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AN INTEGRAL EQUATION VIA WEAKLY PICARD OPERATORS

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Abstract. In this paper we shall study the existence and uniqueness and data dependence: continuity, monotony, differentiability with respect to parameter for the solutions of the following integral equation

$$x(t) = [g_1(t) + \int_a^t K_1(t, s, x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t, s, x(s))ds],\ t \in [a, b].$$

Our results are in connection with some results by E. Brestovanska (Qualitative behaviour of an integral equation related to some epidemic model, Demonstratio Mathematica, **26**(2003), no. 3, 603-609.)

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1. Introduction

G . Griepenberg studied in the paper [2] the qualitative behavior of solutions of the equation

$$x(t) = k[p(t) - \int_0^t A(t-s)x(s)ds] \cdot [f(t) - \int_0^t a(t-s)x(s)ds], \tag{1.1}$$

which arises in the study of the spread of an infectious disease that does not induce the permanent immunity. In the paper [1], Eva Brestovanska obtained the existence result and convergence result for the solution of integral equation

$$x(t) = [g_1(t) + \int_0^t A_1(t-s)F_1(s, x(s))ds] \cdots [g_p(t) + \int_0^t A_p(t-s)F_p(s, x(s))ds] \tag{1.2}$$

More precisely, in the conditions:

(E_1) $g_i, F_i, A_i,\ i = \overline{1,p}$ are continuous function;
(E_2) F_i is Lipchitz with respect to the second argument;

(E_3) there exists $Q > 0$ and a positive number such that

$$|F_i(t,x)| \leq Q|x|^m,$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$

plus some technical conditions she obtained that there exists a unique continuous solution of the equation (1.2) on $\mathbb{R}_+$ with $|x(t)| \leq M$, for all $t \in \mathbb{R}_+$.

In the present paper, we shall study the existence and uniqueness, data dependence: continuity, monotony and smooth dependence on parameter for the solution of following integral equation:

$$x(t) = [g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,x(s))ds], \ t \in [a,b], \tag{1.3}$$

which is a general form of (1.2). First we prove same global and local results of existence and uniqueness of the continuous solution of the equation (1.3). The study of the existence and uniqueness was made through various methods (endowment of the continuous functions set with norm type Bielecki, showing that the iterate of a certain order is a contraction map, or through of comparison functions). The next part of the paper was dedicated to study of continuous dependence of solution of the equation (1.3) with respect to dates. The paper continuous with the study of a inequality of Gronwall type associated to the (1.3). The last part of this paper is dedicated to the smooth dependence of the solution with respect to the parameter. Here I have used the fiber contraction principle.

2. Existence and uniqueness results

We consider the integral equation (1.3) and we search the solutions of the equation (1.3) in the Banach space of continuous functions $C[a,b]$ endowed with Bielecki norm

$$\|x\|_\tau = \max_{t\in[a,b]} |x(t)|e^{-\tau(t-a)}.$$

We have the following result

Theorem 2.1. *We suppose that*

(i) $g_i \in C[a,b]$, $K_i \in C([a,b] \times [a,b] \times \mathbb{R})$, $i = \overline{1,2}$;

(ii) *there exists* $L_{K_i} > 0$ *such that*

$$|K_i(t,s,u) - K_i(t,s,v)| \leq L_{K_i}|u - v|,$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $i = \overline{1,2}$;

(iii) *there exists* $M_{K_i} > 0$ *such that*

$$|K_i(t,s,u)| \leq M_{K_i},$$

for all $t,s \in [a,b]$, $u,v \in \mathbb{R}$, $i = \overline{1,2}$;

Then the equation (1.3) has, in $C([a,b])$, *a unique solution* $x^\star$.

Proof. We consider the operator $A : C[a,b] \to C[a,b]$ defined by:

$$A(x)(t) = [g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,x(s))ds].$$

For all $x, y \in C[a,b]$ we have that

$$|A(x)(t) - A(y)(t)| \leq$$

$$\leq |[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,x(s))ds] -$$

$$-[g_1(t) + \int_a^t K_1(t,s,y(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds]| =$$

$$= |[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,x(s))ds] -$$

$$-[g_1(t) + \int_a^t K_1(t,s,x(s)]ds) \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds] +$$

$$+[g_1(t) + \int_a^t K_1(t,s,x(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds] -$$

$$-[g_1(t) + \int_a^t K_1(t,s,y(s))ds] \cdot [g_2(t) + \int_a^t K_2(t,s,y(s))ds]| \leq$$

$$(\frac{L_{K_2} M(g_1, K_1)}{\tau} + \frac{L_{K_1} M(g_2, K_2)}{\tau}) \|x - y\|_\tau e^{\tau(t-a)},$$

where

$$M(g_1, K_1) := \max_{t \in [a,b]} |g_1(t)| + M_{K_1}(b-a)$$

$$M(g_2, K_2) := \max_{t \in [a,b]} |g_2(t)| + M_{K_2}(b-a)|.$$

Then

$$\|A(x) - A(y)\|_\tau \leq (\frac{L_{K_2} M(g_1, K_1)}{\tau} + \frac{L_{K_1} M(g_2, K_2)}{\tau}) \|x - y\|_\tau$$

Now, the proof it follows from the contraction principle

Example 1. Let us consider the following equation

$$x(t) = [g_1(t) + \int_a^t e^{\alpha_1 |t-s|} \sin x(s) ds] \cdot [g_2(t) + \int_a^t e^{\alpha_2 |t-s|} \sin x(s) ds], \ t \in [a,b] \quad (2.1)$$

where $\alpha_1,\ \alpha_2 \in \mathbb{R}_+$ and $g_1, g_2 \in C[a,b]$ arbitrarily. We remark that we are in the conditions of Theorem 2.1. So, the equation (2.1) has, in $C[a,b]$, a unique solution $x^\star$.

Remark 2.1. If we consider on $C[a,b]$ the supremum norm, $\|\cdot\|_\infty$, then we have

$$|A(x)(t)-A(y)(t)| \le M(g_1,g_2,K_1,K_2)(t-a)\|x-y\|_\infty,$$
$$M(g_1,g_2,K_1,K_2) := L_{K_1}M(g_2,K_2)+L_{K_2}M(g_1,K_1).$$

Then

$$|A^2(x)(t)-A^2(y)(t)| \le M^2(g_1,g_2,K_1,K_2)\frac{(t-a)^2}{2!}\|x-y\|_\infty$$

By induction we prove that

$$|A^n(x)(t)-A^n(y)(t)| \le M^n(g_1,g_2,K_1,K_2)\frac{(t-a)^n}{n!}\|x-y\|_\infty,$$

for all $t \in [a,b]$, $n \in \mathbb{N}$. Then

$$\|A^n(x)-A^n(y)\|_\infty \le M^n(g_1,g_2,K_1,K_2)\frac{(b-a)^n}{n!}\|x-y\|_\infty.$$

Because

$$\lim_{n\to\infty} M^n(g_1,g_2,K_1,K_2)\frac{(b-a)^n}{n!} = 0$$

it follows that there exists $N \in \mathbb{N}$ such that

$$M^N(g_1,g_2,K_1,K_2)\frac{(b-a)^N}{N!} < 1.$$

So, we have a new proof for the Theorem 2.1 by the contraction principle and Lemma 1.3.3 in [4].

Theorem 2.2. *We suppose that*

(i) g_1, g_2, K_1, K_2 *, verify the conditions* $(i), (iii)$ *from the Theorem 2.1;*

(ii) *there exists* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *function such that*

$$|K_i(t,s,u)-K_i(t,s,v)| \le \varphi(|u-v|),$$

for all $t, s \in [a,b]$, $u, v \in \mathbb{R}$, $i=\overline{1,2}$;

(iii) $(M(g_1,K_1)+M(g_2,K_2))(b-a)\varphi$ *is a comparison function.*

Then the equation (1.3) has, in $C[a,b]$*, a unique solution* $x^\star$.

Proof. We consider the operator $A : C[a,b] \to C[a,b]$ defined by the relation

$$A(x)(t) = (g_1(t) + \int_a^t K_1(t,s,x(s))ds)\cdot(g_2(t)+\int_a^t K_2(t,s,x(s))ds).$$

For all $x, y \in C[a,b]$, and for all $t \in [a,b]$ we have that

$$|A(x)(t)-A(y)(t)| \le (M(g_1,K_1)+M(g_2,K_2))(b-a)\varphi(\|x-y\|_\infty).$$

So, by the $\varphi-$contraction principle, A is Picard operator.(I.A. Rus [6])

Next we establish the existence and uniqueness result for the solution of integral equation (1.3) in the some close ball $\overline{B}(0,R) \subset C[a,b]$, $R \in (0,1)$. Let $J \subset \mathbb{R}$ a compact interval such that $x \in \overline{B}(0,R)$ imply $x(t) \in J$, for all $t \in [a,b]$. Then we have

Theorem 2.3. *We suppose that*

(i) $g_i \in C[a,b]$, $K_i \in C([a,b]\times[a,b]\times J,\mathbb{R})$, $i=\overline{1,2}$;

(ii) *there exists* $L_{K_i} > 0$ *such that*

$$|K_i(t,s,u) - K_i(t,s,v)| \leq L_{K_i}|u-v|,$$

for all $t,s \in [a,b]$, $u,v \in J$, $i = \overline{1,2}$;

(iii) $\max\limits_{t\in[a,b]} |g_i(t)| + (b-a) \max\limits_{\substack{t,s\in[a,b]\\ u\in J}} |K_i(t,s,u)| \leq R$, $i = \overline{1,2}$;

Then the equation (1.3) has, in $\overline{B}(0,R)$, *a unique solution* $x^\star$.

Proof. We consider the operator $A : C[a,b] \to C[a,b]$ defined by relation

$$A(x)(t) = (g_1(t) + \int_a^t K_1(t,s,x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t,s,x(s))ds).$$

Let be $x \in \overline{B}(0,R)$. Then

$$|A(x)(t)| \leq \prod_{i=1}^{2}[\max_{t\in[a,b]} |g_i(t)| + (b-a) \max_{\substack{t,s\in[a,b]\\ u\in J}} |K_i(t,s,u)|] \leq R^2 \leq R.$$

It follow that $A(\overline{B}(0,R)) \subseteq \overline{B}(0,R)$. Similarly with the proof of the Theorem 2.1 we have that

$$|A(x)(t) - A(y)(t)| \leq R(\frac{L_{K_2}}{r} + \frac{L_{K_1}}{\tau})\|x-y\|_\tau e^{\tau(t-a)},$$

for all $x,y \in \overline{B}(0,R)$. From here we obtain that A is a Picard operator. In the same mod we obtain the following results

Theorem 2.4. *We suppose that* g_1, g_2, K_1, K_2 *verify the conditions of Theorem 2.3. Then the equation (1.3) has, in* $(\overline{B}(0,R), |\cdot|_\infty)$, *a unique solution* $x^\star$.

Theorem 2.5. *We suppose that:*

(a) $g_i \in C[a,b]$, $K_i \in C([a,b]\times[a,b]\times J, \mathbb{R})$, $i = \overline{1,2}$;;

(b) *there exists* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a function such that*

$$|K_i(t,s,u) - K_i(t,s,v)| \leq \varphi(|u-v|),$$

for all $t,s \in [a,b]$, $u,v \in J$, $i = \overline{1,2}$;

(c) $\max\limits_{t\in[a,b]} |g_i(t)| + (b-a) \max\limits_{\substack{t,s\in[a,b]\\ u\in J}} |K_i(t,s,u)| \leq R$, $i = \overline{1,2}$;

(d) $2R(b-a)\varphi$ *is a comparison function.*

Then the equation (1.3) has, in $\overline{B}(0,R)$, *a unique solution* $x^\star$.

3. Data dependence: continuity

Consider the equation (1.3) in the conditions of the Theorem 2.1. Denote by $x(\cdot; g_1, g_2, K_1, K_2)$ the solution of this equation. We have

Theorem 3.1. *Let* $g_1^j, g_2^j, K_1^j, K_2^j$, $j = 1,2$ *be as in the Theorem 2.1. We suppose that*

(a) *there exists* $\eta_i > 0$ *such that*

$$|g_i^1(t) - g_i^2(t)| \leq \eta_i,$$

for all $t \in [a, b]$, $i = 1, 2$;

(b) *there exists* $\mu_i > 0$ *such that*

$$|K_i^1(t, s, u) - K_i^2(t, s, u)| \leq \mu_i,$$

for all $t, s \in [a, b]$, $u \in \mathbb{R}$, $i = 1, 2$.

Then

$$|x(\cdot; g_1^1, g_2^1, K_1^1, K_2^1) - x(\cdot; g_1^2, g_2^2, K_1^2, K_2^2)| \leq$$
$$\leq \frac{M(g_1^1, K_1^1)(\eta_2 + \mu_2(b-a)) + M(g_2^2, K_2^2)(\eta_1 + \mu_1(b-a))}{1 - \frac{\alpha}{\tau}},$$

where

$$\alpha = \max_{j=1,2}\{L_{K_2^j} M(g_1^j, K_1^j) + L_{K_1^j} M(g_2^j, K_2^j)\}.$$

Proof. We consider the operators $A_j : C[a, b] \to C[a, b]$ defined by

$$A_j(x)(t) = (g_1^j(t) + \int_a^t K_1^j(t, s, x(s))ds) \cdot (g_2^j(t) + \int_a^t K_2^j(t, s, x(s))ds).$$

These operators are Picard operators. Moreover

$$\|A_1(x) - A_2(x)\|_\infty \leq M(g_1^1, K_1^1)(\eta_2 + \mu_2(b-a)) + M(g_2^2, K_2^2)(\eta_1 + \mu_1(b-a)),$$

for all $x \in C[a, b]$. Now the proof follows from the well known result(see I.A. Rus [5]).

4. A Gronwall inequality

In this section we shall use the following result

Lemma 4.1. *(I.A. Rus* [6]*) Let* $(X, d, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator such that:*

(i) *the operator* A *is Picard,* $F_A = \{x_A^\star\}$*;*

(ii) A *is increasing operator.*

Then

(a) $x \leq A(x)$ *implies* $x \leq x_A^\star$*;*

(b) $x \geq A(x)$ *implies* $x \geq x_A^\star$.

We consider the inequalities

$$x(t) \leq (g_1(t) + \int_a^t K_1(t, s, x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t, s, x(s))ds) \quad (4.1)$$

$$x(t) \geq (g_1(t) + \int_a^t K_1(t, s, x(s))ds) \cdot (g_2(t) + \int_a^t K_2(t, s, x(s))ds) \quad (4.2)$$

We have

Theorem 4.1. *We suppose that*

(i) $g_i \in C([a,b],\mathbb{R}_+)$, $K_i \in C([a,b]\times[a,b]\times\mathbb{R}_+,\mathbb{R}_+)$, $i=\overline{1,2}$;

(ii) *there exists* $L_{K_i} > 0$ *such that*

$$|K_i(t,s,u)-K_i(t,s,v)| \le L_{K_i}|u-v|,$$

for all $t,s \in [a,b]$, $u,v\in\mathbb{R}_+$, $i=\overline{1,2}$.

(iii) *there exists* $M_{K_i} > 0$ *such that*

$$|K_i(t,s,u)| \le M_{K_i},$$

for all $t,s \in [a,b]$, $u,v\in\mathbb{R}_+$, $i=\overline{1,2}$;

(iv) *the operators* g_i *and* $K_i(t,s,\cdot):\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$ *are increasing.*

Then

(a) *the equation (1.3) has, in* $C[a,b]$, *a unique solution* $x^\star$;

(b) *for all solution* $x\in C([a,b],\mathbb{R}_+)$ *of inequalities (4.1) we have* $x\le x^\star$;

(c) *for all solution* $x\in C([a,b],\mathbb{R}_+)$ *of inequalities (4.2) we have* $x\ge x^\star$;

Proof.

(a) Similarly with the proof of Theorem 2.1.

(b)+(c) From the condition (iv) we obtain that the operator

$$A: C([a,b],\mathbb{R}_+)\to C([a,b],\mathbb{R}_+),$$

$$A(x)(t) = (g_1(t)+\int_a^t K_1(t,s,x(s))ds)\cdot(g_2(t)+\int_a^t K_2(t,s,x(s))ds).$$

is an increasing operator. Then, from Lemma 4.1 we have the conclusions.

Sometimes is hard to determine a fixed point $x_A^\star$ of operator A and this why we use a abstract lemma of Gronwall type with two operators in order to estimate a solution of inequality (4.1). These being said, we use this procedure in the following example. We consider the inequality

$$x(t) \le (\frac{1}{4}+\frac{1}{4}\int_a^t \frac{x(s)}{1+x(s)}ds)^2, t\in[a,a+1] \tag{4.3}$$

where $x:[a,a+1]\to\mathbb{R}_+$ is a continuous function and $a\ge 0$.

Let be $R\in[\frac{1}{2},1]$. We consider the operator $A: C([a,a+1],\mathbb{R}_+)\to C([a,a+1],\mathbb{R}_+)$ defined by relation

$$A(x)(t) = (\frac{1}{4}+\frac{1}{4}\int_a^t \frac{x(s)}{1+x(s)}ds)^2.$$

We are applying the Theorem 2.4 and we obtain that $A\mid_{\overline{B}(0,R)}$ is Picard operator. Also, we remark that A is increasing and for all $x\in\overline{B}(0,R)$ we have that

$$A(x)(t) \le \frac{1}{8}(1+\int_a^t \frac{x^2(s)}{(1+x(s))^2}ds) \le \frac{1}{8}(1+\int_a^t x^2(s)ds).$$

We denote by

$$B : C([a, a+1], \mathbb{R}_+) \to C([a, a+1], \mathbb{R}_+),$$

$$B(x)(t) = \frac{1}{8}(1 + \int_a^t x^2(s)ds).$$

From the Theorem 2.4 we obtain that $B|_{\overline{B}(0,R)}$ is Picard operator with the unique fixed point $x^\star(t) = \frac{8}{64-t+a}$. Now, we apply abstract lemma of Gronwall with two operators(see I.A. Rus [6]) and we obtain that

Theorem 4.2. *If $x \in \overline{B}(0, R) \subset C([a, a+1], \mathbb{R}_+)$ is a solution of (4.1) then*

$$x(t) \le \frac{8}{64 - t + a}$$

for all $t \in [a, a+1]$.

We remark that these results generalized some results from B.G. Pachpate [3].

5. Smooth dependence on parameter

Next we consider the following integral equation

$$x(t, \lambda) = (g_1(t, \lambda) + \int_a^t K_1(t, s, x(s, \lambda), \lambda)ds) \cdot (g_2(t, \lambda) + \int_a^t K_2(t, s, x(s, \lambda), \lambda)ds), \quad (5.1)$$

for all $t \in [a, b], \lambda \in J \subset \mathbb{R}$. We assume that

(H_1) $J \subset \mathbb{R}$ an compact interval;
(H_2) $g_i \in C^1([a, b] \times J)$, $K_i \in C^1([a, b] \times [a, b] \times \mathbb{R} \times J)$, $i = 1, 2$;
(H_3) there exists $L_{K_i} > 0$ such that

$$|\frac{\partial K_i}{\partial u}(t, s, u, \lambda)| \le L_{K_i}$$

for all $t, s \in [a, b]$, $u \in \mathbb{R}$, $\lambda \in J$, $i = 1, 2$;
(H_4) there exists $M_{K_i} > 0$ such that

$$|K_i(t, s, u, \lambda)| \le M_{K_i}$$

for all $t, s \in [a, b]$, $u \in \mathbb{R}$, $\lambda \in J$, $i = 1, 2$;

We define the operator

$$B : C([a, b] \times J) \to C([a, b] \times J),$$

$$B(x)(t, \lambda) = [g_1(t, \lambda) + \int_a^t K_1(t, s, x(s, \lambda), \lambda)ds] \cdot [g_2(t, \lambda) + \int_a^t K_2(t, s, x(s, \lambda), \lambda)ds]$$

It is clear that, in the conditions $(H_1) - (H_4)$, B is Picard operator. Let $x^\star(\cdot, \lambda)$ the unique fixed point of operator B. Then

$$x^\star(t,\lambda) = [g_1(t,\lambda) + \int_a^t K_1(t,s,x^\star(s,\lambda),\lambda)ds] \cdot [g_2(t,\lambda) + \int_a^t K_2(t,s,x^\star(s,\lambda),\lambda)ds], \tag{5.2}$$

for all $t \in [a,b], \lambda \in J \subset \mathbb{R}$. We suppose that there exists $\dfrac{\partial x^\star}{\partial \lambda}$. Then from relation (5.2) we obtain that

$$\frac{\partial x^\star}{\partial \lambda} =$$

$$= [\frac{\partial g_1}{\partial \lambda}(t,\lambda) + \int_a^t \frac{\partial K_1}{\partial u}(t,s,x^\star(s,\lambda),\lambda) \cdot \frac{\partial x^\star}{\partial \lambda}(s,\lambda)ds + \int_a^t \frac{\partial K_1}{\partial \lambda}(t,s,x^\star(s,\lambda),\lambda)ds] \cdot$$

$$[g_2(t,\lambda) + \int_a^t K_2(t,s,x^\star(s,\lambda),\lambda)ds] + [g_1(t,\lambda) + \int_a^t K_1(t,s,x^\star(s,\lambda),\lambda)ds] \cdot$$

$$[\frac{\partial g_2}{\partial \lambda}(t,\lambda) + \int_a^t \frac{\partial K_2}{\partial u}(t,s,x^\star(s,\lambda),\lambda) \cdot \frac{\partial x^\star}{\partial \lambda}(s,\lambda)ds + \int_a^t \frac{\partial K_2}{\partial \lambda}(t,s,x^\star(s,\lambda),\lambda)ds].$$

This relation suggest us to consider the following operator

$$C : C([a,b] \times J) \times C([a,b] \times J) \to C([a,b] \times J),$$

$$C(x,y)(t,\lambda) :=$$

$$= [\frac{\partial g_1}{\partial \lambda}(t,\lambda) + \int_a^t \frac{\partial K_1}{\partial u}(t,s,x(s,\lambda),\lambda) \cdot y(s,\lambda)ds + \int_a^t \frac{\partial K_1}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds] \cdot$$

$$[g_2(t,\lambda) + \int_a^t K_2(t,s,x(s,\lambda),\lambda)ds] + [g_1(t,\lambda) + \int_a^t K_1(t,s,x(s,\lambda),\lambda)ds] \cdot$$

$$[\frac{\partial g_2}{\partial \lambda}(t,\lambda) + \int_a^t \frac{\partial K_2}{\partial u}(t,s,x(s,\lambda),\lambda) \cdot y(s,\lambda)ds + \int_a^t \frac{\partial K_2}{\partial \lambda}(t,s,x(s,\lambda),\lambda)ds].$$

In this way we have the triangular operator

$$A : C([a,b] \times J) \times C([a,b] \times J) \to C([a,b] \times J) \times C([a,b] \times J),$$

$$A(x,y)(t,\lambda) = (B(x)(t,\lambda), C(x,y)(t,\lambda)).$$

We remark that the operator $C(x,\cdot) : C([a,b] \times J) \to C([a,b] \times J)$ is a $\alpha-$ contraction with $\alpha := \dfrac{M(g_2,K_2)L_{K_1} + M(g_1,K_1)L_{K_2}}{\tau}$. From the theorem of fiber contraction (see I.A. Rus [6]) we have that the operator A is Picard operator. So, the sequences

$$x_{n+1} = B(x_n), n \in \mathbb{N}$$

$$y_{n+1} = C(x_n, y_n)$$

converges uniformly to $(x^\star, y^\star) \in F_A$, for all $x_0, y_0 \in C([a,b] \times J)$.

If we take $x_0 = 0$, $y_0 = \frac{\partial x_0}{\partial \lambda} = 0$ then $y_1 = \frac{\partial x_1}{\partial \lambda}$ and by induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in \mathbb{N}^\star$.

Thus

$$x_n \to x^\star, \quad uniform \ \ as\ n \to \infty$$

$$\frac{\partial x_n}{\partial \lambda} \to y^\star, \quad uniform \ \ as\ n \to \infty$$

These imply that there exists $\frac{\partial x^\star}{\partial \lambda}$ and $\frac{\partial x^\star}{\partial \lambda} = y^\star$

From the above considerations, we have that

Theorem 5.1. *We consider the integral equation (5.1) in the hypothesis* $(H_1)-(H_4)$. *Then*

(i) *the equation (5.1) has, in* $C([a,b] \times J)$, *a unique solution* $x^\star(t,\cdot)$;
(ii) $x^\star(t,\cdot) \in C^1(J)$, *for all* $t \in [a,b]$.

References

[1] E. Brestovanska, *Qualitative behaviour of an integral equation related to some epidemic model*, Demonstratio Mathematica, **36**(2003), no. 3, 603-609.

[2] G. Gripenberg, *On some epidemic models*, Quart. Appl. Math., **39**(1981), 317-327.

[3] B.G. Pachpate, *On a new inequality suggested by the study of certain epidemic models*, J. Math. Anal. Appl., **195**(1995), 638-644.

[4] I.A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj-Napoca, 2001.

[5] I.A. Rus, *Ecuatii diferentiale, Ecuatii integrale, Sisteme Dinamice*, Transilvania Press, Cluj-Napoca, 1996.

[6] I.A. Rus, *Weakly Picard operators and applications*, Seminar on Fixed Point Theory Cluj-Napoca, **2**(2001), 41-58.

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