Fixed Point Theory, 11(2010), No. 1, 113-118
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

COMMON FIXED POINTS OF CONDITIONALLY COMMUTING MAPS

VYOMESH PANT* AND R.P. PANT**

*Corresponding Author
A-24, J.K. Puram, Choti Mukhani
Haldwani - 263 139, Nainital Uttarakhand, India
E-mail: vyomeshpant@yahoo.co.in

** Department of Mathematics, Kumaon University
D.S.B. Campus, Nainital - 263 002,
Uttarakhand, India
E-mail: pant_rp@rediffmail.com

Abstract. In the instant paper, redefining the concept of occasionally weakly compatible mappings, given by Jungck and Rhoades [3], we introduce the notion of conditionally commuting mappings. We first prove a common fixed point theorem (Theorem 1) for a pair of noncompatible mappings without assuming completeness of the space or continuity of the mappings involved. This theorem extends the result of Pant [7]. In Theorem 2, we further generalize the result obtained in Theorem 1. For this purpose we use the property (E.A) given by Aamri and Moutawakil [1].
Key Words and Phrases: Conditionally commuting maps, noncompatible maps, property (E.A) and common fixed points.
2000 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In 1982, Seesa [10] introduced the notion of weakly commuting maps. If (X, d) be a metric space, two selfmaps f and g of X are called weakly commuting provided $d(fgx, gfx) \leq d(fx, gx)$ for each x in X. The mappings f and g are said to be weakly commuting at a point z in X if $d(fgz, gfz) \leq d(fz, gz)$. In 1986, Jungck [2] introduced the notion of compatibility (also called asymptotic commutativity by Tivari and Singh [11] in an independent work), which is more general than weak commutativity. Two selfmaps f and g of a metric space (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. Right since the introduction of compatibility, the study of common fixed points is developed around the compatible mappings and its weaker forms and it has become an area of vigorous research activity. Albeit, the study of noncompatible maps is equally interesting and various fruitful results have been obtained using the aspect of noncompatibility (e.g. [5], [6], [8]).

It is clear from the definition of compatible maps *ibid*, that f and g will be non-compatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either non-zero or non-existent.

It is of worth to mention here that in the study of common fixed points of contractive type mapping pairs we often require assumptions on continuity of the mappings involved and the completeness of the space assumed. However, the study of fixed points of noncompatible mappings can be extended to the class of Lipschitz type mapping pairs even without assuming continuity and completeness.

In 1994, Pant [4] has further generalized the notion of weakly commuting maps and introduced the notion of R- weakly commuting mappings.

Two selfmappings f and g of a metric space X are called R-weakly commuting at a point x in X if $d(fgx, gfx) \leq R\ d(fx, gx)$ for some $R > 0$. The maps f and g are called R-weakly commuting on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \leq R\ d(fx, gx)$.

From the above definition it is obvious that f and g can fail to be pointwise R-weakly commuting only if there exists some x in X such that $fx = gx$ but $fgx \neq gfx$, that is, only if they possess a coincident point at which they do not commute. It may be observed that compatibility implies pointwise R-weak commutativity since compatible maps commute at their coincidence points. The converse, however, is not true (see e.g. in [8]).

Recently, Pathak *et al* [9] have generalized results for bounded convex sets or asymptotically compact sets under some asymptotic I-contraction assumptions. In a relatively recent work Jungck and Rhoades [3] have generalized the notion of weakly compatible maps by introducing the concept of occasionally weakly compatible maps in the study of symmetric spaces.

Two selfmappings f and g of a metric space (X, d) are occasionally weakly compatible [3] iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

It may be observed that the aspects occasionally weak compatibility, as defined by Jungck and Rhoades [3], and noncompatibility are independent from each other. There exist maps which are noncompatible but satisfy the above property, defined as occasionally weak compatibility by Jungck and Rhoades [3]. The following examples illustrate the fact.

Example 1.1. Let $X = [2, 20]$. Define $f,\ g : X \to X$ as follows:

$$fx = 2, \text{ if } x = 2 \text{ or } > 5, \qquad fx = 6, \text{ if } 2 < x \leq 5,$$
$$g2 = 2, \qquad gx = 2x, \text{ if } 2 < x \leq 5, \qquad gx = \frac{x+1}{3}, \text{ if } x > 5.$$

It may be verified in this example that $f2 = 2 = g2$ and $f3 = 6 = g3$. Hence 2 and 3 are coincidence points of f and g. Further, $fg2 = 2 = gf2$. However, $fg3 = f6 = 2, gf3 = g6 = \frac{7}{3}$ and, therefore, $fg3 \neq gf3$. f and g are occasionally weakly compatible, by Jungck and Rhoades [3], as they commute at one coincidence point, that is, 2. It may be seen that f and g are noncompatible. To see that f and g are noncompatible, consider the sequence $\{5 + \frac{1}{n} : n \geq 1\}$ in X. Then $\lim_n fx_n = 2$, $\lim_n gx_n = 2$, $\lim_n fgx_n = 6$, $\lim_n gfx_n = 2$. Hence f and g are noncompatible.

We now give two more examples:

Example 1.2. Let $X = [2, 20]$. Define $f,\ g : X \to X$ as follows:

$$fx = 6, \text{ if } 2 \leq x < 5, \qquad fx = 2, \text{ if } x \geq 5,$$
$$gx = 2, \text{ if } 2 \leq x < 5, \qquad gx = x - 3, \text{ if } x \geq 5.$$

In this example $f5 = 2 = g5$, but $fg5 = 6$, $gf5 = 2$, $fg5 \neq gf5$. Thus f and g do not commute at their only coincidence point $x = 5$. Thus f and g are not occasionally weakly compatible. Let us now consider a sequence $\{5 + \frac{1}{n} : n \geq 1\}$ in X. Then $\lim_n fx_n = 2$, $\lim_n gx_n = 2$, $\lim_n fgx_n = 6$, $\lim_n gfx_n = 2$. Hence f and g are noncompatible.

Example 1.3. Let $X = [2, 20]$. Define $f,\ g : X \to X$ as follows:

$$f2 = 2, \qquad fx = 2x + 1, \text{ if } 2 < x \leq 5, \qquad fx = \tfrac{x-1}{2}, \text{ if } x > 5,$$
$$g2 = 2, \qquad gx = \tfrac{x+8}{2}, \text{ if } 2 < x < 5, \qquad gx = x - 3, \text{ if } x \geq 5.$$

In this example $f2 = 2 = g2$ and $f5 = 2 = g5$. Hence 2 and 5 are coincidence points of f and g. Further, $fg2 = 2 = gf2$ and $fg5 = 2 = gf5$. Thus f and g are occasionally weakly compatible, by Jungck and Rhoades [3]. It may be seen that f and g are compatible maps. Further, f and g are R-weakly commuting maps also as they commute at both the coincidence points.

From the examples illustrated above, it may be observed that the notions of occasionally weakly compatibility and noncompatibility are independent to each other. There are maps which are noncompatible but satisfy the property defined as occasionally weak compatibility by Jungck and Rhoades [3]. In Example 1, above, f and g are noncompatible. Simultaneously, they are occasionally weakly compatible, as defined by Jungck and Rhoades [3]. Thus there appears some degree of ambiguity regarding terminology, as the maps are in one way called occasionally weakly compatible and simultaneously they are noncompatible. To rule out the possibility of ambiguity we define the occasionally weakly compatibility in slight different manner as follows.

Definition 1.1. Two selfmappings f and g of a metric space $(X,\ d)$ are called conditionally commuting if they commute on a nonempty subset of the set of coincidence points whenever the set of their coincidences is nonempty.

For illustration we refer Examples 1, 2 and 3 above. From the definition itself it is clear that if two maps are R-weakly commuting they are necessarily conditionally commuting, however, the conditionally commuting mappings are not necessarily R-weakly commuting. Mappings in Example 1 above illustrates the argument as f and g are conditionally commuting but not R-weakly commuting.

Recently Aamri and Moutawakil [1] introduced the property (E.A) and thus generalized the concept of noncompatible maps. The results obtained in the metric fixed point theory by using the notion of noncompatible maps or the property (E.A) are very interesting.

Definition 1.2. Let f and g be two selfmappings of a metric space $(X,\ d)$. We say that f and g satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \text{ for some } t \in X.$$

Motivated by the work of Jungck and Rhoades [3] and using the conditionally commuting property, as defined above, in the present paper we first prove a common fixed point theorem for a pair of noncompatible mappings without assuming completeness of the space or continuity of the mappings involved. Theorem 1 can be considered as a common fixed theorem for Lipschitz type mapping pairs under minimal commutativity condition. For further generalization of this result we assume the property (E.A) in place of noncompatibility and prove Theorem 2. Our results generalize the results of Pant [7].

Theorem 1.1. Let f and g be conditionally commuting noncompatible selfmappings of a metric space $(X,\ d)$ satisfying

(i) $\overline{fX} \subset gX$, where $\overline{fX}$denotes the closure of range of f.

(ii) $d(fx, fy) \leq k\ d(gx, gy), k \geq 0$, and

(iii) $d(fx, f^2x) \neq max\{d(fx, gfx), d(f^2x, gfx)\}$, *whenever* $fx \neq f^2x$.

Then f and g have a common fixed point.

Proof. Since f and g are noncompatible, there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $gx_n \to t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent. Then, since $t \in \overline{fX}$ and $\overline{fX} \in gX$ there exists u in X such that $t = gu$. By (ii) we now get

$$d(fx_n, fu) \leq k\ d(gx_n, gu).$$

On letting $t \to \infty$, we get, $fu = gu$. Thus u is a coincidence point of f and g and, thus, the set of coincidence points of f and g is non empty. Since f and g are conditionally commuting, two cases arise - f and g may or may not commute at u. If f and g commute at u, $fgu = gfu$. Also, $ffu = fgu = gfu = ggu$. We claim that $ffu = fu$. If not, by virtue of (iii) we get

$$d(fu, ffu) \neq max\{d(fu, gfu), d(ffu, gfu)\} = d(fu, ffu),$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. If f and g do not commute at u, then by virtue of conditional commutativity of f and g, there exists a coincidence point of f and g at which f and g commute, that is, there exists a point v in X such that $fv = gv$ and $fgv = gfv$. Rest of the proof can be completed on the similar lines as has been done in the case when f and g commute at u. This completes the proof of the theorem. □

We now give an example to illustrate the above theorem.

Example 1.4. Let $X = [0, 1]$ and d be the usual metric on X. Define $f,\ g : X \to X$ as follows:

$$fx = \left(\frac{\sqrt{5 - 4(2x-1)^2 - 1}}{4} \right),$$

$$gx = \frac{1-x}{3}.$$

Then f and g satisfy all the conditions of the above theorem and have a common fixed point $x = \frac{1}{4}$. It may be verified in this example that $fX = [0, \frac{\sqrt{5}-1}{4}]$, $gX = [0, \frac{1}{3}]$ and $\overline{fX} \in gX$. f and g are noncompatible conditionally commuting mappings. To see that f and g are noncompatible, let us consider the sequence $\{x_n\}$ given by $\{x_n = 1 - \frac{1}{n}, n \geq 1\}$. Then $fx_n \to 0,\ gx_n \to 0,\ fgx_n \to 0$ and $gfx_n \to \frac{1}{3}$. Hence f and g are noncompatible. It may also be seen that $f(\frac{1}{4}) = \frac{1}{4} = g(\frac{1}{4})$ and $f(1) = 0 = g(1)$. As such, $\frac{1}{4}$ and 1 are two coincidence points of f and g. Further, $fg(\frac{1}{4}) = \frac{1}{4} = gf(\frac{1}{4}), fg(1) = f(0) = 0, gf(1) = g(0) = \frac{1}{3}$, and, therefore, $fg1 \neq gf1$. Thus f and g are conditionally commuting mappings. f and g are not R-weakly commuting as they do not commute at their coincidence point $x = 1$. It may be verified that f and g satisfy the Lipschitz type condition

$$d(fx, fy) \leq 6d(gx, gy),$$

together with the condition

$$d(fx, f^2x) < max\{d(fx, gfx), d(f^2x, gfx)\}.$$

Theorem 1 can be generalized further if we use the property (E.A) instead of the aspect of noncompatibility. We do so in our next theorem.

Theorem 1.2. Let f and g be conditionally commuting selfmappings of a metric space $(X,\ d)$ satisfying

(i) $\overline{fX} \subset gX$, where $\overline{fX}$denotes the closure of range of f.

(ii) $d(fx, fy) \leq k\ d(gx, gy), k \geq 0$, and

(iii) $d(fx, f^2x) \neq max\{d(fx, gfx), d(f^2x, gfx)\}$, *whenever* $fx \neq f^2x$.

If f and g satisfy the property (E.A.), then f and g have a common fixed point.

Proof. Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $gx_n \to t$ for some t in X. Then, since $t \in \overline{fX}$ and $\overline{fX} \in gX$ there exists u in X such that $t = gu.$ By (ii) we now get

$$d(fx_n, fu) \leq k\ d(gx_n, gu).$$

On letting $t \to \infty$, we get, $fu = gu$. Thus u is a coincidence point of f and g and, thus, the set of coincidence points of f and g is non empty. Since f and g are conditionally commuting, two cases arise - f and g may or may not commute at u. If f and g commute at u, $fgu = gfu$. Also, $ffu = fgu = gfu = ggu$. We claim that $ffu = fu$. If not, by virtue of (iii) we get

$$d(fu, ffu) \neq max\{d(fu, gfu), d(ffu, gfu)\} = d(fu, ffu),$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. If f and g do not commute at u, then by virtue of conditional commutativity of f and g, there exists a coincidence point of f and g at which f and g commute, that is, there exists a point v in X such that $fv = gv$ and $fgv = gfv$. Rest of the proof can be completed on the similar lines as has been done in the case when f and g commute at u. This completes the proof of the theorem. □

Example 4 illustrates the above theorem also.

Acknowledgement. The authors are thankful to the referee for giving valuable suggestions to improve the paper. We also extend our gratitude to Prof. B.E. Rhoades for providing reprints of his papers, especially reprint of [3].

References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270**(2002), 181-188.

[2] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9**(1986), 771-779.

[3] G. Jungck and B.E. Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory, **7-2**(2006), 287-296.

[4] R.P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl., **188**(1994), 436-440.

[5] R.P. Pant, *R-weak commutativity and common fixed points of noncompatible maps*, Ganita, **49**(1998), 19-27.

[6] R.P. Pant, *Common fixed point theorems for contractive maps*, J. Math. Anal. Appl., **226**(1998), 251-258.

[7] R.P. Pant, *Common fixed points of Lipschitz type mapping pairs*, J. Math. Anal. Appl., **240**(1999), 280-283.

[8] R.P. Pant, *Noncompatible mappings and common fixed points*, Soochow J. Math., **26**(2000), 29-35.

[9] H.K. Pathak, B.E. Rhoades and M.S. Khan, *Common fixed point theorems for asymptotically I-contractive mappings without convexity*, Fixed Point Theory, **8**(2007), no. 2, 285-296.

[10] S. Sessa, *On a weak commutativity condition of mappings in fixed point consideration*, Publ. Inst. Math., **32**(1982), 149-153.

[11] B.M.L. Tivari and S.L. Singh, *A note on recent generalizations of Jungck contraction principle*, J. U. P. Govt. Colleges Acad. Soc., **3**(1986), 13-18.

Received: 18.12.2008; Accepted: 13.04.2009.