Fixed Point Theory, 11(2010), No. 1, 129-132
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

NEW FIXED POINT THEOREMS FOR PSEUDOCONTRACTIVE MAPPINGS AND ZERO POINT THEOREMS FOR ACCRETIVE OPERATORS IN BANACH SPACES

YONGFU SU* AND MENGQIN LI**

*Department of Mathematics
Tianjin Polytechnic University
Tianjin 300160, China
E-mail: suyongfu@tjpu.edu.cn

**Department of Mathematics
Tianjin Polytechnic University
Tianjin 300160, China

Abstract. The purpose of this article is to prove some new fixed point theorems for continuous pseudocontractive mappings and some zero theorems for continuous accretive operators in the reflexive or uniformly convex Banach spaces.
Key Words and Phrases: Pseudocontractive mapping, fixed point theorem, demi-closed principle.
2000 Mathematics Subject Classification: 47H05, 47H10.

1. Introduction and preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into E^* given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|f\|^2 = \|x\|^2\}, \quad x \in E.$$

Recall that, a mapping T with domain $D(T)$ and range $R(T)$ in a Banach space is said to be pseudocontractive, if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2, \quad \forall\ x, y \in D(T).$$

Recall that, a mapping with domain $D(T)$ and range $R(T)$ in a Banach space is said to be strongly pseudocontractive, if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2, \quad \forall\ x, y \in D(T),$$

where $k \in (0,1)$ is a constant.

In 1974, Deimling [1] had proved the following fixed point theorem.

Theorem 1.1. *Let E be a real Banach space, K a nonempty closed convex subset of E and $T : K \to K$ a continuous strongly pseudocontractive mapping. Then T has a unique fixed point in K.*

Recall that, a Banach space E is said to satisfy Opial's condition, if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall\, y \in E,\ y \neq x.$$

Recall that, a mapping T in a Banach space is said to be demi-closed at the zero if for any sequence $\{x_n\}$ which converges weakly to x_0 and $\{Tx_n\}$ converges strongly to zero, then $Tx_0 = 0$.

In 2007, H. Zhou [2] had proved two demi-closed principles for continuous pseudocontractive mappings. These demi-closed principles are very useful for our main results.

Lemma 1.2.[2]. *Let E be a real reflexive Banach space which satisfies Opial's condition. Let K be a nonempty closed convex subset of E, and $T : K \to K$ be a continuous pseudocontractive mapping. Then $I - T$ is demi-closed at zero.*

Lemma 1.3.[2] *Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E, and $T : K \to K$ be a continuous pseudocontractive mapping. Then, $I - T$ is demi-closed at zero.*

Definition 1.4. Let T be a mapping with domain $D(T)$ and range $R(T)$ in a Banach space.

(1) T is said to be invariant, if $R(T) \subset D(T)$;

(2) T is said to be strong invariant, if the range $R(T)$ is bounded and there exists some constant $t_0 \in (0, 1)$ such that $tR(T) \subset D(T)$ for all $t \in [t_0, 1]$.

It is obvious that, a strong invariant mapping must be invariant. However, the inverse is not true.

Example 1.5. Let E be a Banach space and T be a retraction from $K = \{x : \|x - x_0\| \leq 2\}$ onto $R(T) = \{x : \|x - x_0\| \leq 1\|$, where x_0 is a point of E. Then for all $\alpha \in [1/2, 1]$, we have $\alpha R(T) \subset K$, and hence T is strong invariant.

Example 1.6. Let E be a Banach space, $S(E)$ denote the unit ball of E, and let $T : S(E) \to S(E)$ be a any mapping, then T must be strong invariant.

2. Main results

Theorem 2.1. *Let E be a real reflexive Banach space which satisfying Opial's condition, and K be a closed convex subset of E. Let T be a continuous pseudocontractive mapping from K into itself. If T is strong invariant on the K, then T has fixed point in K.*

Proof. Letting $\{\alpha_n\} \subset (0, 1)$ and $\alpha_n \to 1$, since T is continuous pseudocontractive, then $\alpha_n T : K \to K$ is a continuous strongly pseudocontractive mapping for sufficiently large n. By using Theorem 1.1, there exists $\{x_n\} \subset K$ such that $x_n = \alpha_n T x_n$ for sufficiently large n. Therefore we have

$$\|x_n - Tx_n\| = \|x_n - \frac{1}{\alpha_n} x_n\| = (\frac{1}{\alpha_n} - 1)\|x_n\|,$$

and hence,

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty.$$

Since E is reflexive Banach space, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $x^* \in K$. By using Lemma 1.2, we get $x^* = Tx^*$. The proof is complete. □

The following two theorems are special cases of Theorem 2.1.

Theorem 2.2. *Let E be a real reflexive Banach space which satisfying Opial's condition, and $B(r) = \{x \in E : \|x\| \leq r\}$ be a closed ball with radius $r > 0$, and let $T : B(r) \to B(r)$ be a continuous pseudocontractive mapping. Then T has fixed point in $B(r)$.*

Theorem 2.3. *Let E be a real reflexive Banach space which satisfying Opial's condition, T be a continuous pseudocontractive mapping defined on E with bounded range. Then T has fixed point.*

Next, we prove the theorems in the formwork of uniformly convex Banach spaces.

Theorem 2.4. *Let E be a real uniformly convex Banach space, and K be a closed convex subset of E. let T be a continuous pseudocontractive mapping from K into itself. If T is strong invariant on the K, then T has fixed point in K.*

Proof. Letting $\{\alpha_n\} \subset (0,1)$ and $\alpha_n \to 1$, since T is continuous pseudocontractive, then $\alpha_n T : K \to K$ is a continuous strongly pseudocontractive mapping for sufficiently large n, by using Theorem 1.1, there exists $\{x_n\} \subset K$ such that $x_n = \alpha_n T x_n$ for sufficiently large n. Therefore we have

$$\|x_n - Tx_n\| = \|x_n - \frac{1}{\alpha_n} x_n\| = (\frac{1}{\alpha_n} - 1)\|x_n\|,$$

and then we obtain

$$\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty.$$

Since E is uniformly convex Banach space, then E must be reflexive Banach space, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $x^* \in K$. By using Lemma 1.3 we get $x^* = Tx^*$. The proof is complete. □

The following two theorems are special cases of Theorem 2.4.

Theorem 2.5. *Let E be a real uniformly convex Banach space, and $B(r) = \{x \in E : \|x\| \leq r\}$ be a closed ball with radius $r > 0$, and let $T : B(r) \to B(r)$ be a continuous pseudocontractive mapping. Then T has fixed point in $B(r)$.*

Theorem 2.6. *Let E be a real uniformly convex Banach space, T be a continuous pseudocontractive mapping defined on E with bounded range. Then T has fixed point.*

By using the above results, we can obtain the zero point theorems of accretive operators.

Theorem 2.7. *Let E be a real reflexive Banach space which satisfying Opial's condition, A be a continuous accretive operator with closed convex domain $D(A)$ and range $R(A)$. Assume there exists $a \in (0,1)$ such that $\lambda(D(A) - R(A)) \subset D(A)$ for all $\lambda \in [a,1]$ and $D(A) - R(A)$ is bounded. Then A has zero point in D(T).*

Proof. Define a mapping $T = I - A : D(A) \to D(A) - R(A)$, then T is a strong invariant continuous pseudocontractive mapping from $D(A)$ into itself. By using Theorem 2.1, we know that, T has fixed point $p \in D(A)$, so that p is also a zero point of A. The proof is complete. □

Theorem 2.8. *Let E be a real uniformly convex Banach space, A be a continuous accretive operator with closed convex domain $D(A)$ and range $R(A)$. Assume there exists $a \in (0,1)$ such that $\lambda(D(A) - R(A)) \subset D(A)$ for all $\lambda \in [a,1]$ and $D(A) - R(A)$ is bounded. Then A has zero point in D(T).*

Proof. Define a mapping $T = I - A : D(A) \to D(A) - R(A)$, then T is a strong invariant continuous pseudocontractive mapping from $D(A)$ into itself. By Theorem 2.4, we know that, T has fixed point $p \in D(A)$, so that p is also a zero point of A. The proof is completed. □

Acknowledgements. This project is supported by the National Natural Science Foundation of China under grant (10771050).

References

[1] K. Deimling, *Zeros of accretive operators*, Manuscripta Math., **13**(1974), 365-374.

[2] H. Zhou, *Convergence theorems of common fixed points for a finite family of Lipschitz pseudo-contractions in Banach spaces*, Nonlinear Analysis, 2007, doi:10.1016/j.na.2007.02.041.

Received: 05.01.2009; Accepted: 13.04.2009.