

## METRIC FIXED POINT THEORY: OLD PROBLEMS AND NEW DIRECTIONS

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*Dedicated to Wataru Takahashi on the occasion of his retirement*

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**Abstract.** This is a brief review of some of the things, both past and present, which have motivated the writer's interest in metric fixed point theory.

**Key Words and Phrases:** Nonexpansive mappings, common fixed points,  $\mathbb{R}$ -trees, product spaces  
**2000 Mathematics Subject Classification:** 47H09, 47H10, 54H25, 05C05.

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The paper was presented at The 9th International Conference on Fixed Point Theory and Its Applications, July 16-22, 2009, National Changhua University of Education, Changhua, Taiwan (R.O.C.).

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*Received: 31.12.2009; Accepted:10.02.2010.*