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MANN ITERATIONS VIA STATISTICALLY REGULAR MATRICES

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Abstract. In 1953, W.R. Mann introduced the generalized iteration method by means of a non-negative regular summability matrix. We consider this iteration process in the case when the defining matrix is statistically regular.
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1. Introduction and preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural, real and complex numbers, respectively, and let X be a Banach space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $s(X)$ be the vector space of all X-valued sequences $x = (x_k) = (x_k)_{k\in\mathbb{N}}$. The sets of all bounded, convergent and convergent to zero sequences in X are denoted by $m(X)$, $c(X)$ and $c_0(X)$, respectively. For $X = \mathbb{K}$ we write m, c and c_0 instead of $m(X)$, $c(X)$ and $c_0(X)$. We also use the symbols $\sup_n$, $\lim_n$ and $\sum_k$ instead of $\sup_{n\in\mathbb{N}}$, $\lim_{n\to\infty}$ and $\sum_{k=1}^{\infty}$, respectively. By an index sequence (or index set) we shall mean a strictly increasing sequence of natural numbers.

Let $\lambda(X)$ and $\mu(X)$ be two linear subspaces of $s(X)$. We say that an infinite scalar matrix $A = (a_{nk})$ (or matrix method of summability A) maps $\lambda(X)$ into $\mu(X)$, briefly $A \in (\lambda(X), \mu(X))$, if for any $x = (x_k) \in \lambda(X)$ the series $A_n x = \sum_k a_{nk} x_k \quad (n \in \mathbb{N})$ converge in X and the sequence $Ax = (A_n x)$ is in $\mu(X)$. If the spaces $\lambda(X)$ and $\mu(X)$ are equipped with the limits λ-lim and μ-lim, respectively, and $A \in (\lambda(X), \mu(X))$ satisfies the equality $\mu\text{-}\lim_n A_n x = \lambda\text{-}\lim_k x_k$ for any $x \in \lambda(X)$, then we say that A maps $\lambda(X)$ into $\mu(X)$ regularly and write $A \in (\lambda(X), \mu(X))_{reg}$. The matrices (or matrix methods) $A \in (c(X), c(X))_{reg}$ are called *regular* in X. A matrix $A = (a_{nk})$ is called triangular if $a_{nk} = 0$ for $k > n$.

From the classical theory of summability it is well known that a matrix A is regular in $\mathbb{K}$, i.e., $A \in (c, c)_{reg}$, if and only if (see, e.g., [14, Theorem 1.3.3], [17, Theorem 32.I]

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or [16])

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \tag{1}$$

$$\lim_n a_{nk} = 0 \quad (k \in \mathbb{N}), \tag{2}$$

$$\lim_n \sum_k a_{nk} = 1. \tag{3}$$

The conditions (1) − (3) characterize the regularity of A also in any Banach space (and, generally, in any sequentially complete locally convex Hausdorff topological vector space) X (cf. [7], [13]). Thus we can formulate

Proposition 1. *A matrix $A = (a_{nk})$ is regular in a Banach space X if and only if the conditions* (1) − (3) *are satisfied.*

Analogously, basing on the characterization of matrix class (m, c_0) (see [14, Corollary 2 of Theorem 1.5.2] or [16]), in view of [7, Remark 2] we have

Proposition 2. *For a Banach space X, $A \in (m(X), c_0(X))$ if and only if $\sum_k |a_{nk}|$ converges for every $n \in \mathbb{N}$ and*

$$\lim_n \sum_k |a_{nk}| = 0.$$

Generally speaking, a sequence is statistically convergent to an element l if almost all of its values are close to l where "almost all" is defined using a finitely additive set function. This notion has been in the literature, under different guises, since at least 1913. Fast [4] introduced the definition of statistical convergence of number sequences using the asymptotic density of sets of positive integers. If $K \subseteq \mathbb{N}$, then the *asymptotic density* of K, denoted $\delta(K)$, is given by

$$\delta(K) = \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

where $|N|$ denotes the cardinality of the set N. A sequence $x = (x_k) \in s$ is called *statistically convergent* to a number l if

$$\delta(\{k : |x_k - l| > \varepsilon\}) = 0, \text{ for every } \varepsilon > 0.$$

The notion of statistical convergence, as defined by Fast, has been extended in different ways. For instance, Maddox [11] considered the statistical convergence of sequences taking values in a locally convex space. So, in the case of a Banach space X, a sequence $x = (x_k) \in s(X)$ is said to be *statistically convergent* to $l \in X$ if

$$\text{for every } \varepsilon > 0, \quad \delta(\{k : \|x_k - l\| > \varepsilon\}) = 0.$$

In this note we use an extension of the last definition of statistical convergence where the asymptotic density is replaced with one generated by a non-negative matrix.

Let $A = (a_{nk})$ be a regular matrix with the elements $a_{nk} \geq 0$ $(n, k \in \mathbb{N})$. A set $K \subset \mathbb{N}$ is said to have *A-density* $\delta_A(K)$ if the limit

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{nk}$$

exists (see [5]). A sequence $x = (x_k) \in s(X)$ is called *A-statistically convergent* to a point $l \in X$, briefly $st(A)\text{-}\lim x = l$ (see [8], [1], [9]), if

$$\delta_A(\{k : \|x_k - l\| > \varepsilon\}) = 0, \text{ for every } \varepsilon > 0.$$

Let $st(A, X)$ denote the set of all A-statistically convergent X-valued sequences. We remark that $st(A, X)$ is a linear subspace of $s(X)$ and

$$c(X) \subseteq st(A, X). \tag{4}$$

If A is the identity matrix, then A-statistical convergence is just the ordinary convergence in X and if A is the Cesàro matrix $C_1 = (c_{nk})$ (i.e. $c_{nk} = 1/n$ if $k \le n$ and $c_{nk} = 0$ if $k > n$), then A-statistical convergence is statistical convergence as defined above. It is also known (see [9] or [10]) that the inclusion (4) is strict whenever $\lim_n a_{nk} = 0$ uniformly in $k \in \mathbb{N}$.

Let $bst(A, X)$ be the set of all bounded A-statistically convergent X-valued sequences, i.e., $bst(A, X) = st(A, X) \cap m(X)$. The following definition extends the notion of a regular matrix.

Definition 1. *An infinite scalar matrix $B = (b_{nk})$ is called A-statistically regular in a Banach space X if $B \in (bst(A, X), bst(A, X))_{reg}$, i.e., if for every $x = (x_k) \in bst(A, X)$ the series*

$$B_n x = \sum_k b_{nk} x_k \quad (n \in \mathbb{N})$$

converge and $st(A)\text{-}\lim Bx = st(A)\text{-}\lim x$ in X.

Now let $B = (b_{nk})$ be a non-negative triangular matrix such that

$$\sum_{k=1}^{n} b_{nk} = 1 \quad (n \in \mathbb{N}). \tag{5}$$

Suppose E is a closed convex subset of a Banach space X, $x_1 \in E$ and T is a mapping of E into itself. Then the *Mann iterative process* $M(x_1, B, T)$ is defined by

$$x_{n+1} = T(v_n), \quad v_n = \sum_{k=1}^{n} b_{nk} x_k \quad (n \in \mathbb{N}).$$

W.R. Mann introduced this process and proved the following proposition (see [12, Theorem 1]).

Proposition 3. *If T is continuous, then the convergence of either (x_n) or (v_n) to a point p implies the convergence of the other to p, and also implies the equality $Tp = p$.*

Our purpose is to extend this result to the statistical convergence. We consider Mann iterative process $M(x_1, B, T)$ by the assumption that the matrix B is A-statistically regular.

2. Statistically regular matrices and statistically continuous mappings

Recall that the statistical convergence has the following important characterization [15, 6, 9].

Proposition 4. *A sequence* (x_k) *converges A-statistically to* l *in a Banach space* X *if and only if there exists an index sequence* $K = \{k_i\}$ *with* $\delta_A(K) = 1$ *such that the subsequence* (x_{k_i}) *converges to* l *in* X.

Let $bst_0(A, X) = m(X) \cap st_0(A, X)$ where $st_0(A, X)$ denotes the set of all A-statistically convergent to zero X-valued sequences. For a matrix $B = (b_{nk})$ and a set $K \subset \mathbb{N}$ let $B^{[K]} = (d_{nk})$ where for all $n \in \mathbb{N}$, $d_{nk} = b_{nk}$ if $k \in K$ and $d_{nk} = 0$ otherwise. Because of Proposition 4, similarly to [10, Theorem 4.1] we get

Proposition 5. *A matrix* $B = (b_{nk})$ *is A-statistically regular in a Banach space* X, *i.e.,* $B \in (bst(A, X), bst(A, X))_{reg}$, *if and only if*

$$B \in (c(X), bst(A, X))_{reg},$$
$$B^{[K]} \in (m(X), bst_0(A, X)) \quad (\delta_A(K) = 0).$$

This result allows to give effective sufficient conditions for the statistical regularity.

Theorem 1. *A matrix* $B = (b_{nk})$ *is A-statistically regular in a Banach space* X *if* $\|B\| < \infty$ *and there exists an index sequence* $N = (n_j)$ *with* $\delta_A(N) = 1$ *such that*

$$\lim_j b_{n_j k} = 0 \quad (k \in \mathbb{N}), \tag{6}$$

$$\lim_j \sum_k b_{n_j k} = 1, \tag{7}$$

$$\lim_j \sum_{k \in K} |b_{n_j k}| = 0 \quad (\delta_A(K) = 0). \tag{8}$$

Proof. If $\|B\| < \infty$, then the transformed sequence $Bx = (B_n x)$ is determined for every $x \in m(X)$. Moreover, $B \in (c(X), m(X))$ and $B^{[K]} \in (m(X), m(X))$ for any $K \subset \mathbb{N}$. On the basis of Proposition 5 it remains to prove that $B \in (c(X), st(A, X))_{reg}$ and $B^{[K]} \in (m(X), st_0(A, X))$ for any $K \subset \mathbb{N}$ with $\delta_A(K) = 0$.

Let $x = (x_k) \in c(X)$ with $\lim_k x_k = l$. If (6) and (7) hold, then the submatrix $B_{(N)} = (b_{n_j k})$ is regular in X, so $B_{(N)} x$ converges to l by Proposition 1 and $st_A(X)$-$\lim Bx = l$ by Proposition 4. Consequently, $B \in (c(X), st(A, X))_{reg}$.

Now let $K \subset \mathbb{N}$ with $\delta_A(K) = 0$. Since for the matrix $B^{[K]} = (d_{nk})$ we have $\sum_k |d_{n_j k}| < \infty \quad (j \in \mathbb{N})$ and $\lim_j \sum_k |d_{n_j k}| = 0$, by Proposition 2 we get $(d_{n_j k}) \in (m(X), c_0(X))$ which clearly gives $B^{[K]} \in (m(X), st_0(A, X))$. □

The matrices A and B in Theorem 1 are connected with (8). If the submatrix $B_{(N)} = (b_{n_j k})$ is non-negative, then (8) can be rewritten as

$$\delta_A(K) = 0 \implies \delta_{B_{(N)}}(K) = 0. \tag{9}$$

So from Theorem 1 we immediately get the following two corollaries.

Corollary 1. *A non-negative matrix $B = (b_{nk})$ is A-statistically regular in X if $\|B\| < \infty$ and there exists an index set N with $\delta_A(N) = 1$ such that $B_{(N)}$ is regular and (9) is true.*

Corollary 2. *A matrix $B = (b_{nk})$ is A-statistically regular in X if $\|B\| < \infty$ and there exists an index set N with $\delta_A(N) = 1$ such that $B_{(N)} = A$.*

Mann iteration process $M(x_1, B, T)$ is applicable to continuous mappings T. Following [2], we introduce the notion of statistically continuous mapping.

Definition 2. *Let E be a subset of a Banach space X. A mapping $T : E \to X$ is called A-statistically continuous at the point $l \in E$ if for any sequence $x = (x_k) \in s(E)$ with $st(A)$-$\lim x = l$ the sequence $(T(x_k))$ converges A-statistically to $T(l)$.*

Demirci [3, Lemma 1] (see also [2, Proposition 4]) proved that every continuous real function is A-statistically continuous. We show that this is true also for our mappings T.

Theorem 2. *Let E be a subset of a Banach space X. Every continuous mapping $T : E \to X$ is A-statistically continuous.*

Proof. Let $l \in E$ and let $x = (x_k) \in s(E)$ with $st(A)$-$\lim x = l$. Basing on Proposition 4, we fix an index sequence $K = (k_i)$ such that $\delta_A(K) = 1$ and $\lim_i x_{k_i} = l$. Then, by the continuity of T at l, we have $\lim_i T(x_{k_i}) = T(l)$ which, by Proposition 4, shows that the sequence $(T(x_k))$ is A-statistically convergent to $T(l)$. □

3. Mann iteration process for statistically regular matrices

Let $A = (a_{nk})$ be a non-negative regular matrix. Suppose $B = (b_{nk})$ is a triangular non-negative matrix satisfying (5), (6) and (8), i.e., $b_{nk} = 0 \;\; (k > n)$, $\sum_{k=1}^{n} b_{nk} = 1 \;\; (n \in \mathbb{N})$ and there exists an index sequence $N = (n_j)$ with $\delta_A(N) = 1$ such that $\lim_j b_{n_j k} = 0$ and $\lim_j \sum_{k \in K} b_{n_j k} = 0$ whenever $\delta_A(K) = 0$. By Theorem 1, matrix B is A-statistically regular. We consider Mann iterative process $M(x_1, B, T)$ under the assumption that E is a closed, convex and bounded subset of a Banach space X and the mapping $T : E \to E$ is continuous.

Theorem 3. *Let $x = (x_n)$ and $v = (v_n)$ be sequences defined by Mann iterative process $M(x_1, B, T)$. If either of the sequences x and v converges A-statistically to a point p, then the other sequence also converges A-statistically to p, and $T(p) = p$.*

Proof. Let $st(A)$-$\lim x = p$. Since x is bounded and B is A-statistically regular, $st(A)$-$\lim v = p$. By Theorem 2, T is A-statistically continuous, so the sequence $(T(v_n))$ converges A-statistically to $T(p)$. The equality $T(p) = p$ follows from $T(v_n) = x_{n+1}$.

Conversely, if $st(A)$-$\lim v = p$, then $st(A)$-$\lim(x_{n+1}) = st(A)$-$\lim(T(v_n)) = T(p)$ and by the A-statistical regularity of B, $st(A)$-$\lim v = st(A)$-$\lim(B_n x) = T(p)$. Hence, $T(p) = p$. □

Remark 1. *If A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence in X and Theorem 3 reduces to Proposition 3.*

If $A = C_1$, then the A-density δ_A is just the asymptotic density δ and the A-statistical convergence is the statistical convergence as defined by Fast. For example, the sets $K = \{2^i : i \in \mathbb{N}\}$ and $K' = \{2^i - 1 : i \in \mathbb{N}\}$ have asymptotic density zero, i.e., $\delta(K) = \delta(K') = 0$. Let $N = \mathbb{N} \setminus K$, then $\delta(N) = 1$. Using Cesáro matrix $C_1 = (c_{nk})$, we consider new non-negative regular matrix $H = (h_{nk})$ such that $(h_{nk})_{n\in\mathbb{N},k\in\mathbb{N}\setminus K} = C_1$ and $h_{n,2^i} = 0 \quad (n, i \in \mathbb{N})$, and define triangular matrix $G = (g_{nk})$ by the equalities $(g_{nk})_{n\in\mathbb{N}\setminus K',k\in\mathbb{N}} = H$ and

$$g_{2^i-1,k} = \begin{cases} 1, & \text{if } k = 1,\ i \in \mathbb{N}, \\ 0, & \text{if } k \geq 2,\ i \in \mathbb{N}. \end{cases}$$

Since $\lim_n g_{n1} \neq 0$, the matrix G is not regular. But G is H-statistically regular by Corollary 2 because of $G_{(\mathbb{N}\setminus K')} = H$ and $\delta_H(K') = 0$. We claim that G is also statistically regular. Indeed, a simple calculation shows that

$$\sup_n \sum_k (k+1)|h_{nk} - h_{n,k+1}| < \infty.$$

Thus (see [17, Theorem 52.I]) $\lim_n (C_1)_n x = 0$ implies $\lim_n H_n x = 0$ and G is statistically regular by Corollary 1.

Denoting $N = \{n_j\}$, it is easy to see that the iterations x_n in Mann process $M(x_1, G, T)$ are determined by

$$x_{n_{j+1}} = T(v_{n_j}), \quad v_{n_j} = \frac{1}{j}\sum_{k=1}^{j} x_{n_k} \quad (j \in \mathbb{N})$$

and

$$x_k = T(x_1) \quad (k \in K). \tag{10}$$

Consider now the particular case where $X = \mathbb{R}$ and E is a closed bounded interval $[a, b]$. The following special result is obtained.

Theorem 4. *Let $f : [a, b] \to [a, b]$ be a continuous function. If f has a unique fix point p, then $M(x_1, G, f)$ converges statistically to p for all choices of x_1 on $[a, b]$.*

Proof. As in [12, Theorem 4] we can prove that $\lim_i v_{n_i} = p$. Since $\delta(N) = 1$, the whole sequence (v_n) converges statistically to p in view of Proposition 4. By Theorem 3, the sequence (x_n) converges statistically to p also. □

Remark 2. *By (10) we conclude that the sequence (x_n) diverges if $f(x_1)$ is not the fix point of f.*

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