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FIXED POINTS FOR COMPACT PERMISSIBLE MAPS IN EXTENSION TYPE SPACES

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Abstract. Several new fixed point results for compact permissible self maps in extension type spaces are presented in this paper.

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1. Introduction

In Section 2 we present new results on fixed point theory for permissible maps in extension type spaces, in particular in metric NES(compact metric) and metric $SANES$(compact metric) spaces. These results improve those in the literature; see [1-14] and the references therein.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : \; f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \backslash N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\, \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$.

Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism. We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q.

It is well known [9, pp 53] that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$.

Let X and Y be Hausdorff topological spaces.

Definition 1.1. A multivalued map $F : X \to K(Y)$ ($K(Y)$ denotes the class of nonempty compact subsets of Y) is in the class $\mathcal{A}_m(X,Y)$ if (i). F is continuous, and (ii). for each $x \in X$ the set $F(x)$ consists of one or m acyclic components; here m is a positive integer. We say F is of class $\mathcal{A}_0(X,Y)$ if F is upper semicontinuous and for each $x \in X$ the set $F(x)$ is acyclic.

Definition 1.2. A decomposition $(F_1, ..., F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \overset{F_1}{\to} X_1 \overset{F_2}{\to} X_2 \overset{F_3}{\to} \;\; ..... \;\; \overset{F_{n-1}}{\to} X_{n-1} \overset{F_n}{\to} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ .... \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, ..., F_n)$. The number n is said to be the length of the decomposition $(F_1, ..., F_n)$. We will denote the class of decompositions by $\mathcal{D}(X,Y)$.

Definition 1.3. An upper semicontinuous map $F : X \to K(Y)$ is permissible provided it admits a selector $G : X \to K(Y)$ which is determined by a decomposition $(G_1, ..., G_n) \in \mathcal{D}(X,Y)$. We denote the class of permissible maps from X into Y by $\mathcal{P}(X,Y)$.

Let X be a Hausdorff topological space and let a map Φ be determined by $(\Phi_1, ..., \Phi_k) \in \mathcal{D}(X,X)$. Then Φ is said to be a Lefschetz map if the induced homology homomorphism [7, pp 27] $(\Phi_1, ..., \Phi_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [7]) $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\mathbf{\Lambda}(\Phi) = \Lambda((\Phi_1, ..., \Phi_k)_\star).$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class $\mathcal{D}$) provided every compact $\Phi : X \to K(X)$ determined by a decomposition $(\Phi_1, ..., \Phi_k) \in \mathcal{D}(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies Φ has a fixed point.

Example 1.1. If X is a metric ANR then X is a Lefschetz space (for the class $\mathcal{D}$) (see [7, pp 42] or [9, Section 50-53]).

A map $\Phi \in \mathcal{P}(X,X)$ is said to be a Lefschetz map provided every selector $G : X \to K(X)$ of Φ which is determined by $(G_1, ..., G_k) \in \mathcal{D}(X,X)$ is such that $(G_1, ..., G_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi \in \mathcal{P}(X,X)$ is a Lefschetz map as described above then we define the Lefschetz set $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\begin{aligned}\mathbf{\Lambda}(\Phi) \;=\; & \{\Lambda((G_1,...,G_k)_\star) : \ (G_1,...,G_k) \in \mathcal{D}(X,X) \\ & \quad \text{and } \ (G_1,....,G_k) \ \text{ determines a selection of } \ \Phi\}.\end{aligned}$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class $\mathcal{P}$) provided every compact $\Phi \in \mathcal{P}(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies Φ has a fixed point.

Example 1.2. If X is a metric ANR then X is a Lefschetz space (for the class $\mathcal{P}$) (see [7, pp 43]).

We now recall some ideas and results which we will use in Section 2. Recall the Hilbert cube I^∞ is the subset of l^2 consisting of points $(x_1,x_2,...)$ with $|x_i| \leq \frac{1}{2^i}$ for all i. It is well known (see [12] and the references therein) that any compact metric space admits an embedding into the Hilbert cube I^∞. Also we know [9] that I^∞ is an AR (so an ANR).

For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α-fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov(Y)$, f and g are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \leq t \leq 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0,1]$.

The following results can be found in [4, Lemma 1.2 and 4.7].

Theorem 1.1. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α-fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.1. From Theorem 1.1 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [5 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity so such refinements form a cofinal family of open covers). Note also uniform spaces are regular (in fact completely regular). Note in Theorem 1.1 if F is compact valued then the assumption that X is regular can be removed. In this paper we will apply Theorem 1.1 when the space is metric.

2. Fixed Point Theory

In this section we present two new results which guarantee that the space X is a Lefschetz space (for the class $\mathcal{P}$). Of course these results can trivially be adjusted to guarantee that our spaces X below are Lefschetz space (for the class $\mathcal{D}$).

By a space we mean a Hausdorff topological space. Let X and Y be spaces. A space Y is an neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q$,

$\forall K \subseteq X$ closed in X, and for any continuous function $f_0 : K \to Y$, there exists a continuous extension $f : U \to Y$ of f_0 over a neighbourhood U of K in X.

Let X be a metric space, $X \in NES$(compact metric) and $F \in \mathcal{P}(X,X)$ a compact map (from [7] we know that we might as well assume that X is not a metric ANR since we know in this situation that metric ANR's are Lefschetz spaces (for the class $\mathcal{P}$)). Now let $K = \overline{F(X)}$. We know [12] that K can be embedded as a closed subset $K^\star$ of I^∞; let $s : K \to K^\star$ be a homeomorphism. Also let $i : K \hookrightarrow X$ be an inclusion. Let U be an open neighbourhood of $K^\star$ in I^∞ and $h_U : U \to X$ be a continuous extension of $i\, s^{-1} : K^\star \to X$ on U (guaranteed since $X \in NES$(compact metric)). Let $j_U : K^\star \hookrightarrow U$ be the natural embedding so $h_U\, j_U = i\, s^{-1}$. Finally let $J = j_U\, s\, F\, h_U$. Notice $J \in \mathcal{P}(U,U)$.

We now assume the following condition:

$$(2.1)\qquad \begin{cases} \text{for every selector } G : X \to K(X) \text{ of } F \text{ which is} \\ \text{determined by } (G_1,...,G_n) \in \mathcal{D}(X,X) \text{ there exists} \\ \text{a } (H_1,...,H_k) \in \mathcal{D}(U,X) \text{ which determines } F\, h_U \text{ with} \\ (H_1,...,H_k)_\star = (G_1,...,G_n)_\star\, (h_U)_\star \text{ and} \\ (j_U)_\star\, (s)_\star\, (H_1,...,H_k)_\star = (H_1,...,H_k,s,j_U)_\star \\ \text{(note automatically } (H_1,...,H_k,s,j_U) \in \mathcal{D}(U,U)). \end{cases}$$

Let $G : X \to K(X)$ be a selector of F which is determined by $(G_1,...,G_n) \in \mathcal{D}(X,X)$. Then by (2.1) there exists $(H_1,...,H_k) \in \mathcal{D}(U,X)$ which determines $F\, h_U$ with

$$(H_1,...,H_k)_\star = (G_1,...,G_n)_\star\, (h_U)_\star$$

and

$$(j_U)_\star\, (s)_\star\, (H_1,...,H_k)_\star = (H_1,...,H_k,s,j_U)_\star.$$

Note

$$(H_1,...,H_k)_\star\, (j_U)_\star\, (s)_\star = (G_1,...,G_n)_\star\, (h_U)_\star\, (j_U)_\star\, (s)_\star = (G_1,...,G_n)_\star$$

since $h_U\, j_U\, s = i\, s^{-1}\, s$. Next note $J = j_U\, s\, F\, h_U \in \mathcal{P}(U,U)$ and in fact is determined by $(H_1,...,H_k,s,j_U) \in \mathcal{D}(U,U)$. Now U is a Lefschetz space (see Example 1.2 and note any open subset of an ANR is an ANR) and as a result $(H_1,...,H_k,s,j_U)_\star$ is a Leray endomorphism. Notice from (2.1) that

$$(j_U)_\star\, s_\star\, (H_1,...,H_k)_\star = (H_1,...,H_k,s,j_U)_\star$$

and also from above

$$(H_1,...,H_k)_\star\, (j_U)_\star\, (s)_\star = (G_1,...,G_n)_\star$$

so [8, page 214, see (1.3)] (here $E' = U'$, $E'' = X'$, $u = (H_1,...,H_k)_\star$, $v = (j_U)_\star\, s_\star$, $f' = (H_1,...,H_k,s,j_U)_\star$ and $f'' = (G_1,...,G_n)_\star$) guarantees that $(G_1,...,G_n)_\star$ is a Leray endomorphism and $\Lambda\,((G_1,...,G_n)_\star) = \Lambda\,((H_1,...,H_k,s,j_U)_\star)$. Thus $\mathbf{\Lambda}\,(F)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F) \neq \{0\}$.

Then there exists a $(G_1,...,G_n)$ (with $\Lambda\,((G_1,...,G_n)_\star) \neq 0$) and $(H_1,...,H_k)$ as described above (and of course with $\Lambda\,((H_1,...,H_k,s,j_U)_\star) \neq 0$ since we have as above $\Lambda\,((G_1,...,G_n)_\star) = \Lambda\,((H_1,...,H_k,s,j_U)_\star)$). Now since U is a Lefschetz space there exists $x \in U$ with $x \in J\, x$. Let $y = h_U(x)$, so $y \in h_U\, j_U\, s\, F\,(y)$ i.e.

$y = h_U\, j_U\, s\,(q)$ for some $q \in F\,(y)$. Since $h_U\, j_U\,(z) = i\, s^{-1}(z)$ for $z \in K^\star$, we have $h_U\, j_U\, s\,(q) = i\,(q)$, so $y \in F(y)$.

As a result we have the following result.

Theorem 2.1. *Let X be a metric space, $X \in NES($compact metric$)$ and $F \in \mathcal{P}(X,X)$ a compact map. Also assume* (2.1) *holds with K, $K^\star$, U, s, i, j_U and h_U as described above. Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall\, \alpha \in Cov\,(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α close and α-homotopic.

Let X be a metric space, $X \in SANES($compact metric$)$ and $F \in \mathcal{P}(X,X)$ a compact map. Also let $\alpha \in Cov_X\,(K)$ where $K = \overline{F(X)}$. To show F has a fixed point it suffices (Theorem 1.1 with Remark 1.1) to show F has an α-fixed point. Let $\alpha' = \alpha \cup \{X \setminus K\}$ and let $K^\star$, s and i be as above. Since $X \in SANES($compact metric$)$ there exists an open neighborhood U_α of $K^\star$ in I^∞ and $f_\alpha : U_\alpha \to X$ a continuous function such that $f_\alpha|_{K^\star}$ and s^{-1} are α'-close and α'-homotopic and as a result $f_\alpha\, j_{U_\alpha}\, s : K \to X$ and $i : K \to X$ are α-close and α-homotopic; here $j_{U_\alpha} : K^\star \hookrightarrow U_\alpha$ is the natural imbedding. Finally let $J_\alpha = j_{U_\alpha}\, s\, F\, f_\alpha$. Notice $J_\alpha \in \mathcal{P}(U_\alpha, U_\alpha)$.

We now assume the following condition for each $\alpha \in Cov_X\,(\overline{F(X)})$:

$$(2.2)\qquad \begin{cases} \text{for every selector } G : X \to K(X) \text{ of } F \text{ which is} \\ \text{determined by } (G_1, ..., G_n) \in \mathcal{D}(X,X) \text{ there exists} \\ \text{a } (H_{1,\alpha}, ..., H_{k,\alpha}) \in \mathcal{D}(U_\alpha, X) \text{ which determines } F\, f_\alpha \text{ with} \\ (H_{1,\alpha}, ..., H_{k,\alpha})_\star = (G_1, ..., G_n)_\star\,(f_\alpha)_\star \text{ and} \\ (j_{U_\alpha})_\star\,(s)_\star\,(H_{1,\alpha}, ..., H_{k,\alpha})_\star = (H_{1,\alpha}, ..., H_{k,\alpha}, s, j_{U_\alpha})_\star \\ \text{(note automatically } (H_{1,\alpha}, ..., H_{k,\alpha}, s, j_{U_\alpha}) \in \mathcal{D}(U_\alpha, U_\alpha)). \end{cases}$$

Fix $\alpha \in Cov_X\,(\overline{F(X)})$. Let $G : X \to K(X)$ be a selector of F which is determined by $(G_1, ..., G_n) \in \mathcal{D}(X,X)$. Then (2.2) guarantees that there exists $(H_{1,\alpha}, ..., H_{k,\alpha}) \in \mathcal{D}(U_\alpha, X)$ which determines $F\, f_\alpha$ with

$$(H_{1,\alpha}, ..., H_{k,\alpha})_\star = (G_1, ..., G_n)_\star\,(f_\alpha)_\star$$

and

$$(j_{U_\alpha})_\star\,(s)_\star\,(H_{1,\alpha}, ..., H_{k,\alpha})_\star = (H_{1,\alpha}, ..., H_{k,\alpha}, s, j_{U_\alpha})_\star.$$

Note

$$(H_{1,\alpha}, ..., H_{k,\alpha})_\star\,(j_{U_\alpha})_\star\,(s)_\star = (G_1, ..., G_n)_\star\,(f_\alpha)_\star\,(j_{U_\alpha})_\star\,(s)_\star = (G_1, ..., G_n)_\star$$

since $f_\alpha\, j_{U_\alpha}\, s$ is α homotopic to i (note $f_\alpha|_{K^\star}$ and s^{-1} are α'-homotopic by definition). Next note $J_\alpha = j_{U_\alpha}\, s\, F\, f_\alpha \in \mathcal{P}(U_\alpha, U_\alpha)$ and in fact is determined by $(H_{1,\alpha}, ..., H_{k,\alpha}, s, j_{U_\alpha}) \in \mathcal{D}(U_\alpha, U_\alpha)$. Now U_α is a Lefschetz space (see Example 1.2 and note any open subset of an ANR is an ANR) and as a result $(H_{1,\alpha}, ..., H_{k,\alpha}, s, j_{U_\alpha})_\star$ is a Leray endomorphism. Notice from (2.2) that $(j_{U_\alpha})_\star\,(s)_\star\,(H_{1,\alpha}, ..., H_{k,\alpha})_\star = (H_{1,\alpha}, ..., H_{k,\alpha}, s, j_{U_\alpha})_\star$ and also from above

$(H_{1,\alpha},...,H_{k,\alpha})_\star\,(j_{U_\alpha})_\star\,(s)_\star = (G_1,...,G_n)_\star$ so [8, page 214, see (1.3)] guarantees that $(G_1,...,G_n)_\star$ is a Leray endomorphism and we have $\Lambda\left((G_1,...,G_n)_\star\right) = \Lambda\left((H_{1,\alpha},...,H_{k,\alpha},s,j_{U_\alpha})_\star\right)$. Thus $\mathbf{\Lambda}\,(F)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F) \neq \{0\}$. Fix $\alpha \in Cov_X\,(\overline{F(X)})$. Then there exists a $(G_1,...,G_n)$ (with $\Lambda\left((G_1,...,G_n)_\star\right) \neq 0$) and $((H_{1,\alpha},...,H_{k,\alpha})$ as described above (and with $\Lambda\left((H_{1,\alpha},...,H_{k,\alpha},s,j_{U_\alpha})_\star\right) \neq 0$ since we have as above $\Lambda\left((G_1,...,G_n)_\star\right) = \Lambda\left((H_{1,\alpha},...,H_{k,\alpha},s,j_{U_\alpha})_\star\right)$). Now since U_α is a Lefschetz space there exists $x \in U$ with $x \in J_\alpha\, x$. Now let $y = f_\alpha(x)$ so $y \in f_\alpha\, j_{U_\alpha}\, s\, F(y)$ i.e. $y = f_\alpha\, j_{U_\alpha}\, s(q)$ for some $q \in F(y)$. Now since $f_\alpha\, j_{U_\alpha}\, s$ and i are α-close there exists $U \in \alpha$ with $f_\alpha\, j_{U_\alpha}\, s(q) \in U$ and $i(q) \in U$ i.e. $q \in U$ and $y = f_\alpha\, j_{U_\alpha}\, s(q) \in U$. Thus $q \in U$ and $y \in U$ so

$$y \in U \quad \text{and} \quad F(y) \cap U \neq \emptyset \quad \text{since} \quad q \in F\,(y).$$

As a result we have the following result.

Theorem 2.2. *Let X be a metric space, $X \in SANES($compact metric$)$ and $F \in \mathcal{P}(X,X)$ a compact map. Also assume* (2.2) *holds with K, $K^\star$, U_α, s, j_{U_α}, i and f_α as described above. Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

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