exist $f \in L^1(D;\mathbb{R}^n)$, $g_1 \in L^1(D_1;\mathbb{R}^n)$, $g_2 \in L^1(D_2;\mathbb{R}^n)$, $g_3 \in L^1(D_3;\mathbb{R}^n)$, $h_1 \in L^1([0,a];\mathbb{R}^n)$, $h_2 \in L^1([0,b];\mathbb{R}^n)$, $h_3 \in L^1([0,c];\mathbb{R}^n)$, such that*

$$\begin{aligned} u(x,y,z) = &\int_0^x \int_0^y \int_0^z f(r,s,t)\,dr\,ds\,dt + \int_0^x \int_0^y g_1(r,s)\,dr\,ds+ \\ &+ \int_0^y \int_0^z g_2(s,t)\,ds\,dt + \int_0^x \int_0^z g_3(r,t)\,dr\,dt+ \\ &+ \int_0^x h_1(r)\,dr + \int_0^y h_2(s)\,ds + \int_0^z h_3(t)\,dt + u(0,0,0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense on D by $C^*(D;\mathbb{R}^n)$ [14].

Theorem 2.6. *The space $C^*(D;\mathbb{R}^n)$ endowed with the norm*

$$\begin{aligned} \|u(\cdot,\cdot,\cdot)\| = &\int_0^a \int_0^b \int_0^c \|u_{xyz}(r,s,t)\|\,dr\,ds\,dt + \int_0^a \int_0^b \|u_{xy}(r,s,0)\|\,dr\,ds+ \\ &+ \int_0^b \int_0^c \|u_{yz}(0,s,t)\|\,ds\,dt + \int_0^a \int_0^c \|u_{xz}(r,0,t)\|\,dr\,dt+ \\ &+ \int_0^a \|u_x(r,0,0)\|\,dr + \int_0^b \|u_y(0,s,0)\|\,ds+ \end{aligned}$$

$$+\int_0^c \|u_z(0,0,t)\|\,dt + \|u(0,0,0)\|,$$

where $D=[0,a]\times[0,b]\times[0,c]\subset\mathbb{R}^3$*, and* $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

We denote by $d(x,y)$ the Euclidean distance from x to y, $x,y\in\mathbb{R}^n$, where $\mathbb{R}^n$ is the Euclidean space. $B[x,r]$ is the closed ball of radius $r>0$ centered at $x\in\mathbb{R}^n$. If $A,B\subset\mathbb{R}^n$, $d(x,A)=\inf\{d(x,y)\mid y\in A\}$, $d^*(A,B)=\sup\{d(y,B)\mid y\in A\}$, $d(x,\emptyset)=\infty$.

Let (X,d) be a metric space. We denote by d_H the Hausdorff-Pompeiu pseudometric on 2^X. The function d_H defines a metric on the space $\mathcal{F}(X)$ of the non-empty and closed subsets of X, called the Hausdorff-Pompeiu metric.

3. Continuous approximate selections

Let $F:D\times B\to \operatorname{comp}A$ be a multifunction with non-empty compact values, where A is the closed ball centered at the origin of $\mathbb{R}^n$ with radius M, $D=[0,a]\times[0,b]\times[0,c]\subset\mathbb{R}^3$ and B is the closed ball centered at the origin of $\mathbb{R}^n$ with radius $r=M_1+Mabc$, where M_1 is given by (3.2).

Let $C(D;\mathbb{R}^n)$ be the Banach space of continuous functions from D into $\mathbb{R}^n$ and $\mathcal{L}^1(D;\mathbb{R}^n)$ the Banach space of equivalence classes of Lebesgue-integrable functions on D and taking values in $\mathbb{R}^n$.

Let the following hypotheses be satisfied:

- (H_1) $F:D\times B\to\operatorname{comp}A$ is a continuous multifunction;
- (H_2) $\varphi\in C^*(D_1;\mathbb{R}^n)$, $\psi\in C^*(D_2;\mathbb{R}^n)$, $\chi\in C^*(D_3;\mathbb{R}^n)$ and it satisfy the conditions (1.3).

Remark 3.1. The function $\alpha:D\to\mathbb{R}^n$ defined by

$$\begin{aligned}\alpha(x,y,z)&=\varphi(x,y)+\psi(y,z)+\chi(x,z)-\varphi(x,0)-\varphi(0,y)-\psi(0,z)+\psi(0,0)=\\&=\varphi(x,y)+\psi(y,z)+\chi(x,z)-v^1(x)-v^2(y)-v^3(z)+v^0,\end{aligned}\tag{3.1}$$

is an absolutely continuous in Carathéodory's sense function on D.

Suppose that the following hypothesis holds:

- (H_3) The function $\alpha:D\to\mathbb{R}^n$ defined by (3.1)is bounded, that is

$$\|\alpha(x,y,z)\|\le M_1,\quad (x,y,z)\in D,\ M_1>0.\tag{3.2}$$

Define $\mathcal{K}$ to be the set of absolutely continuous in Carathéodory's sense functions $u:D\to\mathbb{R}^n$ satisfying

$$\left\|\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z}\right\|\le M,\quad \text{for a.e.}\quad (x,y,z)\in D,\ M>0,\tag{3.3}$$

and the conditions (1.2).

Proposition 3.3. *The set* $\mathcal{K}$ *is a non-empty compact and convex subset of* $C(D;\mathbb{R}^n)$*.*

Proof. The relation $u \in \mathcal{K}$ implies $u \in C(D;\mathbb{R}^n)$. Integrating $\dfrac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z}$ on D and using the conditions (1.2) we obtain

$$\begin{aligned}
u(x,y,z) &= u(x,y,0) + u(x,0,z) - u(x,0,0) + u(0,y,z) - \\
&\quad - u(0,y,0) - u(0,0,z) + u(0,0,0) + \int_0^x\int_0^y\int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\,\partial s\,\partial t}\,dr\,ds\,dt = \\
&= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \\
&\quad - \psi(0,z) + u(0,0,0) + \int_0^x\int_0^y\int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\,\partial s\,\partial t}\,dr\,ds\,dt = \\
&= \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0 + \\
&\quad + \int_0^x\int_0^y\int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\,\partial s\,\partial t}\,dr\,ds\,dt = \\
&= \alpha(x,y,z) + \int_0^x\int_0^y\int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\,\partial s\,\partial t}\,dr\,ds\,dt, \quad (x,y,z) \in D.
\end{aligned} \tag{3.4}$$

The compactness of the set $\mathcal{K}$ results using the theorem of Arzelà-Ascoli. The set $\mathcal{K}$ is equibounded. From (3.2), (3.3) and (3.4) we have

$$\begin{aligned}
\|u(x,y,z)\| &\le \|\alpha(x,y,z)\| + \int_0^x\int_0^y\int_0^z \left\|\frac{\partial^3 u(r,s,t)}{\partial r\,\partial s\,\partial t}\right\| dr\,ds\,dt \le \\
&\le M_1 + \int_0^x\int_0^y\int_0^z M\,dr\,ds\,dt = \\
&= M_1 + Mxyz \le M_1 + Mabc = r, \quad r > 0, \quad (x,y,z) \in D.
\end{aligned}$$

The set $\mathcal{K}$ is equicontinuous. Using the absolute continuity of the integral follows that

$$\begin{gathered}
\|u(x+h,y+k,z+l) - u(x,y,z)\| \le \varepsilon \quad \text{for} \quad h,k,l \in \mathbb{R} \\
\text{with} \quad |h|,|k|,|l| < \delta(\varepsilon),\ \delta(\varepsilon) > 0,\ \varepsilon > 0.
\end{gathered}$$

The set $\mathcal{K}$ is convex. Indeed let be $u_1, u_2 \in \mathcal{K}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 \le \lambda_1 \le 1$, $0 \le \lambda_2 \le 1$, $\lambda_1 + \lambda_2 = 1$. From (3.3) and (1.2) we have

$$\left\|\frac{\partial^3 u_i(x,y,z)}{\partial x\,\partial y\,\partial z}\right\| \le M, \quad (x,y,z) \in D, \quad i = \overline{1,2},$$

and

$$\left\{\begin{array}{ll}
u_i(x,y,0) = \varphi(x,y), & (x,y) \in D_1 = [0,a] \times [0,b], \\
u_i(0,y,z) = \psi(y,z), & (y,z) \in D_2 = [0,b] \times [0,c], \\
u_i(x,0,z) = \chi(x,z), & (x,z) \in D_3 = [0,a] \times [0,c].
\end{array}\right. \qquad i = \overline{1,2}$$

Using the properties of absolutely continuous in Carathéodory's sense functions, it follows that $\lambda_1 u_1 + \lambda_2 u_2 \in C^*(D;\mathbb{R}^n)$.

The relations (3.3) and (1.2) for the function $\lambda_1 u_1 + \lambda_2 u_2$ hold.

$$\left\| \frac{\partial^3 (\lambda_1 u_1 + \lambda_2 u_2)(x,y,z)}{\partial x\, \partial y\, \partial z} \right\| = \left\| \lambda_1 \frac{\partial^3 u_1 (x,y,z)}{\partial x\, \partial y\, \partial z} + \lambda_2 \frac{\partial^3 u_2 (x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \leq$$

$$\leq \lambda_1 \left\| \frac{\partial^3 u_1 (x,y,z)}{\partial x\, \partial y\, \partial z} \right\| + \lambda_2 \left\| \frac{\partial^3 u_2 (x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \leq$$

$$\leq \lambda_1 M + \lambda_2 M = (\lambda_1 + \lambda_2) M = M, \quad (x,y,z) \in D,$$

and

$$\begin{aligned}(\lambda_1 u_1 + \lambda_2 u_2)(x,y,0) &= \lambda_1 u_1 (x,y,0) + \lambda_2 u_2 (x,y,0) = \lambda_1 \varphi (x,y) + \lambda_2 \varphi (x,y) = \\ &= (\lambda_1 + \lambda_2) \varphi (x,y) = \varphi (x,y), \quad (x,y) \in D_1, \\ (\lambda_1 u_1 + \lambda_2 u_2)(0,y,z) &= \lambda_1 u_1 (0,y,z) + \lambda_2 u_2 (0,y,z) = \lambda_1 \psi (y,z) + \lambda_2 \psi (y,z) = \\ &= (\lambda_1 + \lambda_2) \psi (y,z) = \psi (y,z), \quad (y,z) \in D_2, \\ (\lambda_1 u_1 + \lambda_2 u_2)(x,0,z) &= \lambda_1 u_1 (x,0,z) + \lambda_2 u_2 (x,0,z) = \lambda_1 \chi (x,z) + \lambda_2 \chi (x,z) = \\ &= (\lambda_1 + \lambda_2) \chi (x,z) = \chi (x,z), \quad (x,z) \in D_3.\end{aligned}$$

Hence $\lambda_1 u_1 + \lambda_2 u_2 \in \mathcal{K}$ and the set $\mathcal{K}$ is convex.

Remark 3.2. The membership $u \in \mathcal{K}$ implies $(x,y,z,u(x,y,z)) \in D \times B$ for each $(x,y,z) \in D$. In view of the fact that each $u \in \mathcal{K}$ generates a multifunction $(x,y,z) \to F(x,y,z,u(x,y,z))$ from D into $\operatorname{comp} A$, we shall denote this multifunction by $G(u)$

$$G(u)(x,y,z) = F(x,y,z,u(x,y,z)), \quad (x,y,z) \in D. \tag{3.5}$$

Proposition 3.4. *Let be $F : D \times B \to \operatorname{comp} A$ a continuous multifunction (the hypothesis (H_1)). Then for each $\varepsilon > 0$ there exists a continuous function $g : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$ such that for any $u \in \mathcal{K}$,*

$$d(g(u)(x,y,z), G(u)(x,y,z)) < \varepsilon \quad \textit{for a.e.} \quad (x,y,z) \in D. \tag{3.5}$$

Proof. Let $\varepsilon > 0$ be given. In view of the fact that F is continuous on $D \times B$ and $D \times B$ is compact, F is uniformly continuous on $D \times B$ and there exists $\delta(\varepsilon) > 0$ such that

$$d_H(F(x,y,z,u), F(r,s,t,\tilde{u})) < \varepsilon, \tag{3.7}$$

for any two points (x,y,z,u), $(r,s,t,\tilde{u})$ in $D \times B$ with

$$\|(x,y,z) - (r,s,t)\| < \delta(\varepsilon), \quad \|u - \tilde{u}\| < \delta(\varepsilon), \ \delta(\varepsilon) > 0.$$

Let $\{U_l\}_{1 \leq l \leq N}$ be a finite open covering of $\mathcal{K}$ such that $\operatorname{diam} U_l < \delta(\varepsilon)$ for any $l = \overline{1,N}$. Let $\{p_l\}_{1 \leq l \leq N}$ be the continuous partition of unity subordinate to $\{U_l\}$; for each l select a point $u_l \in U_l$ and let $\{v_l\}_{1 \leq l \leq N}$ be a sequence of Lebesgue measurables functions $v_l : D \to \mathbb{R}^n$ such that, for every l, $v_l(x,y,z) \in G(u_l)(x,y,z)$, for a.e. $(x,y,z) \in D$. Such functions v_l exist because each $G(u_l)$ is continuous and measurable in D [1]; $v_l \in \mathcal{L}^1(D;\mathbb{R}^n)$ for every l. We can take $N = N_1 N_2 N_3$. Denote $p_l(u) = p_{ijk}(u)$ and suppose

$$p_{ijk}(u) = q_i(u)\, r_j(u)\, s_k(u), \quad i = \overline{1,N_1}, \ j = \overline{1,N_2}, \ k = \overline{1,N_3}.$$

The functions $p_{ijk} : \mathcal{K} \to \mathbb{R}$, $i = \overline{1,N_1}$, $j = \overline{1,N_2}$, $k = \overline{1,N_3}$, satisfy the properties:

a) $0 \leq p_{ijk}(u) \leq 1$ for $u \in \mathcal{K}$, $i = \overline{1,N_1}$, $j = \overline{1,N_2}$, $k = \overline{1,N_3}$;

b) $p_{ijk}(u) = 0$ if $u \notin U_{ijk}$, $i = \overline{1,N_1}$, $j = \overline{1,N_2}$, $k = \overline{1,N_3}$;

c) $\sum\limits_{i=1}^{N_1}\sum\limits_{j=1}^{N_2}\sum\limits_{k=1}^{N_3} p_{ijk}(u) = \sum\limits_{i=1}^{N_1}\sum\limits_{j=1}^{N_2}\sum\limits_{k=1}^{N_3} q_i(u)\, r_j(u)\, s_k(u) = 1$ for $u \in \mathcal{K}$.

For each $u \in \mathcal{K}$ define the continuous functions $\lambda_{ijk} : \mathcal{K} \to \mathbb{R}$, $\lambda_{ijk}(u) = x_i(u)\, y_j(u)\, z_k(u)$, $i = \overline{1,N_1}$, $j = \overline{1,N_2}$, $k = \overline{1,N_3}$, where

$$\begin{cases} x_0(u) = 0 \\ x_i(u) = x_{i-1}(u) + a q_i(u) \sum\limits_{j=1}^{N_2} r_j(u) \sum\limits_{k=1}^{N_3} s_k(u), \quad i = \overline{1,N_1}, \end{cases}$$

$$\begin{cases} y_0(u) = 0 \\ y_j(u) = y_{j-1}(u) + b r_j(u) \sum\limits_{i=1}^{N_1} q_i(u) \sum\limits_{k=1}^{N_3} s_k(u), \quad j = \overline{1,N_2}, \end{cases}$$

$$\begin{cases} z_0(u) = 0 \\ z_k(u) = z_{k-1}(u) + c s_k(u) \sum\limits_{i=1}^{N_1} q_i(u) \sum\limits_{j=1}^{N_2} r_j(u), \quad k = \overline{1,N_3}. \end{cases}$$

For each $u \in \mathcal{K}$ define the parallelepipeds

$$D_{ijk}(u) = [x_{l-1}(u), x_i(u)[\times [y_{j-1}(u), y_j(u)[\times [z_{k-1}(u), z_k(u)[, \\ i = \overline{1,N_1},\ j = \overline{1,N_2},\ k = \overline{1,N_3},$$

which constitute a partition of D excepting the surface $x = a$, $y = b$, $z = c$.

We construct the desired function $g : \mathcal{K} \to \mathcal{L}^1(D; \mathbb{R}^n)$,

$$\begin{cases} g(u)(x,y,z) = \sum\limits_{i=1}^{N_1}\sum\limits_{j=1}^{N_2}\sum\limits_{k=1}^{N_3} \chi[D_{ijk}(u)](x,y,z)\, v_{ijk}(x,y,z), \\ \qquad\qquad 0 \le x < a, 0 \le y < b, 0 \le z < c, \\ g(u)(a,y,z) = v_{m,N_2,N_3}(a,y,z), \quad m = \min\{i \ge 1, x_i(u) = a\}, \\ g(u)(x,b,z) = v_{N_1,n,N_3}(x,b,z), \quad n = \min\{j \ge 1, y_j(u) = b\}, \\ g(u)(x,y,c) = v_{N_1,N_2,r}(x,y,c), \quad r = \min\{k \ge 1, z_k(u) = c\}, \end{cases} \tag{3.8}$$

where $\chi[D_{ijk}(u)]$ is the characteristic function of the set $D_{ijk}(u)$.

Obviously, g maps $\mathcal{K}$ into $\mathcal{L}^1(D; \mathbb{R}^n)$. Moreover, for a given $u \in \mathcal{K}$ and any fixed $(x,y,z) \in D$, these exists a unique (i,j,k) such that $(x,y,z) \in D_{ijk}(u)$ and this implies $u \in U_{ijk}$.Thus, $g(u)(x,y,z) = v_{ijk}(x,y,z)$ and $\|u(x,y,z) - u_{ijk}(x,y,z)\| < \delta(\varepsilon)$ so that

$$d(g(u)(x,y,z), G(u)(x,y,z)) \le d(v_{ijk}(x,y,z), G(u_{ijk})(x,y,z)) + \\ d_H(G(u_{ijk})(x,y,z), G(u)(x,y,z)) < d(v_{ijk}(x,y,z), G(u_{ijk})(x,y,z)) + \varepsilon.$$

It follows that, for each $u \in \mathcal{K}$, $d(g(u)(x,y,z), G(u)(x,y,z)) < \varepsilon$, for a.e. $(x,y,z) \in D$. We show that g is continuous on $\mathcal{K}$.

Then, for any points u, w in $\mathcal{K}$ and any $(x,y,z) \in D$, $0 \le x < a$, $0 \le y < b$, $0 \le z < c$,

$$\|g(u)(x,y,z) - g(w)(x,y,z)\| \le$$
$$\le \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\sum_{k=1}^{N_3} \chi\left[D_{ijk}(u) \triangle D_{ijk}(w)\right](x,y,z)\,\|v_{ijk}(x,y,z)\|, \tag{3.9}$$

where $D_{ijk}(u) \triangle D_{ijk}(w) = (D_{ijk}(u) - D_{ijk}(w)) \cup (D_{ijk}(w) - D_{ijk}(u))$ is the symmetric difference of two sets.

Since the set $\mathcal{K}$ is compact, $\{\lambda_{ijk}\}_{\substack{1\le i\le N_1\\ 1\le j\le N_2\\ 1\le k\le N_3}}$ is a uniformly equicontinuous family of real valued functions. Thus, for every $\eta > 0$, there exists a $\gamma > 0$ such that, for any $u \in \mathcal{K}$, $w \in \mathcal{K}$ satisfying

$$\|u(x,y,z) - w(x,y,z)\| < \gamma \quad \text{at every} \quad (x,y,z) \in D,$$

we have

$$|\lambda_{ijk}(u) - \lambda_{ijk}(w)| < \eta_{/2MN},$$

and hence

$$\mu\left(D_{ijk}(u) \triangle D_{ijk}(w)\right) < \eta_{/MN},$$

so that (3.8) implies

$$\|g(u) - g(w)\|_{\mathcal{L}^1} = \iiint\limits_D \|g(u)(x,y,z) - g(w)(x,y,z)\|\,dx\,dy\,dz < \eta. \tag{3.10}$$

Therefore $g : \mathcal{K} \to \mathcal{L}^1(D, \mathbb{R}^n)$ is uniformly continuous.

4. Continuous selections

We are able to state our first main result.

Theorem 4.1. *If the hypotheses* (H_1) *is satisfied, that is* $F : D \times B \to \operatorname{comp} A$ *is a continuous multifunction, then there exists a continuous function* $g : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$ *such that, for every* $u \in \mathcal{K}$, $g(u)(x,y,z) \in G(u)(x,y,z)$ *for a.e.* $(x,y,z) \in D$, *that* $g(u)$ *is a continuous selection of* $G(u)$ *given by* (3.5).

Proof. By the Proposition 3.4 there exists a continuous function $g_0 : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$ such that, for each $u \in \mathcal{K}$,

$$d\left(g_0(u)(x,y,z), G(u)(x,y,z)\right) < \frac{1}{2}, \quad \text{for a.e.} \quad (x,y,z) \in D.$$

We will use the induction to show that for every $n \ge 1$, there exists a continuous function $g_n : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$ with the properties that, for each $u \in \mathcal{K}$,

$$d\left(g_n(u)(x,y,z), G(u)(x,y,z)\right) < \frac{1}{2^{n-1}}, \quad \text{for a.e.} \quad (x,y,z) \in D, \tag{4.1}$$

and

$$\mu\left(\left\{(x,y,z) \in D \mid \|g_n(u)(x,y,z) - g_{n-1}(u)(x,y,z)\| \ge \frac{1}{2^{n-1}}\right\}\right) < \frac{1}{2^n}. \tag{4.2}$$

As a result there will exists, for each $u \in \mathcal{K}$, a measurable function $g(u)$ from D into A such that the sequence $\{g_n(u)\}$ converges to $g(u)$ a.e. in measure and that a subsequence of $\{g_n(u)\}$ converges to $g(u)$ a.e. in D. Thus, by (4.1), we will have that, for each $u \in \mathcal{K}$, $g(u)(x,y,z) \in G(u)(x,y,z)$ for a.e. $(x,y,z) \in D$. The details are analogue to those of [1], [3].

5. THE DARBOUX PROBLEM FOR THIRD ORDER HYPERBOLIC INCLUSIONS

We present first the concept of solution for the Darboux problem associated to a third order hyperbolic inclusion.

Definition 5.2. A function $U : D \to \mathbb{R}^n$ is called a *solution* of the Darboux problem $(1.1) + (1.2)$ if it is absolutely continuous in Carathéodory's sense on D and it satisfies (1.1) for a.e. $(x,y,z) \in D$, and also the initial conditions (1.2) for all $(x,y) \in D_1$, all $(y,z) \in D_2$, all $(x,z) \in D_3$.

Theorem 5.1. *Assume that the hypotheses* $(H_1) - (H_3)$ *are satisfied. Then there exists an absolutely continuous in Caratheodory's sense function* $\hat{u} : D \to \mathbb{R}^n$ *which is a solution of Darboux Problem* $(1.1) + (1.2)$.

Proof. Using Theorem 4.1, there is a continuous selection $g : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$ for $G(u)$ given by (3.5). Let $h(u)$, for each $u \in \mathcal{K}$, be the continuous function $h(u) : D \to \mathbb{R}^n$ defined by

$$h(u)(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z g(u)(r,s,t)\,dr\,ds\,dt, \quad (x,y,z) \in D. \tag{5.1}$$

Using (3.1) we have

$$\begin{aligned} h(u)(x,y,z) = {} & \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) + \\ & +\psi(0,0) + \int_0^x \int_0^y \int_0^z g(u)(r,s,t)\,dr\,ds\,dt, \quad (x,y,z) \in D, \end{aligned} \tag{5.2}$$

which can be rewritten, for $(x,y,z) \in D$, as

$$\begin{aligned} h(u)(x,y,z) = {} & \int_0^x \int_0^y \int_0^z g(u)(r,s,t)\,dr\,ds\,dt + \int_0^x \int_0^y \frac{\partial^2 \varphi(r,s)}{\partial r\,\partial s} dr\,ds + \\ & + \int_0^y \int_0^z \frac{\partial^2 \psi(s,t)}{\partial s\,\partial t} ds\,dt + \int_0^x \int_0^z \frac{\partial^2 \chi(r,t)}{\partial r\,\partial t} dr\,dt - \\ & + \int_0^x \frac{\partial \varphi(r,0)}{\partial r} dr + \int_0^y \frac{\partial \varphi(0,s)}{\partial s} ds + \int_0^z \frac{\partial \psi(0,t)}{\partial t} dt + u(0,0,0). \end{aligned} \tag{5.3}$$

Indeed, we have

$$\begin{aligned} \int_0^x \int_0^y \frac{\partial^2 \varphi(r,s)}{\partial r\,\partial s} dr\,ds = {} & \int_0^x \left[\int_0^y \frac{\partial^2 \varphi(r,s)}{\partial r\,\partial s} ds \right] dr = \int_0^x \frac{\partial \varphi(r,s)}{\partial r} \Big|_{s=0}^{s=y} dr = \\ = {} & \int_0^x \left[\frac{\partial \varphi(r,y)}{\partial r} - \frac{\partial \varphi(r,0)}{\partial r} \right] dr = \varphi(r,y) \Big|_{r=0}^{r=x} - \varphi(r,0) \Big|_{r=0}^{r=x} = \\ = {} & [\varphi(x,y) - \varphi(0,y)] - [\varphi(x,0) - \varphi(0,0)] = \\ = {} & \varphi(x,y) - \varphi(0,y) - \varphi(x,0) + \varphi(0,0), \end{aligned} \tag{5.4}$$

$$\begin{aligned}
\int_0^y \int_0^z \frac{\partial^2 \psi(s,t)}{\partial s\,\partial t}\,ds\,dt &= \int_0^y \left[\int_0^z \frac{\partial^2 \psi(s,t)}{\partial s\,\partial t} dt\right] ds = \int_0^y \frac{\partial \psi(s,t)}{\partial s}\Big|_{t=0}^{t=z} ds = \\
&= \int_0^y \left[\frac{\partial \psi(s,z)}{\partial s} - \frac{\partial \psi(s,0)}{\partial s}\right] ds = \psi(s,z)\Big|_{s=0}^{s=y} - \psi(s,0)\Big|_{s=0}^{s=y} = \\
&= [\psi(y,z) - \psi(0,z)] - [\psi(y,0) - \psi(0,0)] = \\
&= \psi(y,z) - \psi(0,z) - \psi(y,0) + \psi(0,0), \qquad (5.5)
\end{aligned}$$

$$\begin{aligned}
\int_0^x \int_0^z \frac{\partial^2 \chi(r,t)}{\partial r\,\partial t}\,dr\,dt &= \int_0^x \left[\int_0^z \frac{\partial^2 \chi(r,t)}{\partial r\,\partial t} dt\right] dr = \int_0^x \frac{\partial \chi(r,t)}{\partial r}\Big|_{t=0}^{t=z} dr = \\
&= \int_0^x \left[\frac{\partial \chi(r,z)}{\partial r} - \frac{\partial \chi(r,0)}{\partial r}\right] dr = \chi(r,z)\Big|_{r=0}^{r=x} - \chi(r,0)\Big|_{r=0}^{r=x} = \\
&= [\chi(x,z) - \chi(0,z)] - [\chi(x,0) - \chi(0,0)] = \\
&= \chi(x,z) - \chi(0,z) - \chi(x,0) + \chi(0,0), \qquad (5.6)
\end{aligned}$$

$$\int_0^x \frac{\partial \varphi(r,0)}{\partial r} dr = \varphi(r,0)\Big|_{r=0}^{r=x} = \varphi(x,0) - \varphi(0,0), \qquad (5.7)$$

$$\int_0^y \frac{\partial \varphi(0,s)}{\partial s} ds = \varphi(0,s)\Big|_{s=0}^{s=y} = \varphi(0,y) - \varphi(0,0), \qquad (5.8)$$

$$\int_0^z \frac{\partial \psi(0,t)}{\partial t} dt = \psi(0,t)\Big|_{t=0}^{t=z} = \psi(0,z) - \psi(0,0). \qquad (5.9)$$

From (1.3) we have

$$u(0,0,0) = \varphi(0,0) = \psi(0,0) = \chi(0,0). \qquad (5.10)$$

Replacing (5.4) – (5.10) in (5.3) it results (5.2).

Using Theorem 2.5, from (5.3) it follows that $h(u) \in C^*(D;\mathbb{R}^n)$ for each $u \in \mathcal{K}$, i.e. $h(u)$ is an absolutely continuous in Carathéodory's sense function. One obtains $h(u) \in \mathcal{K}$ and $h(\mathcal{K}) \subseteq \mathcal{K}$.

Indeed, from (5.2) it results $\dfrac{\partial^3 h(u)(x,y,z)}{\partial x\,\partial y\,\partial z} = g(u)(x,y,z)$, $(x,y,z) \in D$, but $g(u)(x,y,z) \in G(u)(x,y,z) = F(x,y,z,u(x,y,z))$, $(x,y,z) \in D$. Hence $\zeta = g(u)(x,y,z)$ is an element of the ball A, and consequently $\|\zeta\| = \|g(u)(x,y,z)\| \le M$, i.e. $\left\|\dfrac{\partial^3 h(u)(x,y,z)}{\partial x\,\partial y\,\partial z}\right\| \le M$, and from (5.2) $h(u)$ satisfies (1.2). Using the definition of the set $\mathcal{K}$, this inequality shows that $h(u) \in \mathcal{K}$. From $u \in \mathcal{K}$ implying that $h(u) \in \mathcal{K}$, we conclude that $h(\mathcal{K}) \subseteq \mathcal{K}$.

Now, we can apply the Schauder's fixed point theorem and conclude that there exists $\hat{u} \in \mathcal{K}$ such that $\hat{u} = h(\hat{u})$, i.e. $\hat{u}(x,y,z) = h(\hat{u})(x,y,z)$ at every $(x,y,z) \in D$.

This implies that $\hat{u}$ satisfies (1.1),

$$\frac{\partial^3 \hat{u}(x,y,z)}{\partial x\,\partial y\,\partial z} = g(\hat{u})(x,y,z) \in F(x,y,z,\hat{u}(x,y,z)) \quad \text{for a.e.} \quad (x,y,z) \in D,$$

and the relations (1.2). Therefore $\hat{u}(x,y,z) = h(\hat{u})(x,y,z)$ is a solution of Darboux Problem (1.1) + (1.2).

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