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SOME EXISTENCE RESULTS FOR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider a semilinear functional differential equation of a fractional order in a Banach space assuming that its linear part is the generator of a noncompact semigroup. It is assumed that the nonlinearity satisfies a regularity condition expressed in terms of the measures of noncompactness. The theory of condensing maps is used to obtain local and global existence results. The same approach is applied to a neutral functional differential equation.
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1. Introduction

In this paper we consider existence results for fractional semilinear functional differential equations in a Banach space. At the present time the differential equations with fractional order have been proved to be valuable tools in the investigation of many phenomena in various fields of physics and engineering (viscoelasticity, electrochemistry, electromagnetism, etc.) and they attract the attention of many researchers (see, e.g., the monographs [5] - [9]).

In a separable Banach space E we study the functional differential equation of the form

$$D^{\alpha}y\,(t) = Ay\,(t) + f\,(t, y_t)\,, \quad t \in [0, T],$$

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where D^α, $0 < \alpha < 1$, is the Riemann-Liouville fractional derivative, A is a linear closed (non necessarily bounded) operator in E, and $f : [0,T]\times\mathcal{C} \to E$ is a continuous map. Here $\mathcal{C} = C([-\tau,0];E)$, $\tau > 0$ and, for $t \in [0,T]$, the function $y_t \in \mathcal{C}$ is defined as $y_t(\theta) = y(t+\theta)$, $-\tau \le \theta \le 0$.

It should be noted that certain existence theorems for equations of such type were obtained in the recent paper [2] under assumption that the nonlinearity f satisfies a Lipschitz type condition or the semigroup e^{At} generated by A is compact. Contrary to this approach, we do not suppose the compactness of e^{At}, assuming instead that f satisfies the Ambrosetti - Sadovskii type regularity condition expressed in terms of the measures of noncompactness. Notice that this condition includes the Lipschitz and compactness conditions as particular cases (see Remark 1). This assumption allows us to apply the theory of condensing maps and to obtain local and global existence results for the Cauchy problem (Section 3).

In conclusion we extend the same approach to obtain the existence theorem for a neutral fractional semilinear functional differential equation (Section 4).

2. Preliminaries

We start with the following notions (see [8], [9]). Let E be a Banach space.

Definition 1. *The Riemann-Liouville fractional primitive of order $\alpha \in (0,1)$ of a continuous function $g : [0,h] \to E$ is defined by*

$$I_0^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds,$$

where Γ is the gamma function.

Definition 2. *The Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ of a continuous function $g : [0,h] \to E$ is defined by*

$$D^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} g(s)ds = \frac{d}{dt} I_0^{1-\alpha} g(t).$$

We will need some facts from the theory of measures of noncompactness and condensing maps (see, e.g., [1], [4]).

Definition 3. *Let $\mathcal{E}$ be a Banach space and $(\mathcal{A},\ge)$ a partially ordered set.*

A function $\beta : P(\mathcal{E}) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\,\Omega) = \beta(\Omega) \quad \text{for every } \Omega \in P(\mathcal{E}).$$

A MNC β is called:

- (*i*) *monotone, if $\Omega_0, \Omega_1 \in P(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$;*
- (*ii*) *nonsingular, if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in \mathcal{E}$, $\Omega \in P(\mathcal{E})$;*
- (*iii*) *invariant with respect to union with compact sets, if $\beta(\{K\} \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset \mathcal{E}$, $\Omega \in P(\mathcal{E})$.*

If $\mathcal{A}$ is a cone in a normed space, we say that the MNC β is

- (*iv*) *algebraically semiadditive, if $\beta(\Omega_0 + \Omega_1) \le \beta(\Omega_0) + \beta(\Omega_1)$ for each $\Omega_0, \Omega_1 \in P(\mathcal{E})$;*

(v) *regular, if* $\beta(\Omega) = 0$ *is equivalent to the relative compactness of* Ω;
(vi) *real, if* $\mathcal{A}$ *is* $[0, +\infty)$ *with the natural order.*

As an example of MNC satisfying all above properties we can consider *the Hausdorff MNC*

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\} .$$

Let us mention the following property of the MNC χ:

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega), \text{ for each } \lambda \in \mathbb{R}, \Omega \in P(\mathcal{E}).$$

Another examples can be presented by the following measures of noncompactness defined on the space of continuous functions $C([a, b]; E)$ with the values in a Banach space E:

(i) *the modulus of fiber noncompactness*

$$\varphi(\Omega) = \sup_{t\in[a,b]} \chi_E(\Omega(t)),$$

where χ_E is the Hausdorff MNC in E and $\Omega(t) = \{y(t) : y \in \Omega\}$;

(ii) *the modulus of equicontinuity* defined as

$$\operatorname{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\leq\delta} \|y(t_1) - y(t_2)\| .$$

It should be mentioned that these MNCs satisfy all above-mentioned properties except regularity.

If $\mathcal{L} : E \to E$ is a bounded linear operator, then its χ-*norm* is defined as

$$\|\mathcal{L}\|^{(\chi)} = \chi(\mathcal{L}(B)),$$

where $B \subset E$ is a unit ball. It is easy to see that $\|\mathcal{L}\|^{(\chi)} \leq \|\mathcal{L}\|$.

Now, let $G : [0, h] \to P(E)$ be a multifunction. It is called:

(i) *integrable, if it admits a Bochner integrable selection* $g : [0, h] \to E$, $g(t) \in G(t)$ *for a.e.* $t \in [0, h]$;
(ii) *integrably bounded, if there exits a function* $\zeta \in L^1_+[0, h]$ *such that*

$$\|G(t)\| := \sup\{\|g\| : g \in G(t)\} \leq \zeta(t) \text{ a.e. } t \in [0, h].$$

For an integrable multifunction, let $\mathbb{S}_G$ denote the set of all integrable selections of G. Then, for any $t \in [0, h]$ the Aumann integral of G is defined as

$$\int_0^t G(s)ds = \Big\{ \int_0^t g(s)ds : g \in \mathbb{S}_G \Big\}.$$

In the sequel we will need the following assertion about χ-estimates for a multivalued integral (Theorem 4.2.3 of [4]).

Proposition 1. *For an integrable, integrably bounded multifunction* $G : [0, h] \to P(E)$ *where* E *is a separable Banach space, let*

$$\chi(G(t)) \leq q(t) \text{ for a.e. } t \in [0, h],$$

where $q \in L^1_+[0, h]$. *Then* $\chi\left(\int_0^t G(s)ds\right) \leq \int_0^t q(s)ds$ *for all* $t \in [0, h]$.

Let $\mathcal{E}$ be a Banach space, β a monotone nonsingular MNC in $\mathcal{E}$.

Definition 4. *A continuous map $\mathfrak{F} : X \subseteq \mathcal{E} \to \mathcal{E}$ is called condensing with respect to a MNC β (or β-condensing) if for every bounded set $\Omega \subseteq X$ which is not relatively compact, we have*

$$\beta(\mathfrak{F}(\Omega)) \ngeq \beta(\Omega).$$

The application of the topological degree theory for condensing maps (see, e.g., [1], [4]) yields the following fixed point principles which we will use in the sequel.

Theorem 1. *Let $\mathfrak{M}$ be a bounded convex closed subset of $\mathcal{E}$ and $\mathfrak{F} : \mathfrak{M} \to \mathfrak{M}$ a β-condensing map. Then $Fix\mathfrak{F} = \{x : x = \mathfrak{F}(x)\}$ is a nonempty compact set.*

Theorem 2. *Let $\mathcal{V} \subset \mathcal{E}$ be a bounded open neighborhood of zero and $\mathfrak{F} : \overline{\mathcal{V}} \to \mathcal{E}$ a β-condensing map satisfying the boundary condition*

$$x \neq \lambda \mathfrak{F}(x)$$

for all $x \in \partial\mathcal{V}$ and $0 < \lambda \leq 1$. Then $Fix\mathfrak{F}$ is a nonempty compact set.

3. Local and Global Existence Results

We will consider the following Cauchy problem for a fractional semilinear functional differential equation in a separable Banach space E :

$$D^{\alpha} y(t) = Ay(t) + f(t, y_t), \ t \in [0,T]; \tag{3.1}$$

$$y(t) = \widetilde{\psi}(t), \quad -\tau \leq t \leq 0, \quad \widetilde{\psi}(0) = 0. \tag{3.2}$$

It will be supposed that:

(A) *$A : D(A) \subset E \to E$ is a closed (not necessarily bounded) linear operator generating an immediately norm-continuous semigroup $\{e^{At}\}_{t\geq 0}$, i.e., it is supposed to be C_0-continuous and norm-continuous for $t > 0$;*

($\widetilde{\psi}$) *$\widetilde{\psi} \in \mathcal{C}$, $\widetilde{\psi}(0) = 0$ is a given initial function.*

We will assume that the nonlinearity f obeys the following conditions:

($f1$) *the map $f : [0,T] \times \mathcal{C} \to E$ is continuous;*

($f2$) *for each nonempty, bounded set $\mathcal{Q} \subset \mathcal{C}$, there exists a continuous function $\mu_{\mathcal{Q}} : [0,T] \to \mathbb{R}_+$ such that*

$$\|f(t,\psi)\|_E \leq \mu_{\mathcal{Q}}(t)$$

for all $t \in [0,T]$ and $\psi \in \mathcal{Q}$;

($f3$) *there exists a continuous function $k : [0,T] \to \mathbb{R}_+$ such that for each nonempty, bounded set $\mathcal{Q} \subset \mathcal{C}$*

$$\chi(f(t,\mathcal{Q})) \leq k(t)\varphi(\mathcal{Q})$$

for all $t \in [0,T]$, where χ is the Hausdorff MNC in E and $\varphi(\mathcal{Q})$ is the modulus of fiber noncompactness of the set $\mathcal{Q}$.

Remark 1. *It is known (see, e.g., [1], [4]) that condition ($f3$) is fulfilled if f may be represented as*

$$f(t,\psi) = f_0(t,\psi) + f_1(t,\psi),$$

where $f_0, f_1 : [0,T] \times \mathcal{C} \to E$ *are continuous functions,* f_0 *is Lipschitz in the second argument:*

$$\|f_0(t,\psi_2) - f_0(t,\psi_1)\|_E \le k(t)\|\psi_2 - \psi_1\|_{\mathcal{C}}, \qquad \forall t \in [0,T], \quad \psi_1, \psi_2 \in \mathcal{C},$$

where $k : [0,T] \to \mathbb{R}_+$ *is a continuous function, and* f_1 *is compact in the second argument, i.e., for each* $t \in [0,T]$ *and bounded* $\mathcal{Q} \subset \mathcal{C}$*, the set* $f_1(t,\mathcal{Q})$ *is relatively compact in* E*.*

For $0 < h \le T$, by the symbol $\mathcal{D}_h$ we will denote the closed subspace of $C([0,h];E)$ defined as

$$\mathcal{D}_h = \{y \in C([0,h];E) : y(0) = 0\}.$$

Further, for any $y \in \mathcal{D}_h$ we define the function $y[\widetilde{\psi}] \in C((-\tau,h];E)$:

$$y[\widetilde{\psi}](t) = \begin{cases} \widetilde{\psi}(t), & -\tau < t < 0, \\ y(t), & 0 \le t \le h\,. \end{cases} \tag{3.3}$$

Then, clearly for $t \in [0,h]$, $t \le \tau$:

$$y[\widetilde{\psi}]_t(\theta) = \begin{cases} \widetilde{\psi}(t+\theta), & -\tau \le \theta < -t, \\ y(t+\theta), & -t \le \theta \le 0 \end{cases}$$

and

$$y[\widetilde{\psi}]_t(\theta) = y(t+\theta), \quad -\tau \le \theta \le 0, \text{ provided } t > \tau.$$

Definition 5. *For a given* $0 < h \le T$*, a function* $y \in C((-\tau,h];E)$ *is called a mild solution to problem (3.1)-(3.2) on interval* $[-\tau,h] \subseteq [-\tau,T]$ *if* y *satisfies initial condition (3.2) and on interval* $[0,h]$ *it has the form*

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{A(t-s)} f(s, y[\widetilde{\psi}]_s)ds, \qquad t \in [0,h].$$

Let us consider the operator $\mathcal{F}_h : \mathcal{D}_h \to \mathcal{D}_h$ defined as

$$\mathcal{F}_h\,(x)\,(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{A(t-s)} f(s, x[\widetilde{\psi}]_s)ds.$$

It is clear that each mild solution $y \in C((-\tau,h];E)$ of problem (3.1)-(3.2) is determined by a fixed point $x = \mathcal{F}_h(x)$ by the rule:

$$y = x[\widetilde{\psi}]$$

(see (3.3)).

It is easy to see that the operator $\mathcal{F}_h$ is well-defined and continuous. Let us investigate its further properties.

Proposition 2. *The operator* $\mathcal{F}_h$ *transforms bounded sets into equicontinuous ones.*

Proof. From condition (A) it follows that there exists $C \ge 1$ such that

$$\|e^{At}\|_{L(E)} \le C, \text{ for all } t \in [0,T].$$

Now, let $\Omega \subset \mathcal{D}_h$ be any nonempty, bounded set. Condition $(f2)$ implies that there exists a constant $\mu_\Omega \geq 0$ such that

$$\|f(t, x[\widetilde{\psi}]_t)\|_E \leq \mu_\Omega, \qquad \forall t \in [0,h],\ x \in \Omega.$$

For a given $x \in \Omega$, take $0 < t_1 < t_2 \leq h$ and let ε, $0 < \varepsilon < t_1$ be an arbitrary number. Then we have

$$\left\|\mathcal{F}_h(x)(t_2) - \mathcal{F}_h(x)(t_1)\right\|_E \leq$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_2}(t_2-s)^{\alpha-1}e^{A(t_2-s)}f(s,x[\widetilde{\psi}]_s)ds - \int_0^{t_1}(t_1-s)^{\alpha-1}e^{A(t_1-s)}f(s,x[\widetilde{\psi}]_s)ds\right\|_E \leq$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1-\varepsilon}\left[(t_2-s)^{\alpha-1}e^{A(t_2-s)} - (t_1-s)^{\alpha-1}e^{A(t_1-s)}\right]f(s,x[\widetilde{\psi}]_s)ds\right\|_E +$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_{t_1-\varepsilon}^{t_1}\left[(t_2-s)^{\alpha-1}e^{A(t_2-s)} - (t_1-s)^{\alpha-1}e^{A(t_1-s)}\right]f(s,x[\widetilde{\psi}]_s)ds\right\|_E +$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e^{A(t_2-s)}f(s,x[\widetilde{\psi}]_s)ds\right\|_E \leq$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1-\varepsilon}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]e^{A(t_1-s)}f(s,x[\widetilde{\psi}]_s)ds\right\|_E +$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1-\varepsilon}(t_2-s)^{\alpha-1}e^{A(t_1-\varepsilon-s)}\left[e^{A(t_2-t_1+\varepsilon)} - e^{A\varepsilon}\right]f(s,x[\widetilde{\psi}]_s)ds\right\|_E +$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_{t_1-\varepsilon}^{t_1}\left[(t_2-s)^{\alpha-1}e^{A(t_2-s)} - (t_1-s)^{\alpha-1}e^{A(t_1-s)}\right]f(s,x[\widetilde{\psi}]_s)ds\right\|_E +$$

$$\frac{1}{\Gamma(\alpha)}\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}e^{A(t_2-s)}f(s,x[\widetilde{\psi}]_s)ds\right\|_E \leq$$

$$\frac{C\mu_\Omega}{\Gamma(\alpha)}\Big(\int_0^{t_1-\varepsilon}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]ds +$$

$$\left\|e^{A(t_2-t_1+\varepsilon)} - e^{A\varepsilon}\right\|_{L(E)}\int_0^{t_1-\varepsilon}(t_2-s)^{\alpha-1}\,ds +$$

$$\int_{t_1-\varepsilon}^{t_1}(t_2-s)^{\alpha-1}\,ds + \int_{t_1-\varepsilon}^{t_1}(t_1-s)^{\alpha-1}\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\Big)$$

Since ε may be taken arbitrarily small, we can see that the above expression tends to zero while $t_2 - t_1 \to 0$ uniformly with respect to $x \in \Omega$. □

Now, let us consider the MNC ν in the space $C([0,h];E)$ with the values in the cone $\mathbb{R}^2_+$ of the following form:

$$\nu(\Omega) = (\varphi(\Omega), \mathrm{mod}_C\,\Omega), \tag{3.4}$$

where $\varphi(\Omega)$ is the modulus of fiber noncompactness of Ω and $\mathrm{mod}_C\,\Omega$ is its modulus of equicontinuity (see Section 2).

It is easy to see that the MNC ν obeys all the properties mentioned in Section 2, including, by the Arzela-Ascoli theorem, the regularity.

We would like to indicate conditions under which the operator $\mathcal{F}_h$ is ν-condensing.

Let

$$\widetilde{k}_h = \max_{0 \le t \le h} k(t),$$

where $k(\cdot)$ is the function from condition $(f3)$. Let $C_h^{(\chi)} \ge 0$ be a constant such that

$$\|e^{At}\|^{(\chi)} \le C_h^{(\chi)} \quad \text{for all } t \in (0, h].$$

It is clear that $C_h^{(\chi)} \le C$. In case $h = T$ we will denote $\widetilde{k}_h = \widetilde{k}$, $C_h^{(\chi)} = C^{(\chi)}$.

Proposition 3. *Let*

$$l = \frac{1}{\Gamma(\alpha+1)} \widetilde{k}_h C_h^{(\chi)} h^\alpha < 1. \tag{3.5}$$

Then the operator $\mathcal{F}_h$ is ν-condensing.

Proof. Let $\Omega \subset \mathcal{D}_h$ be a nonempty, bounded set for which

$$\nu(\mathcal{F}_h(\Omega) \ge \nu(\Omega). \tag{3.6}$$

Let us estimate $\varphi(\mathcal{F}_h(\Omega))$. For any $t \in [0, h]$ we have

$$\mathcal{F}_h(\Omega)(t) = \Big\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{A(t-s)} f(s, x[\widetilde{\psi}]_s) ds : \ x \in \Omega \Big\}.$$

Consider the multifunction $s \in [0, t] \multimap G(s)$,

$$G(s) = \Big\{ (t-s)^{\alpha-1} e^{A(t-s)} f(s, x[\widetilde{\psi}]_s) : x \in \Omega \Big\}.$$

It is clear that G is integrable, and from condition $(f2)$ it follows that it is integrably bounded. Further, denoting $\Omega_s = \{x[\widetilde{\psi}]_s : x \in \Omega\}$ and applying condition $(f3)$ we have the following estimate for a.e. $s \in [0, t]$:

$$\chi\big(G(s)\big) \le (t-s)^{\alpha-1} \big\| e^{A(t-s)} \big\|^{(\chi)} \chi\big(\{ f(s, x[\widetilde{\psi}]_s) : x \in \Omega\}\big) \le$$
$$(t-s)^{\alpha-1} C_h^{(\chi)} \chi\big(f(s, \Omega_s)\big) \le (t-s)^{\alpha-1} C_h^{(\chi)} \widetilde{k}_h \varphi(\Omega).$$

Applying Proposition 1 we have

$$\begin{aligned} \chi\left(\mathcal{F}_h\left(\Omega\right)(t)\right) &= \frac{1}{\Gamma(\alpha)} \chi\left(\int_0^t G\left(s\right) ds \right) \le \\ &\frac{1}{\Gamma(\alpha)} C_h^{(\chi)} \widetilde{k}_h \int_0^t (t-s)^{\alpha-1} \, ds \cdot \varphi\left(\Omega\right) \le \\ &\frac{1}{\Gamma(\alpha+1)} C_h^{(\chi)} \widetilde{k}_h h^\alpha \varphi\left(\Omega\right) = l \varphi\left(\Omega\right). \end{aligned} \tag{3.7}$$

But then

$$\varphi\left(\mathcal{F}_h\left(\Omega\right)\right) \le l \varphi\left(\Omega\right)$$

which implies, by (3.6),

$$\varphi\left(\Omega\right) = 0.$$

Further, from Proposition 2 we know that $mod_C\left(\mathcal{F}_h\left(\Omega\right)\right) = 0$ and (3.6) yields $mod_C\left(\Omega\right) = 0$, hence

$$\nu\left(\Omega\right) = 0,$$

implying, by the regularity property of ν, the relative compactness of Ω. □

Remark 2. *Notice that condition (3.5) is obviously fulfilled if the semigroup e^{At} is compact ($C_h^{(\chi)} = 0$) or f is compact in the second argument ($\widetilde{k}_h = 0$).*

Now we are in a position to formulate the following local existence result.

Theorem 3. *Under conditions (A), $(\widetilde{\psi})$ and $(f1)$-$(f3)$, there exists h, $0 < h \leq T$ such that problem (3.1)- (3.2) has a mild solution on the interval $[-\tau, h]$.*

Proof. Choose h_1, $0 < h_1 \leq T$ such that

$$\frac{1}{\Gamma(\alpha+1)}\widetilde{k}C^{(\chi)}h_1^{\alpha} < 1.$$

Take an arbitrary number $r > 0$ and let $\mathcal{Q} \subset \mathcal{C}$ be a closed ball with the radius $\|\widetilde{\psi}\| + r$ centered at the origin and $\mu_{\mathcal{Q}}$ the corresponding function from condition $(f2)$.

Let $\widetilde{\mu}_{\mathcal{Q}} = \max_{0 \leq t \leq T} \mu_{\mathcal{Q}}(t)$. Take $0 < h_2 \leq T$ such that

$$\frac{1}{\Gamma(\alpha+1)}C\widetilde{\mu}_{\mathcal{Q}}h_2^{\alpha} \leq r.$$

Take $h = \min\{h_1, h_2\}$ and consider the closed convex subset $\overline{B}_r \subset \mathcal{D}_h$, defined as

$$\overline{B}_r = \{x \in \mathcal{D}_h : \|x\| \leq r\}.$$

It is clear that $x[\widetilde{\psi}]_t \in \mathcal{Q}$ for each $x \in \overline{B}_r$ and $t \in [0, h]$.

The operator $\mathcal{F}_h$ transforms the set $\overline{B}_r$ into itself. In fact, if $x \in \overline{B}_r$, then we have for $t \in [0, h]$:

$$\left\|\mathcal{F}_h(\Omega)(t)\right\| = \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}e^{A(t-s)}f(s, x[\widetilde{\psi}]_s)ds\right\| \leq$$

$$\frac{1}{\Gamma(\alpha)}C\widetilde{\mu}_{\mathcal{Q}}\int_0^t (t-s)^{\alpha-1}ds \ \leq \ \frac{1}{\Gamma(\alpha+1)}C\widetilde{\mu}_{\mathcal{Q}}h^{\alpha} \ \leq r.$$

From Proposition 3 it follows that $\mathcal{F}_h$ is ν-condensing and to complete the proof we apply Theorem 1. □

To prove the global existence result, we will need the following generalization of the Gronwall lemma for singular kernels (see [3]).

Lemma 1. *Let $u, w : [0, b] \to [0, +\infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $a > 0$ and $0 < \gamma < 1$ such that*

$$u(t) \leq w(t) + a\int_0^t \frac{u(s)}{(t-s)^{\gamma}}ds,$$

then there exists a constant $q = q(\gamma)$ such that

$$u(t) \leq w(t) + qa\int_0^t \frac{w(s)}{(t-s)^{\gamma}}ds$$

for every $t \in [0, b]$.

Now, strengthening the boundedness condition $(f2)$, we can obtain the following result on the existence of a solution on the whole $[0, T]$.

Theorem 4. *Assume that conditions* (A), $(\widetilde{\psi})$, $(f1)$, *and* $(f3)$ *are satisfied and that*

$$\frac{1}{\Gamma(\alpha+1)}\widetilde{k}C^{(\chi)}T^{\alpha} < 1, \tag{3.8}$$

and also

$(f2')$ *there exists a continuous function* $\eta : [0,T] \to [0,+\infty)$ *such that*

$$\|f(t,\psi)\| \le \eta\,(t)\,(1+\|\psi\|_{\mathcal{C}}) \quad for \quad t \in [0,T].$$

Then the set $\Sigma_{\widetilde{\psi}}$ *of all mild solutions of problem (3.1)- (3.2) on* $[-\tau,T]$ *is a nonempty compact subset of the space* $C([-\tau,T];E)$.

Proof. Denoting $\mathcal{F}_T = \mathcal{F}$, consider the following one-parameter family of maps $\Psi : [0,1]\times\mathcal{D}_T \to \mathcal{D}_T$:

$$\Psi(\lambda,x) = \lambda\mathcal{F}(x).$$

We will demonstrate that the fixed point set of the family Ψ,

$$Fix\Psi = \{x \in \Psi(\lambda,x) \quad \text{for some} \quad \lambda \in (0,1]\}$$

is a priori bounded. Indeed, let $x \in Fix\Psi$. Then

$$x(t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{A(t-s)}f(s,x[\widetilde{\psi}]_s)ds$$

and hence

$$\begin{aligned}\|x(t)\| \le \frac{1}{\Gamma(\alpha)}\Big\|\int_0^t (t-s)^{\alpha-1}e^{A(t-s)}f(s,x[\widetilde{\psi}]_s)ds\Big\| \le \\ \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|f(s,x[\widetilde{\psi}]_s)\big\|ds \le \\ \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\left(1+\|x[\widetilde{\psi}]_s\|_{\mathcal{C}}\right)ds \le \\ \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}+\sup_{0\le\vartheta\le s}\|x(\vartheta)\|\right)ds\end{aligned}$$

Since the last expression is a nondecreasing function of t, we have for the function $u(t) = \sup_{0\le\vartheta\le t}\|x(\vartheta)\|$ the following estimate for each $t\in[0,T]$:

$$\begin{aligned}u(t) \le \frac{C}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\eta(s)\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}+u(s)\right)ds \le \\ \frac{C}{\Gamma(\alpha+1)}\widetilde{\eta}\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}\right)T^{\alpha} + \frac{C}{\Gamma(\alpha)}\widetilde{\eta}\int_0^t (t-s)^{\alpha-1}u(s)ds,\end{aligned}$$

where $\widetilde{\eta} = \max_{0\le t\le T}\eta(t)$. Denoting

$$\mathcal{N} = \frac{C}{\Gamma(\alpha+1)}\widetilde{\eta}\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}\right)T^{\alpha}$$

and applying Lemma 1 we obtain

$$u(t) \le \mathcal{N} + \frac{qC}{\Gamma(\alpha)}\widetilde{\eta}\mathcal{N}\int_0^t (t-s)^{\alpha-1}ds \le \mathcal{N}\left(1+\frac{qC}{\Gamma(\alpha+1)}\widetilde{\eta}T^{\alpha}\right)$$

yielding the desired a priori boundedness.

Now take a closed ball $B_R \subset \mathcal{D}_T$ centered at the origin with radius $R > 0$ large enough to contain the set $Fix\Psi$ inside itself. By Proposition 3, condition (3.8) implies that $\mathcal{F} : B_R \to \mathcal{D}_T$ is ν-condensing and it remains to apply Theorem 2. □

4. Neutral Fractional Functional Differential Equations

In this section we will consider the following Cauchy problem:

$$D^{\alpha}[y\,(t) - \varrho\,(t, y_t)] = A[y\,(t) - \varrho\,(t, y_t)] + f\,(t, y_t)\ \ ,\ t \in [0, T]\,; \tag{4.1}$$

$$y\,(t) = \widetilde{\psi}(t), \quad -\tau \le t \le 0, \quad \widetilde{\psi}(0) = 0, \tag{4.2}$$

where A, f, and $\widetilde{\psi}$ are as in the previous section and $\varrho : [0, T] \times \mathcal{C} \to E$ is a given function.

Definition 6. *A function $y \in C((-\tau, T]; E)$ is called a mild solution to problem (4.1)-(4.2) on interval $(-\tau, T]$ if y satisfies initial condition (4.2) and on interval $[0, T]$ it has the form*

$$y(t) = \varrho(t, y_t) - e^{At}\varrho(0, \widetilde{\psi}) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{A(t-s)} f(s, y[\widetilde{\psi}]_s) ds.$$

We will assume that the function ϱ satisfies the following conditions:

(ϱ1) $\varrho : [0, T] \times \mathcal{C} \to E$ *is a continuous function;*

(ϱ2) *for each nonempty, bounded set $\mathcal{Q} \subset \mathcal{C}$ the family of functions $\{t \to \varrho(t, \psi) : \psi \in \mathcal{Q}\}$ is equicontinuous;*

(ϱ3) *there exists a continuous function $\omega : [0, T] \to [0, +\infty)$ such that*

$$\|\varrho(t, \psi)\| \le \omega\,(t)\,(1 + \|\psi\|_{\mathcal{C}}) \quad \text{for} \quad t \in [0, T];$$

(ϱ4) *there exists a continuous function $j : [0, T] \to \mathbb{R}_+$ such that for each nonempty, bounded set $\mathcal{Q} \subset \mathcal{C}$*

$$\chi(\varrho(t, \mathcal{Q})) \le j(t)\varphi(\mathcal{Q})$$

for all $t \in [0, T]$, where χ is the Hausdorff MNC in E and $\varphi(\mathcal{Q})$ is the modulus of fiber noncompactness of the set $\mathcal{Q}$.

Denote $\widetilde{\omega} = \max_{0 \le t \le T} \omega(t)$, $\widetilde{j} = \max_{0 \le t \le T} j(t)$.

We may present the following existence result for problem (4.1)-(4.2).

Theorem 5. *Under conditions (A), ($\widetilde{\psi}$), ($f1$), ($f2'$), ($f3$) and (ϱ1) - (ϱ4), suppose that*

$$l' = \widetilde{j} + \frac{1}{\Gamma(\alpha + 1)}\widetilde{k}C^{(\chi)}T^{\alpha} < 1, \tag{4.3}$$

and also

$$\widetilde{\omega} < 1. \tag{4.4}$$

Then the set $\Sigma_{\widetilde{\psi}}$ of all mild solutions of problem (4.1)- (4.2) on $[-\tau, T]$ is a nonempty compact subset of the space $C([-\tau, T]; E)$.

Proof. Consider the continuous operator $\mathcal{P} : \mathcal{D}_T \to \mathcal{D}_T$ defined as

$$\mathcal{P}\,(x)\,(t) = \varrho(t, x[\widetilde{\psi}]_t) - e^{At}\varrho(0, \widetilde{\psi}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{A(t-s)} f(s, x[\widetilde{\psi}]_s) ds.$$

Each mild solution $y \in C((-\tau, h]; E)$ of problem (4.1)-(4.2) is determined by a fixed point $x = \mathcal{P}(x)$ by the same rule as before:

$$y = x[\widetilde{\psi}].$$

Notice that $\mathcal{P}$ may be represented as

$$\mathcal{P}(x)(t) = \varrho(t, x[\widetilde{\psi}]_t) - e^{At}\varrho(0, \widetilde{\psi}) + \mathcal{F}(x)(t)$$

and hence from condition (ϱ2) and Proposition 2 it follows that $\mathcal{P}$ transforms bounded sets into equicontinuous ones.

To prove that $\mathcal{P}$ is ν-condensing it is sufficient to use the property of algebraic semiadditivity of the MNC χ, condition (ϱ4) and estimate (3.7). We obtain for a nonempty, bounded set $\Omega \subset \mathcal{D}_T$ the following estimate:

$$\chi\big(\mathcal{P}(\Omega)(t)\big) \le \chi\Big(\big\{\varrho(t, x[\widetilde{\psi}]_t) :\ x \in \Omega\big\}\Big) + \chi\big(\mathcal{F}(\Omega)(t)\big) \le$$

$$\widetilde{j}\varphi(\Omega) + \frac{1}{\Gamma(\alpha+1)}\widetilde{k}C^{(\chi)}T^{\alpha}\varphi(\Omega)$$

implying

$$\varphi(\mathcal{P}(\Omega)) \le l'\varphi(\Omega).$$

After that, we may follow the same reasonings as in the proof of Proposition 3.

At last, let us demonstrate that the set of all solutions of the family of inclusions

$$x \in \lambda\mathcal{P}(x), \quad 0 < \lambda \le 1 \tag{4.5}$$

is a priori bounded.

For any such solution x we have

$$x(t) = \lambda\varrho(t, x[\widetilde{\psi}]_t) - \lambda e^{At}\varrho(0, \widetilde{\psi}) + \lambda\mathcal{F}(x)(t), \quad t \in [0, T].$$

Applying property (ϱ3) and the estimate obtained in the proof of Theorem 4 we come to the following:

$$\|x(t)\| \le \omega(t)\left(1 + \|x[\widetilde{\psi}]_t\|_{\mathcal{C}}\right) + C\omega(0)\left(1 + \|\widetilde{\psi}\|_{\mathcal{C}}\right) + \|\mathcal{F}(x)(t)\| \le$$

$$\widetilde{\omega}\left(1 + \|\widetilde{\psi}\|_{\mathcal{C}} + \max_{0\le\vartheta\le t}\|x(\vartheta)\|\right) + C\widetilde{\omega}\left(1 + \|\widetilde{\psi}\|_{\mathcal{C}}\right) +$$

$$\frac{C}{\Gamma(\alpha+1)}\widetilde{\eta}\left(1 + \|\widetilde{\psi}\|_{\mathcal{C}}\right)T^{\alpha} + \frac{C}{\Gamma(\alpha)}\widetilde{\eta}\int_0^t (t-s)^{\alpha-1}\max_{0\le\vartheta\le s}\|x(\vartheta)\| ds.$$

The last expression is again a nondecreasing function of t, so denoting

$$u(t) = \max_{0\le\vartheta\le t}\|x(\vartheta)\|$$

we obtain

$$u(t) \le \widetilde{\omega}u(t) + (1 + C)\,\widetilde{\omega}\left(1 + \|\widetilde{\psi}\|_{\mathcal{C}}\right) +$$

$$\frac{C}{\Gamma(\alpha+1)}\widetilde{\eta}\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}\right)T^{\alpha}+\frac{C}{\Gamma(\alpha)}\widetilde{\eta}\int_0^t(t-s)^{\alpha-1}u(s)ds.$$

Therefore

$$u(t)\leq\frac{1}{1-\widetilde{\omega}}\Big[(1+C)\,\widetilde{\omega}\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}\right)+$$
$$\frac{C}{\Gamma(\alpha+1)}\widetilde{\eta}\left(1+\|\widetilde{\psi}\|_{\mathcal{C}}\right)T^{\alpha}+\frac{C}{\Gamma(\alpha)}\widetilde{\eta}\int_0^t(t-s)^{\alpha-1}u(s)ds\Big].$$

Applying Lemma 1 we obtain the following bound for $u(t)$:

$$u(t)\leq\mathcal{M}\Big[1+\frac{qC}{(1-\widetilde{\omega})\Gamma(\alpha+1)}\widetilde{\eta}T^{\alpha}\Big],$$

where

$$\mathcal{M}=\frac{1+\|\widetilde{\psi}\|_{\mathcal{C}}}{1-\widetilde{\omega}}\Big[(1+C)\widetilde{\omega}+\frac{C}{\Gamma(\alpha+1)}\widetilde{\eta}T^{\alpha}\Big],$$

giving the desired a priori boundedness for solutions of family of inclusions (4.5).

Now the proof is completed by the application of Theorem 2. □

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