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ON NONLINEAR VARIATIONAL INEQUALITIES FOR P-CONVEX MAPS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper we prove some results on minimax inequalities for p - convex mappings and we also show the existence of solutions of certain nonlinear variational inequalities. As corollaries, some results on the fixed points, coincidence points and best approximations are obtained.
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1. Introduction and Preliminaries

The famous Brouwer fixed point theorem (any continuous function $f : K \to K$ has at least one fixed point, where $K \subseteq R^n$ is non-empty, compact and convex set of R^n) for $n = 3$ was proved by him in 1909; equivalent results were established earlier by Henri Poincaré in 1883 and P. Bohl in 1904. It was Hadamard who in 1910 gave (using the Kronecker index) the first proof for an arbitrary n. In 1912, Brouwer gave another proof using the simplicial approximation technique, and notions of degree. In 1927 Schauder obtained first infinite dimensional generalization of Brouwer fixed point theorem and gave its applications in the theory of elliptic equations of Mathematical Physic. In 1928, J. von Neumann by using Brouwer theorem proved existence of solution of matrix zero sums games.

A short and simple proof of Brouwer theorem was given in 1929 by Knaster, Kuratowski and Mazurkiewicz. This proof is based on one corollary of the Sperner's lemma which is known as KKM lemma. First infinite dimensional generalization of this statement (so called KKM principle) was obtained by Ky Fan [8] in 1961. This statement, which is an infinite dimensional generalization of classical KKM lemma, is known as KKM principle. It has many applications in modern nonlinear functional analysis (see [9],[13] and [14]). Fixed point formulation of Fans result, which is also very applicable, so-called Fan-Browders theorem was obtained by Felix Browder in 1968.

In 1930's, earlier works of Nikodym, Mazur, Schauder initiated the abstract approach to problems in calculus of variations. For further development of this theory the most important results are existence of solutions of the nonlinear variational inequalities obtained by Hartman and Stamppacia in 1966 and minimax inequality presented by Ky Fan in 1972. Its result has been used in a large variety of problems in nonlinear analysis, convex analysis, partial differential equations, mechanics, physics, optimization and control theory.

In this paper we prove some results on minimax inequalities for p - convex mappings and we also show the existence of solutions of certain nonlinear variational inequalities. As corollaries, some results on the fixed points, coincidence points and best approximations are obtained.

2. Preliminaries

Let X and Y be non-empty sets; we denote by 2^X a family of all non-empty subsets of X and $\mathcal{F}(X)$ a family of all non-empty finite subsets of X. A multifunction G from X into Y is a map $G : X \longrightarrow 2^Y$.

Let K be a convex subset of a linear space X and $p \in (0, 1]$, a map $f : K \to \mathbb{R}$ is called a p-convex map (see [6]) if

$$f(\lambda_1 x_1+\lambda_2 x_2) \leq \lambda_1^p f(x_1)+\lambda_2^p f(x_2), \text{ for all } x_1, x_2 \in K,\ \lambda_1, \lambda_2 \in [0, 1] \text{ with } \lambda_1^p+\lambda_2^p = 1.$$

Every positive convex map is $p-$convex, but the converse is not true. For example, map $f(x) = x^p, 0 < p < 1, x > 0$, is $p-$convex but not convex.

The set $C \subseteq X$ is said to be p-convex set (see for example [7]) if

$$\lambda_1 x_1 + \lambda_2 x_2 \in C, \text{ for all } x_1, x_2 \in C,\ \lambda_1, \lambda_2 \in [0, 1] \text{ with } \lambda_1^p + \lambda_2^p = 1.$$

Recall that a real valued function f defined on a topological space X is said to be a *upper semi-continuous* if for every real number t the set $\{x : f(x) > t\}$ is open. f is *lower semi-continuous* if $-f$ is upper semi-continuous.

In paper [4] Charles Horvath (see also [5]), gave a notion of pseudo-convex topological spaces. Let X be a topological space and $h : X^2 \times [0, 1] \longrightarrow X$ such that $h(x, y, 0) = y$ and $h(x, y, 1) = x$ for any $x, y \in X$. The pair (X, h) is pseudo-convex space if for any non-empty finite $A \subseteq X$ restriction of h on $[A]^2 \times I$ is continuous. $A \subseteq X$ is pseudo-convex set if and only if $h(x, y, \lambda) \in A$ for each $x, y \in A$ and any $\lambda \in [0, 1]$. Pseudo-convex hull of $A \subseteq X$ is intersection of all pseudo-convex subset of X which contains A and denote it with $[A]$. A real valued function f defined on a pseudo-convex space (X, h) is said to be *pseudo quasi-concave* if for every real number t the set $\{x : f(x) > t\}$ is pseudo-convex. A map f is *pseudo quasi convex* if $-f$ is pseudo quasi concave.

Pseudo-convex spaces are include in more general definition of KKM space (Park [11]).

We need the following minimax inequality of Horvath [5]. For more general result see also [11].

Theorem 1. - Ch. Horvath [5] *Let (X,h) be a compact pseudo-convex space. Let f be a real valued function defined on $X\times X$ such that:*
a) for each fixed $x\in X$, $f(x,y)$ is a lower semi-continous function of y on X;
b) for each fixed $y\in X$, $f(x,y)$ is a pseudo quasi-concave function of x on X;
Then, the following minimax inequality holds

$$\min_{y\in X} sup_{x\in X} f(x,y)\leq sup_{x\in X} f(x,x).$$

Let K be a subset of a topological vector space X. Then a map $G:K\to 2^X$ is called a KKM map if for each $A\in\mathbb{F}(K)$

$$\text{conv}(A)\subseteq\cup\{G(a):a\in A\}.$$

The following famous result of Ky Fan [8] (KKM Principle) will be used to prove the main result of this paper.

Theorem 2. - Ky Fan [8] *Let X be a topological vector space, K a nonempty subset of X and $G:K\to 2^X$ a KKM map with closed values. If $G(x)$ is compact for at least one $x\in K$, then*

$$\bigcap_{x\in K} G(x)\neq\emptyset.$$

3. Results

We start with the following variational inequality:

Theorem 3. *Let K be a nonempty convex and weakly compact subset of a reflexive Banach space X, $0<p\leq 1$ and $f:K\times K\to\mathbb{R}$ map such that:*
(i) $f(x,x)=0$ for all $x\in K$,
(ii) $x\mapsto f(x,y)$ is upper semicontinuous in the weak topology and p-convex for all $y\in K$,
(iii) $y\mapsto f(x,y)$ is $p-$ convex for all $x\in K$,
Then there exists $x_0\in K$ such that $f(x_0,y)\geq 0$.

Proof. Define $h:K^2\times[0,1]\longrightarrow X$ by

$$h(x,y,\lambda)=\lambda^{\frac{1}{p}}x_1+(1-\lambda)^{\frac{1}{p}}x_2,$$

for all $x_1,x_2\in K$ and $\lambda_1\in[0,1]$. Then (K,h) is pseudo-convex space, which is compact in weak topology.

Let g be a real valued function defined on $K\times K$ by $g(x,y)=-f(x,y)$. Then
i) for each fixed $x\in X$ set $\{y\in K:f(x,y)\leq 0\}$ is compact in weak topology, which implies that $g(x,y)$ is a lower semi-continuous function of y in weak topology of on K;
ii) for each fixed $y\in X$, $g(x,y)$ is a pseudo quasi concave function of x on (X,h).
From Theorem 1 it follows that

$$\min_{y\in K} sup_{x\in K} f(x,y)\leq sup_{x\in K} f(x,x)=0$$

which implies that there exists $x_0\in K$ such that

$$g(x_0,y)\leq 0,\ \text{for any } y\in K,$$

which implies that

$$f(x_0, y) \geq 0, \text{ for any } y \in K.$$

□

The following theorem is version of Minty's lemma for $p-$convex maps.

Theorem 4. *Let K be a nonempty closed convex subset of a topological vector space X, $0 < p \leq 1$ and $f : K \times K \to \mathbb{R}$ map such that:*
(i) $f(x, x) = 0$ for all $x \in K$,
(ii) $x \mapsto f(x, y)$ is upper semicontinuous for all $y \in K$,
(iii) $y \mapsto f(x, y)$ is $p-$convex for all $x \in K$,
(iv) $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in K$.
Then the following assertions are equivalent:
(a) there exists $x_0 \in K$ such that $f(x_0, y) \geq 0,$ for all $y \in K$;
(b) there exists $x_0 \in K$ such that $f(y, x_0) \leq 0$ for all $y \in K$.

Proof. Suppose that there exists a $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$. Then from condition (iv) it follows that (b) holds.

Conversely, suppose that there exists a $x_0 \in K$ such that

$$f(y, x_0) \leq 0 \text{ for all } y \in K.$$

For an arbitrary $x \in K$, letting $y_\lambda = \lambda y + (1 - \lambda)x_0, \lambda \in (0, 1)$, we have $y_\lambda \in K$ by the convexity of K. Hence

$$f(y_\lambda, x_0) \leq 0, \text{ for all } \lambda \in (0, 1). \tag{1}$$

From the conditions (i) and (iii) we obtain

$$0 = f(y_\lambda, y_\lambda) \leq \lambda^p f(y_\lambda, y) + (1 - \lambda)^p f(y_\lambda, x_0),$$

and hence,

$$f(y_\lambda, y) \geq -\left(\frac{1 - \lambda}{\lambda}\right)^p f(y_\lambda, x_0).$$

Therefore from (1) we obtain $f(y_\lambda, y) \geq 0$. Since the map $x \mapsto f(x, y)$ is upper semicontinuous for all $y \in K$, we obtain $f(x_0, y) \geq 0$ for all $y \in K$. □

Now we present our main result.

Theorem 5. *Let K be a nonempty closed convex and bounded subset of a reflexive Banach space X, $0 < p \leq 1$ and $f : K \times K \to \mathbb{R}$ map such that:*
(i) $f(x, x) = 0$ for all $x \in K$,
(ii) $x \mapsto f(x, y)$ is upper semicontinuous for all $y \in K$, $y \mapsto f(x, y)$ is lower semicontinuous for all $x \in K$,
(iii) $y \mapsto f(x, y)$ is $p-$convex for all $x \in K$,
(iv) $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in K$.
Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$.

Proof. Define $F : K \to 2^K$ by

$$F(y) = \{x \in K : f(x, y) \geq 0\}, \ y \in K.$$

Then $F(y)$ is nonempty for each $y \in K$ since $y \in F(y)$. We claim that F is a KKM-map. Suppose that there exists $A \in K$ such that $u \in \text{conv}(A)$ and real numbers $\lambda_a \in [0, 1]$, $a \in A$ such that $\sum_{a \in A} \lambda_a = 1$ and

$$u = \sum_{a \in A} \lambda_a a.$$

If

$$u \notin F(a) \text{ for all } a \in A,$$

we have

$$f(u, a) < 0 \text{ for all } a \in A.$$

From the $p-$convexity of the map $y \to f(u, y)$, it follows that

$$f(u, u) \leq \sum_{a \in A} \lambda_a^p f(u, a).$$

Thus, $f(u, u) < 0$, which contradicts (i). Therefore F is a KKM-map.

Now let us define

$$G : K \to 2^K \text{ by } G(y) = \{x \in K : f(y, x) \leq 0\}.$$

Then by condition (iv), $F(y) \subseteq G(y)$ for each $y \in K$. ¿From conditions (ii) and (iii) we have $G(y)$ closed and p-convex. Since K is weakly compact, so is $G(y)$ for all $y \in K$. Hence, by Theorem 2, there exists $x_0 \in K$ such that $f(y, x_0) \leq 0$ for all $y \in K$. Then it follows that there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$ for all $y \in K$. □

Corollary 1. *Let K be a nonempty closed convex and bounded subset of a reflexive Banach space X and $f : K \times K \to \mathbb{R}$ continuous map, such that:*

(i) $f(x, x) = 0$ for all $x \in K$,

(ii) $y \mapsto f(x, y)$ is a convex for all $x \in K$,

(iii) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in K$.

Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$.

Let X be a reflexive Banach space with its dual $X^\star$ and K closed convex subset of X. We denote the pairing between $X^\star$ and X by $\langle \cdot, \cdot \rangle$.

If in Theorem 4 and Corollary 1, one take

$$f(x, y) = \langle T(x), \theta(y, x) \rangle \text{ for all } x, y \in K,$$

where $T : K \to X^\star$ and $\theta : K \times K \to X$, we obtain result of A. Behera, G.K. Panda, [3] (Theorem 5. 1) and result of A. Behera, L. Nayak, [2] (Theorem 2.3).

If in Theorem 4 and Corollary 1 one take

$$f(x, y) = \langle T(x), \theta(y, x) \rangle - \langle A(x), \theta(y, x) \rangle \text{ for all } x, y \in K,$$

where $A : K \to X^\star$, we obtain result of G. K. Panda, N. Dash, [10] (Lema 2. 4 and Theorem 2.6).

Note that, in Theorem 3 the assumption, (ii) can be replaced by the following conditions,

$$\{x \in K : f(x,y) \geq 0\} \text{ is closed for all } y \in K,$$
$$\{x \in K : f(x,y) \leq 0\} \text{ is closed for all } y \in K,$$

and the assumptions,

$$y \mapsto f(x,y) \text{ is a p-convex for all } x \in K,$$
$$f(x,y) \geq 0 \text{ implies } f(y,x) \leq 0 \text{ for all } x, y \in K,$$

can be replaced by the conditions:

(a) $\{x \in K : F(x,y) \geq 0\}$ is convex for all $y \in K$,

(b) $\{y \in K : F(x,y) < 0\}$ is convex for all $x \in K$.

In this case as corollary we obtain result of Q.H. Ansari and J.C. Yao, [1] (Theorem 2.1).

Example 1. Let $X = \mathbb{R}, K = [0,1]$ and define a map $f : K \times K \to \mathbb{R}$ with

$$f(x) = \sqrt{y} - \sqrt{x}.$$

Then $y \mapsto f(x,y)$ is a $\frac{1}{2}-$convex for all $x \in K$ and f satisfies all hypotheses in Theorem 5. Note that set $\{y \in K : f(x,y) < 0\}$ is not convex and hence Theorem 2.1 in [1] is not applicable.

Lemma 1. *Let X be a Hilbert space, K a nonempty subset of X and $G : K \to X$ a KKM map with convex closed values. If $G(x)$ is bounded for at least one $x \in K$, then* $\bigcap_{x \in K} G(x) \neq \emptyset$.

Proof. If $G(x)$ is convex, closed and bounded subset of Hilbert space it is compact in the weak topology. □

When K is not bounded, using Lemma 2 and Theorem 4, we obtain the following version of Theorem 5 for Hilbert spaces.

Theorem 6. *Let K be a nonempty closed convex subset of a Hilbert space X, $0 < p \leq 1$ and $f : K \times K \to \mathbb{R}$ a continuous map such that*

(i) $f(x,x) = 0$ for all $x \in K$,

(ii) $\{y \in K : f(x,y) \leq 0\}$ is bounded for at least one $x \in K$,

(iii) $y \mapsto f(x,y)$ is a $p-$convex for all $x \in K$,

(iv) $f(x,y) \geq 0$ implies $f(y,x) \leq 0$ for all $x, y \in K$.

Then there exists $x_0 \in K$ such that $f(x_0, y) \geq 0$.

4. Applications

From Theorem 4 we have the following fixed point theorem and coincidence point theorem for a Hilbert spaces.

Theorem 7. *Let K be a nonempty closed convex and bounded subset of a Hilbert space X, $0 < p \leq 1$ and $f, g : K \to X$ continuous maps such that:*

(i) for each $x \in K$ with $fx \neq gx$ the line segment $[fx, gx]$ contains at least two points of $g(K)$,

(ii) $y \mapsto \langle x, gy \rangle$ *is a* $p-$*convex for all* $x \in K$*,*
(iii) $\langle gx - fx, gy - gx \rangle \geq 0$ *implies* $\langle gy - fy, gx - gy \rangle \leq 0$ *for all* $x, y \in K$*.*
Then there exists $x_0 \in K$ *such that* $f(x_0) = g(x_0)$*.*

Proof. Define
$$f(x, y) = \langle gx - fx, gy - gx \rangle, \text{ for } x, y \in K.$$
Then, from Theorem 1, it follows that there exists $x_0 \in K$ such that
$$\langle gx_0 - fx_0, gy - gx_0 \rangle \geq 0, \text{ for all } y \in K.$$
If $fx_0 \neq gx_0$, by condition (i), we conclude that for some $y_0 \in K$ and $\lambda \in (0, 1)$ we have $gy_0 = \lambda fx_0 + (1 - \lambda)gx_0$. Thus, $-\lambda||gx_0 - fx_0||^2 \geq 0$, and hence $gx_0 = fx_0$. □

Corollary 2. *Let* K *be a nonempty closed convex and bounded subset of a Hilbert space* X *and* $f, g : K \to K$ *continuous maps such that:*
(i) $g(K) = K$*,*
(ii) $y \mapsto \langle x, gy \rangle$ *is a convex for all* $x \in K$*,*
(iii) $\langle gy - gx, fy - fx \rangle \leq ||gy - gx||^2$ *for all* $x, y \in K$*.*
Then there exists $x_0 \in K$ *such that* $f(x_0) = g(x_0)$*.*

Corollary 3. *Let* K *be a nonempty closed convex and bounded subset of a Hilbert space* X *and* $f : K \to K$ *continuous map such that*
$$\langle fy - fx, y - x \rangle \leq ||y - x||^2 \text{ for all } x, y \in K.$$
Then there exists $x_0 \in K$ *such that:* $f(x_0) = x_0$*.*

Note that, Corollary 3 is the famous Browder-Göhde-Kirk fixed point theorem, see for example [13], [14].

Finally we give the following existence of the best approximations point in the Hilbert spaces, see for example [13].

Theorem 8. *Let* K *be a nonempty closed convex subset of a Hilbert space* X*,* $0 < p \leq 1$ *and* $f, g : K \to X$ *continuous maps such that:*
(i) $g(K) = K$*,*
(ii) $\{y \in K : ||gy - fx|| \leq ||gx - fx||\}$ *is bounded for at least one* $x \in K$*,*
(ii) $y \mapsto ||gy - fx||^2$ *is a* $p-$*convex for all* $x \in K$*,*
(iii) $\langle fy - fx, gy - gx \rangle \leq 0$ *for all* $x, y \in K$*.*
Then there exists $x_0 \in K$ *such that* $||gx_0 - fx_0|| = \inf_{y \in K} ||y - fx_0||$*.*

Proof. Take $f(x, y) = ||gy - fx||^2 - ||gx - fx||^2$ for all $x, y \in K$ in Theorem 2. □

Corollary 4. *Let* K *be a nonempty closed convex subset of a Hilbert space* X *and* $f : K \to X$ *continuous map, such that* $\langle fy - fx, y - x \rangle \leq 0$ *for all* $x, y \in K$*. Then, there exists a* $x_0 \in K$ *such that* $||x_0 - fx_0|| = \inf_{y \in K} ||y - fx_0||$*.*

Proof. Take $f(x, y) = ||y - fx||^2 - ||x - fx||^2$ for all $x, y \in K$ in Theorem 8. □

Corollary 5. *Let* K *be a nonempty closed convex subset of a Hilbert space* X *and* $g : K \to X$ *continuous map, such that* $x + gx \in K$ *for all* $x \in K$ *and*
$$\langle gy - gx, y - x \rangle \leq -||y - x||^2 \text{ for all } x, y \in K.$$

Then there exists a unique $x_0 \in K$ *such that* $gx_0 = 0$.

Proof. Take $fx = x + gx$ for all $x \in K$ in Corollary 4. □

5. An open problem

Famous BanachAlaoglu theorem [12] states that every closed bounded and convex subset of subset of a reflexive Banach space is weakly compact.

Problem 1. Let $0 < p \leq 1$. Is every closed bounded and p-convex subset of a reflexive Banach space *weakly compact*?

A positive answer to this question give many new applications of Theorem 3.

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