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STRICT FIXED POINT RESULTS FOR MULTIVALUED CONTRACTIONS ON GAUGE SPACES

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Abstract. The purpose of this article is to present some strict fixed point results for multivalued δ_α-contractions on complete gauge space. Our theorems generalize and extend some recent results in the literature.

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1. Introduction

Throughout this paper $\mathbb{E}$ will denote a nonempty set E endowed with a separating gauge structure $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$, where Λ is a directed set (see [6] for definitions). Let $\mathbb{N} := \{0, 1, 2, \cdots\}$ and let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

A sequence (x_n) of elements in E is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in \Lambda$, there is an N with $p_\alpha(x_n, x_{n+p}) \leq \varepsilon$ for all $n \geq N$ and $p \in \mathbb{N}^*$. The sequence (x_n) is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in \Lambda$, there is an $N \in \mathbb{N}^*$ with $p_\alpha(x_0, x_n) \leq \varepsilon$, for all $n \geq N$.

A gauge space $\mathbb{E}$ is called complete if any Cauchy sequence is convergent. A subset of X is said to be closed if it contains the limit of any convergent sequence of its elements. See also J. Dugundji [6] for other definitions and details.

Let $\mathcal{P}(\mathbb{E})$ be the set of all subsets of E. We will use the following symbols:

$$P(\mathbb{E}) := \{Y \in \mathcal{P}(\mathbb{E}) :\ Y \neq \emptyset\};\ P_{cl}(\mathbb{E}) := \{Y \in P(\mathbb{E}) :\ Y \text{ is closed }\}.$$

Let us define the gap functional between Y and Z in the $(E, \mathcal{D})$ gauge space

$$D_\alpha : P(\mathbb{E}) \times P(\mathbb{E}) \to \mathbb{R}_+ \cup \{+\infty\},\ D_\alpha(Y, Z) = \inf\{d_\alpha(y, z) \mid y \in Y,\ z \in Z\}$$

and the (generalized) Pompeiu-Hausdorff functional

$$H_\alpha : P(\mathbb{E}) \times P(\mathbb{E}) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H_\alpha(Y, Z) = \max\{\sup_{y\in Y} D_\alpha(y, Z), \sup_{z\in Z} D_\alpha(Y, z)\}.$$

Denote also by $\delta_\alpha : P_b(\mathbb{E}) \times P_b(\mathbb{E}) \to \mathbb{R}_+ \cup \{+\infty\}$, by

$$\delta_\alpha(Y,Z) := \sup\{d_\alpha(y,z) \mid y \in Y,\ z \in Z\}, \alpha \in \Lambda.$$

Notice that $\delta_\alpha(Y,Y) := \delta_\alpha(Y)$.

Let $Y \in P(\mathbb{E})$ be a nonempty set. Then, the set Y is bounded with respect to the gauge structure $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$ if

$$\delta_\alpha(Y) := \sup\{d_\alpha(y,z) \mid y, z \in Y\} < +\infty, \text{ for each } \alpha \in \Lambda.$$

We will denote by $P_b(\mathbb{E})$ the family of all nonempty and bounded (with respect to the gauge structure $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$) subsets of $\mathbb{E}$.

If $T : E \to P(E)$ is a multivalued operator, then $x \in E$ is called fixed point for T if and only if $x \in T(x)$. The set $F_T := \{x \in E | x \in T(x)\}$ denotes the fixed point set of T, while the symbol $(SF)_T := \{x \in E | \{x\} = T(x)\}$ denotes the set of all strict fixed points of T.

Recall also that the multivalued operator T is called closed if its graph $Graph(T) := \{(x,y) \in \mathbb{E} \times \mathbb{E}\ : y \in T(x)\}$ is a closed subset of $\mathbb{E} \times \mathbb{E}$.

Following Frigon [8], we recall the notion of admissible contraction with respect to $\{H_\alpha\}_{\alpha\in\Lambda}$, as follows:

Definition 1.1. Let $\mathbb{E} := (E, \{d_\alpha\}_{\alpha\in\Lambda})$ be a gauge space endowed with a complete gauge structure $\{d_\alpha\}_{\alpha\in\Lambda}$. A multivalued operator $T : \mathbb{E} \to P_{cl}(\mathbb{E})$ is called an admissible multivalued a_α-contraction if $\{a_\alpha\}_{\alpha\in\Lambda} \in]0,1[^\Lambda$ and, for each $\alpha \in \Lambda$, the following conditions are satisfied:

i) $H_\alpha(T(x), T(y)) \leq a_\alpha \cdot d_\alpha(x,y)$, for each $x, y \in \mathbb{E}$

ii) for every $x \in \mathbb{E}$ and every $\{\epsilon_\alpha\}_{\alpha\in\Lambda} \in]1, +\infty[^\Lambda$ there exists $y \in T(x)$ such that $d_\alpha(x,y) \leq D_\alpha(x, T(x)) + \epsilon_\alpha$.

Frigon [8] (see also [9] for an excellent synthesis) proved the following fixed point result:

Theorem 1.2. *Let $\mathbb{E}$ be a complete gauge space and let $T : \mathbb{E} \to P_{cl}(\mathbb{E})$ be an admissible multivalued a_α-contraction. Then F_T is nonempty and closed.*

The aim of this paper is to present some strict fixed point theorems for admissible multivalued contractions with respect to $\{\delta_\alpha\}_{\alpha\in\Lambda}$, on complete gauge space. The results of the paper extend and generalize some previous theorems given in R.P. Agarwal, J. Dshalalow, D. O'Regan [1], S. Reich [14], I.A. Rus, A. Petruşel and G. Petruşel [19], A. Petruşel, I.A. Rus and M.A. Şerban [12] and they are related to the works of A. Chiş, R. Precup [3], R. Espinola, A. Petruşel [7], I.A. Rus [16], C. Chifu, G. Petruşel [2] and G. Petruşel [13].

Some results of this type were proved by D.H. Tan in [20], where the existence and the uniqueness of the strict fixed point for a multivalued operator satisfying a Ćirić type condition (see Ćirić [5] and Minh [10]) with respect to δ_α was obtained in complete gauge spaces satisfying the follwing additional condition:

$$d_\alpha(x,y) = 0, \text{ for each } \alpha \in \Lambda \ \Rightarrow\ x = y.$$

2. The main results

We start this section by presenting the notion of admissible multivalued δ_α-contraction of Reich type, as follows:

Definition 2.1. Let $\mathbb{E} := (E, \{d_\alpha\}_{\alpha\in\Lambda})$ be a complete gauge space. A multivalued operator $T : \mathbb{E} \to P_b(\mathbb{E})$ is called an admissible multivalued δ-contraction of Reich type with coefficients (a, b, c) if $a := \{a_\alpha\}_{\alpha\in\Lambda}, b := \{b_\alpha\}_{\alpha\in\Lambda}, c := \{c_\alpha\}_{\alpha\in\Lambda} \in]0, +\infty[^\Lambda$ and for each $\alpha \in \Lambda$ we have $a_\alpha + b_\alpha + c_\alpha < 1$ and the following conditions are satisfied:
i) $\delta_\alpha(T(x), T(y)) \le a_\alpha d_\alpha(x, y) + b_\alpha \delta_\alpha(x, T(x)) + c_\alpha \delta_\alpha(y, T(y))$, for each $x, y \in \mathbb{E}$;
ii) for every $x \in \mathbb{E}$ and every $\{q_\alpha\}_{\alpha\in\Lambda} \in]1, +\infty[^\Lambda$ there exists $y \in T(x)$ such that $\delta_\alpha(x, T(x)) \le q_\alpha d_\alpha(x, y)$.

Let us start this section by presenting our first strict fixed point result.

Theorem 2.2. *Let $\mathbb{E} := (E, \{d_\alpha\}_{\alpha\in\Lambda})$ be a complete gauge space and $T : \mathbb{E} \to P_b(\mathbb{E})$ be a closed admissible multivalued δ-contraction of Reich type. Then*

$$F_T = (SF)_T = \{x^*\}.$$

Proof. Let $x_0 \in \mathbb{E}$ be arbitrary and $\{q_\alpha\}_{\alpha\in\Lambda} \in]1, +\infty[^\Lambda$. Then, there exists $x_1 \in T(x_0)$ such that $\delta_\alpha(x_0, T(x_0)) \le q_\alpha d_\alpha(x_0, x_1)$, for each $\alpha \in \Lambda$. Hence, for each $\alpha \in \Lambda$, we have:

$$\delta_\alpha(x_1, T(x_1)) \le \delta_\alpha(T(x_0), T(x_1)) \le a_\alpha d_\alpha(x_0, x_1) + b_\alpha \delta_\alpha(x_0, T(x_0)) + c_\alpha \delta_\alpha(x_1, T(x_1))$$

$$\le a_\alpha d_\alpha(x_0, x_1) + b_\alpha q_\alpha d_\alpha(x_0, x_1) + c_\alpha \delta_\alpha(x_1, T(x_1)).$$

Thus

$$\delta_\alpha(x_1, T(x_1)) \le \frac{a_\alpha + b_\alpha q_\alpha}{1 - c_\alpha} \cdot d_\alpha(x_0, x_1), \text{ for all } \alpha \in \Lambda.$$

As before, for x_1 already chosen, there exists $x_2 \in T(x_1)$ such that $\delta_\alpha(x_1, T(x_1)) \le q_\alpha d_\alpha(x_1, x_2)$. Then, we also have:

$$d_\alpha(x_1, x_2) \le \delta_\alpha(x_1, T(x_1)) \le \frac{a_\alpha + b_\alpha q_\alpha}{1 - c_\alpha} \cdot d_\alpha(x_0, x_1), \text{ for all } \alpha \in \Lambda.$$

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{E}$ having the following properties:
(i) $(x_0, x_1) \in Graph(T)$ is arbitrary and $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;
(ii) $d_\alpha(x_n, x_{n+1}) \le (\frac{a_\alpha + b_\alpha q_\alpha}{1 - c_\alpha})^n \cdot d_\alpha(x_0, x_1)$, for all $\alpha \in \Lambda$ and $n \in \mathbb{N}$.

We will prove now that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{E}$. Denote $l_\alpha := \frac{a_\alpha + b_\alpha q_\alpha}{1 - c_\alpha}$. Then, for $n, p \in \mathbb{N}$ with $p \ge 1$, we successively have:

$$d_\alpha(x_n, x_{n+p}) \le d_\alpha(x_n, x_{n+1}) + d_\alpha(x_{n+1}, x_{n+2}) + \cdots + d_\alpha(x_{n+p-1}, x_{n+p})$$
$$\le l_\alpha^n \cdot \frac{1 - l_\alpha^p}{1 - l_\alpha} \cdot d_\alpha(x_0, x_1).$$

Let us chose $1 < q_\alpha < \frac{1 - a_\alpha - c_\alpha}{b_\alpha}$. Then $l_\alpha \in [0, 1[$, for each $\alpha \in \Lambda$. Letting $n \to +\infty$, we get that $d_\alpha(x_n, x_{n+p}) \to 0$, for each $\alpha \in \Lambda$. Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{E}$ and hence it is convergent to a certain element $x^* \in \mathbb{E}$. Since the $Graph(T)$ is closed, we immediately get that $x^* \in T(x^*)$. Thus $x^* \in F_T$.

We will prove now that $F_T \subset (SF)_T$. For this purpose, let us consider $x = y := x^* \in F_T$ in the δ-contraction condition. Then we obtain:

$$\begin{aligned}\delta_\alpha(T(x^*)) &:= \delta_\alpha(T(x^*), T(x^*)) \\ &\leq b_\alpha \delta_\alpha(x^*, T(x^*)) + c_\alpha \delta_\alpha(x^*, T(x^*)) \\ &= (b_\alpha + c_\alpha)\delta_\alpha(T(x^*)),\end{aligned}$$

for each $\alpha \in \Lambda$. Thus, for each $\alpha \in \Lambda$, we get that $\delta_\alpha(T(x^*)) = 0$, which means that $T(x^*)$ is a singleton. Since $x^* \in T(x^*)$, we obtain that $x^* \in (SF)_T$. Thus $F_T \subset (SF)_T$.

For the uniqueness of the strict fixed point, let $y \in (SF)_T$, $y \neq x^*$ be another strict fixed point. Then, we have:

$$\begin{gathered}d_\alpha(x^*, y) = \delta_\alpha(T(x^*), T(y)) \\ \leq a_\alpha d_\alpha(x^*, y) + b_\alpha \delta_\alpha(x^*, T(x^*)) + c_\alpha \delta_\alpha(y, T(y)) = a_\alpha d_\alpha(x^*, y),\end{gathered}$$

which represents a contradiction. Thus $F_T = (SF)_T = \{x^*\}$. The proof is now complete. □

Let us present now a concept of well-posedness for the fixed point problem in a complete gauge space.

Definition 2.3. Let $\mathbb{E} = (E, \{d_\alpha\}_{\alpha\in\Lambda})$ be a gauge space and $T : \mathbb{E} \to P_b(\mathbb{E})$ be a multivalued operator. We say that the fixed point problem is well-posed with respect to $(H_\alpha)_{\alpha\in\Lambda}$, if:

(a) $(SF)_T = \{x^*\}$;

(b) if $(x_n)_{n\in\mathbb{N}}$ is in $\mathbb{E}$ such that $H_\alpha(x_n, T(x_n)) \to 0$ as $n \to \infty$, for each $\alpha \in \Lambda$, imply that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x^* as $n \to \infty$.

This concept is important with respect to the approximation of the strict fixed point of a multivalued operator.

We have the following result.

Theorem 2.4. *Let $\mathbb{E} := (E, \{d_\alpha\}_{\alpha\in\Lambda})$ be a complete gauge space and $T : \mathbb{E} \to P_b(\mathbb{E})$ be a closed admissible multivalued δ-contraction of Reich type. Then, the fixed point problem is well-posed with respect to $(H_\alpha)_{\alpha\in\Lambda}$.*

Proof. By Theorem 2.2 we have that $(SF)_T = \{x^*\}$. Let us show that assertion (b) in Definition 2.3 holds. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{E}$ such that $H_\alpha(x_n, T(x_n)) \to 0$ as $n \to \infty$, for each $\alpha \in \Lambda$. Then, for each $\alpha \in \Lambda$, we have:

$$\begin{aligned}d_\alpha(x_n, x^*) &\leq \delta_\alpha(x_n, T(x_n)) + \delta_\alpha(T(x_n), x^*) = \delta_\alpha(x_n, T(x_n)) + \delta_\alpha(T(x_n), T(x^*)) \\ &\leq \delta_\alpha(x_n, T(x_n)) + a_\alpha d_\alpha(x_n, x^*) + b_\alpha \delta_\alpha(x_n, T(x_n)) + c_\alpha \delta_\alpha(x^*, T(x^*)) \\ &= (1 + b_\alpha) \cdot \delta_\alpha(x_n, T(x_n)) + a_\alpha d_\alpha(x_n, x^*).\end{aligned}$$

Thus, for each $\alpha \in \Lambda$, we get that:

$$d_\alpha(x_n, x^*) \leq \frac{1 + b_\alpha}{1 - a_\alpha} \cdot \delta_\alpha(x_n, T(x_n)) = H_\alpha(x_n, T(x_n)) \to 0 \text{ as } n \to +\infty.$$

Hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x^* as $n \to \infty$. □

We will present now a data dependence result for the strict fixed point of a closed admissible multivalued δ-contraction of Reich type.

Theorem 2.5. *Let* $\mathbb{E} := (E, \{d_\alpha\}_{\alpha\in\Lambda})$ *be a complete gauge space,* $T_1 : \mathbb{E} \to P_b(\mathbb{E})$ *be a closed admissible multivalued* δ*-contraction of Reich type, with coefficients* (a, b, c) *and* $T_2 : \mathbb{E} \to P_b(\mathbb{E})$ *be a multivalued operator with the property that* $(SF)_{T_2} \neq \emptyset$*. Suppose there exists* $\eta := (\eta_\alpha)_{\alpha\in\Lambda}$ *such that* $\delta_\alpha(T_1(x), T_2(x)) \leq \eta_\alpha$*, for each* $x \in \mathbb{E}$ *and* $\alpha \in \Lambda$*. Then, the following conclusions hold:*

(i) $F_{T_1} = (SF)_{T_1} = \{x_1^*\}$*;*

(ii) if $x_2^* \in (SF)_{T_2}$*, then* $d_\alpha(x_1^*, x_2^*) \leq \frac{1+c_\alpha}{1-a_\alpha} \cdot \eta_\alpha$*, for each* $\alpha \in \Lambda$*.*

Proof. The assertion (i) follows by Theorem 2.2. let us show that (ii) holds too. For arbitrary $x_2^* \in (SF)_{T_2}$ and for each $\alpha \in \Lambda$, we have:

$$d_\alpha(x_1^*, x_2^*) \leq \delta_\alpha(T_1(x_1^*), T_2(x_2^*)) \leq \delta_\alpha(T_1(x_1^*), T_1(x_2^*)) + \delta_\alpha(T_1(x_2^*), T_2(x_2^*))$$

$$\leq a_\alpha d_\alpha(x_1^*, x_2^*) + b_\alpha \delta_\alpha(x_1^*, T_1(x_1^*)) + c_\alpha \delta_\alpha(x_2^*, T_1(x_2^*)) + \eta_\alpha$$

$$= a_\alpha d_\alpha(x_1^*, x_2^*) + c_\alpha \delta_\alpha(x_2^*, T_1(x_2^*)) + \eta_\alpha \leq a_\alpha d_\alpha(x_1^*, x_2^*) + c_\alpha \delta_\alpha(T_2(x_2^*), T_1(x_2^*)) + \eta_\alpha$$

$$= a_\alpha d_\alpha(x_1^*, x_2^*) + (1 + c_\alpha)\eta_\alpha.$$

Hence, we get that

$$d_\alpha(x_1^*, x_2^*) \leq \frac{1 + c_\alpha}{1 - a_\alpha} \cdot \eta_\alpha, \text{ for each } \alpha \in \Lambda. \ \square$$

The existence of the strict fixed point (without uniqueness) can be obtained by the following contractive type condition.

Theorem 2.6. *Let* $\mathbb{E} := (E, \{d_\alpha\}_{\alpha\in\Lambda})$ *be a complete gauge space and* $T : \mathbb{E} \to P_b(\mathbb{E})$ *be a multivalued operator. Suppose that:*

(i) there exist $a := \{a_\alpha\}_{\alpha\in\Lambda}, b := \{b_\alpha\}_{\alpha\in\Lambda} \in]0, +\infty[^\Lambda$ *such that, for each* $\alpha \in \Lambda$*,* $a_\alpha + b_\alpha < 1$ *and for each* $x \in \mathbb{E}$ *there exists* $y \in T(x)$ *with the property*

$$\delta_\alpha(y, T(y)) \leq a_\alpha d_\alpha(x, y) + b_\alpha \delta_\alpha(x, T(x));$$

(ii) for every $x \in \mathbb{E}$ *and every* $\{q_\alpha\}_{\alpha\in\Lambda} \in]1, +\infty[^\Lambda$ *there exists* $y \in T(x)$ *such that* $\delta_\alpha(x, T(x)) \leq q_\alpha d_\alpha(x, y)$*, for each* $\alpha \in \Lambda$*;*

(iii) the functional $f_\alpha : \mathbb{E} \to \mathbb{R}$ *given by* $f_\alpha(x) = \delta_\alpha(x, T(x))$ *is lower semicontinuous for each* $\alpha \in \Lambda$*.*

Then, $(SF)_T \neq \emptyset$*.*

Proof. Let $x_0 \in \mathbb{E}$ be arbitrary and $\{q_\alpha\}_{\alpha\in\Lambda} \in]1, +\infty[^\Lambda$. Then, there exists $x_1 \in T(x_0)$ such that $\delta_\alpha(x_0, T(x_0)) \leq q_\alpha d_\alpha(x_0, x_1)$, for each $\alpha \in \Lambda$. Hence, by (i), we get that $\delta_\alpha(x_1, T(x_1)) \leq a_\alpha d_\alpha(x_0, x_1) + b_\alpha \delta_\alpha(x_0, T(x_0)) \leq (a_\alpha + b_\alpha q_\alpha) \cdot d_\alpha(x_0, x_1)$. Similarly, there exists $x_2 \in T(x_1)$ such that $\delta_\alpha(x_1, T(x_1)) \leq q_\alpha d_\alpha(x_1, x_2)$, for each $\alpha \in \Lambda$. Thus, $d_\alpha(x_1, x_2) \leq \delta_\alpha(x_1, T(x_1)) \leq (a_\alpha + b_\alpha q_\alpha) \cdot d_\alpha(x_0, x_1)$. By induction, we get a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{E}$ having, for each $\alpha \in \Lambda$, the following properties:

(i) $(x_0, x_1) \in Graph(T)$ is arbitrary and $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

(ii) $\delta_\alpha(x_n, T(x_n)) \leq (a_\alpha + b_\alpha q_\alpha)^n \cdot d_\alpha(x_0, x_1)$, for each $n \in \mathbb{N}$;

(iii) $d_\alpha(x_n, x_{n+1}) \leq (a_\alpha + b_\alpha q_\alpha)^n \cdot d_\alpha(x_0, x_1)$, for each $n \in \mathbb{N}$.

Let us choose $1 < q_\alpha < \frac{1-a_\alpha}{b_\alpha}$. Then $p_\alpha := a_\alpha + b_\alpha q_\alpha \in [0, 1[$, for each $\alpha \in \Lambda$.

By a similar procedure as in the proof of Theorem 2.2, we immediately obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{E}$ and hence it is convergent to a certain element $x^* \in \mathbb{E}$.

Now, by the lower semicontinuity of the functional f_α we get that

$$0 \leq f_\alpha(x^*) \leq \liminf_{n\to+\infty} f_\alpha(x_n) = 0.$$

Hence $f_\alpha(x^*) = 0$ and thus $x^* \in (SF)_T$. $\square$

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