Fixed Point Theory, 11(2010), No. 1, 45-58
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

METRIC FIXED POINT THEORY: OLD PROBLEMS AND NEW DIRECTIONS

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. This is a brief review of some of the things, both past and present, which have motivated the writer's interest in metric fixed point theory.
Key Words and Phrases: Nonexpansive mappings, common fixed points, $\mathbb{R}$-trees, product spaces
2000 Mathematics Subject Classification: 47H09, 47H10, 54H25, 05C05.

1. Introduction

Mathematical research often focuses on problems lying on the boundary of an area of known results, with a view to enlarging that area. This strategy is designed to assure some success. However there are two other strategies that can be even more rewarding when successful. One is the study of old and seemingly difficult problems. Another is the search for genuinely new research directions. In this paper we discuss how both approaches have have influenced the writer's career.

This paper is largely expository, and much of the material in Section 3 has already appeared in [28].

2. Old Problems

In this section we discuss three papers that had a strong impact in stimulating the writer's early interest in fixed point theory. The first is a paper by Ralph DeMarr [14] about commuting continuous mappings on the unit interval, the second is the fundamental paper of Brodskii and Milman [9] that gave us the concept of 'normal structure', and the third is another paper by Ralph DeMarr [13] about commuting families of nonexpansive mappings. These papers triggered a huge amount of further research in the past and to this day leave many related questions open.

The paper was presented at The 9th International Conference on Fixed Point Theory and Its Applications, July 16-22, 2009, National Changhua University of Education, Changhua, Taiwan (R.O.C.).

2.1. **Commuting maps.** In 1954 Eldon Dyer posed the following question: If two continuous mappings $f, g : [0,1] \to [0,1]$ commute, do they necessarily have a common fixed point? A. L. Shields posed the same question in 1955, as did Lester Dubins in 1956 and the problem first appeared in the literature as part of a more general question posed by J. R. Isbell in 1957 [23]. This question immediately attracted a lot of attention, in part because it had long been known that the answer is yes for polynomial functions. In 1963 Ralph DeMarr [14] gave a partial positive answer, and another partial answer to the question was established by Schwartz in 1965, who showed in [40] that if f has a continuous derivative, then there is a common fixed point of f and some iterate of g. However the general question resisted a complete solution for over ten years. In 1967 both W. M. Boyce [7] and H. Huneke [22] gave counterexamples to the problem. Thus the answer to the original topological question is negative, but it gave rise to an interesting positive result. (For a more thorough survey of this topic, see Chapter 7 of [39].)

We now state DeMarr's result. On the surface DeMarr's result appears to be a metric theorem. However an inspection of the proof reveals that it has a strong order-theoretic component as well. For the convenience of the reader we include the proof.

Theorem 1 (DeMarr [14])**.** *Let f and g be two commuting continuous mappings of a closed interval I into itself having respective Lipschitz constants α and β satisfying $\beta(\alpha - 1) < (\alpha + 1)$. Then f and g have a common fixed point.*

Proof ([14]*).* Let N be the set of fixed points of g. Then N is closed and nonempty, and by commutativity, $f : N \to N$. Assume f does not have a fixed point in N. Let $a = \min N$ and $b = \max N$. Then $a < f(a)$ and $f(b) < b$. Now pick $x_0, x_1 \in N$ such that $x_0 < f(x_0)$, $f(x_1) < x_1$, $x_0 < x_1$ and $x_1 - x_0$ is minimal. This is possible because N is compact and f is continuous.

Set $y_0 = f(x_0)$ and $y_1 = f(x_1)$. We will show that $y_0 \geq x_1$ and $y_1 \leq x_0$. Suppose $z \in N$ satisfies $x_0 < z < x_1$. Then $f(z) < z$ or $f(z) > z$, Suppose $f(z) < z$. Then $x_0 < z$, $f(z) < z$, and $x_0 < z$. However $z - x_0 < x_1 - x_0$, contradicting minimality of $x_1 - x_0$. A similar contradiction arises if $f(z) > z$. Hence no point of N lies in the open interval (x_0, x_1). This in turn implies that $g(x) < x$ for all $x \in (x_0, x_1)$ or $x < g(x)$ for all $x \in (x_0, x_1)$. Without loss of generality assume $x < g(x)$ for all $x \in (x_0, x_1)$.

Since $x_0 < f(x_0)$ and $f(x_1) < x_1$, it must be the case that f has a fixed point in (x_0, x_1). Let s be the largest fixed point of f in (x_0, x_1), and let $t = g(s)$. Then $f(t) = t$ because g maps the fixed point set of f into itself, and since $s < t$, it must be the case that $x_1 < t$.

We now assume $\alpha > 1$. Since $f(x_1) \leq x_0$ we now have

$$t - x_0 \leq f(t) - f(x_1) \leq \alpha(t - x_1). \tag{1}$$

Also,

$$t - x_0 = g(s) - g(x_0) \leq \beta(s - x_0); \tag{2}$$

$$s - x_0 \leq f(s) - f(x_1) \leq \alpha(s - x_1). \tag{3}$$

From (1)

$$\alpha x_1 - x_0 \leq (\alpha - 1)\, t. \tag{4}$$

From (2) and (4):

$$\alpha x_1 - x_0 \leq (\alpha - 1)\left[\beta\,(s - x_0) + x_0\right];$$

hence $\alpha x_1 \leq (\alpha - 1)\,\beta\,(s - x_0) + \alpha x_0$, from which

$$\alpha\,(x_1 - x_0) \leq \beta\,(\alpha - 1)\,(s - x_0)\,. \tag{5}$$

From (3) $s - x_0 \leq \alpha\left[(x_1 - x_0) - (s - x_0)\right]$; hence

$$(\alpha + 1)\,(s - x_0) \leq \alpha\,(x_1 - x_0)\,. \tag{6}$$

From (5) and (6)

$$\alpha + 1 \leq \beta\,(\alpha - 1)$$

and this contradicts our initial assumption

$$\beta\,(\alpha - 1) < (\alpha + 1)\,.$$

□

The preceding proof assumed $\alpha > 1$. If $\alpha < 1$ then f is a contraction mapping which has a unique fixed point. If g commutes with f then this point is also fixed under g. In this case no further assumptions on g are needed. The remaining case is $\alpha = 1$. In this case f is nonexpansive and its fixed point set is a closed interval which is invariant under g. Thus if g is merely continuous, then f and g have a common fixed point.

There seems to be very little known about commuting maps on the unit interval beyond DeMarr's result, but two papers might be worth mentioning. In [33] it is shown that commuting continuous mappings f, g of $I \to I$ have a common fixed point if either of the following two conditions is satisfied: (1) the fixed points of f^2 coincide with those of f; (2) there exists an infinite sequence $\{I_n\}$ of disjoint open intervals each of which has as endpoints consecutive points in the set of fixed points of f, and there is a fixed point c of g, such that for each I_n there is some integer m such that $f^m\,(c) \in I_m$. Another result involving Lipschitz-type conditions is given in [24].

It is also interesting to note that the mappings in Huneke's counterexample have Lipschitz constant $3 + \sqrt{6}$, so the restriction on the Lipschitz constants in DeMarr's result is significant.

The first question we raise in this paper is whether DeMarr's Theorem is strictly a result about a real line interval, or do analogs exist in a wider context. We consider possible extensions below.

2.2. **$\mathbb{R}$-trees.** In [6] it is shown that if X is a finite tree (in the topological sense), if $f_\alpha : X \to X$ is a commuting family of open maps, and if $g : X \to X$ is a continuous map which commutes with each f_α, then they have a common fixed point. For corresponding metric results we look to $\mathbb{R}$-trees.

Definition 1. An $\mathbb{R}$-*tree* is a metric space M such that for every x and y in M there is a unique arc between x and y and this arc is isometric to an interval in $\mathbb{R}$ (i.e., is a geodesic segment).

Standard examples of $\mathbb{R}$-trees include the 'radial' and 'river' metrics on $\mathbb{R}^2$. For the radial metric, consider all rays emanating from the origin in $\mathbb{R}^2$. Define the radial distance d_r between $x, y \in \mathbb{R}^2$ as follows:

$$d_r(x, y) = d(x, 0) + d(0, y).$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) For the river metric ρ, if two points x, y are on the same vertical line, define $\rho(x, y) = d(x, y)$. Otherwise define $\rho(x, y) = |x_2| + |y_2| + |x_1 - y_1|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

More subtle examples of $\mathbb{R}$-trees exist; e.g., the real tree of Dress and Terhalle [15].

The following result follows from a more general 1946 result of G. S. Young (see [45], Theorem 16). Young explicitly points out in [46] that compactness is not needed. There is a more constructive metric proof of this result is given in [27].

Theorem 2. *A complete geodesically bounded $\mathbb{R}$-tree has the fixed point property for continuous maps.*

Question 1. *Let X be a compact or, more generally, a geodesically bounded and complete $\mathbb{R}$-tree, and let $f, g : X \to X$ be commuting and lipschitzian. Then if the respective Lipschitz constants of f and g are sufficiently near* 1, *do f and g have a common fixed point?*

A very special case of Question 1 has an affirmative answer.

Theorem 3. *Let X be a geodesically bounded complete $\mathbb{R}$-tree, let g be a nonexpansive mapping of X into itself, and suppose $f : X \to X$ is continuous and commutes with g. Then f and g have a common fixed point.*

Proof. By Theorem 2 g has a nonempty fixed point set N, and it is easy to see that N is closed and convex. Since f commutes with g, $f : N \to N$, and again by Theorem 2 f has a fixed point in N. □

Theorem 3 can be extended a little further. For this result it will be necessary to discuss another concept.

A metric space X is said to be *hyperconvex* (Aronszajn and Panitchpakdi [1]) if every family $\{B(y_\alpha; r_\alpha)\}_{\alpha \in A}$ of closed balls centered at $y_\alpha \in X$ with radii $r_\alpha \geq 0$ has nonempty intersection whenever

$$d(y_\alpha, y_\beta) \leq r_\alpha + r_\beta \ \forall \ \alpha, \beta \in A.$$

It is known that compact hyperconvex spaces (often called *Helly spaces*) are contractible and locally contractible; hence they have the fixed point property for continuous mappings (see [36]). It is important to note that if a hyperconvex metric space X is a subset of a metric space Y then there is always a *nonexpansive retraction* of Y onto X. A discussion of this fact and many more facts about hyperconvex spaces can be found in [16]. We also refer the reader to [5] for a survey that emphasizes applications of $\mathbb{R}$-trees.

For the remarks that follow we shall need the following fact.

Theorem 4 ([27])**.** *For a metric space X the following are equivalent:* (i) *X is a complete $\mathbb{R}$-tree;* (ii) *X is hyperconvex and has unique metric segments.*

For a mapping $g : X \to X$ and each $\delta > 0$, let

$$F_\delta(g) = \{x \in X : d(x, g(x)) \leq \delta\}.$$

Theorem 5. *Let X be a complete $\mathbb{R}$-tree and let g be commuting nonexpansive mapping of X into itself, and suppose $f : X \to X$ is a continuous mapping which commutes with g. If $F_\delta(g)$ is geodesically bounded for some $\delta > 0$, then f and g have a common fixed point.*

Proof. By Theorem 4 X is hyperconvex, and it is known that approximate fixed point sets of nonexpansive mappings in hyperconvex spaces are themselves hyperconvex (Sine[42]). Thus $F_\delta(g)$ is hyperconvex. Also, again by Theorem 4, a hyperconvex subspace of a complete $\mathbb{R}$-tree is also a complete $\mathbb{R}$-tree. Thus $F_\delta(g)$ is a geodesically bounded complete $\mathbb{R}$-tree. Since $g : F_\delta(g) \to F_\delta(g)$, g has a nonempty fixed point set N, which must be convex and geodesically bounded (since it lies in $F_\delta(g)$). Since $f : N \to N$ the conclusion follows from Theorem 2. □

It should be noted that the assumption that $F_\delta(g)$ is geodesically bounded is strictly weaker than the assumption that X is geodesically bounded.

In [17] it is proved (Theorem 4.3) that if X is a geodesically bounded $\mathbb{R}$-tree then every commuting family of nonexpansive mappings of $X \to X$ has a nonempty common fixed point set. The above approach yields the following (slight) extension of that result.

Theorem 6. *Let X be a complete $\mathbb{R}$-tree and let $\mathfrak{F}$ be a commuting family of nonexpansive mappings of $X \to X$. If $F_\delta(g)$ is bounded for some $g \in \mathfrak{F}$, then $\mathfrak{F}$ has a nonempty common fixed point set.*

Proof. Proceed as in the preceding proof and conclude that $f : N \to N$ for each $f \in \mathfrak{F}$. The conclusion now follows from Theorem 4.3 of [17] applied to N. □

If the answer to Question 1 is negative then another question arises, namely to what extent does DeMarr's result character a real line interval among compact $\mathbb{R}$-trees?

Question 2. *If X is a compact $\mathbb{R}$-tree which is not an interval, do there always exist two continuous maps $f, g : X \to X$, each having Lipschitz constant arbitrarily near one, which fail to have a common fixed point?*

2.3. **Isometries and the Brodskii-Milman paper.** We now turn to the original paper of Brodskii and Milman [9] in which the concept of 'normal structure' was introduced. This paper too was motivated by a common fixed point result.

Let K be a convex bounded set in a Banach space X. A point p_0 in K is said to be *diametral* if $\sup_{p \in K} \|p - p_0\| = d$, the diameter of K.

Definition 2. The set K is said to have *normal structure* if every convex subset of K which contains more than one point contains at least one nondiametral point.

A sequence $\{x_n\}$ of points of X is called a *diametral sequence* if the distance d_n from x_{n+1} to the convex hull of $\{x_i : i \leq n\}$ tends to the $\text{diam}\{x_i : 1 \leq i < \infty\}$. It was first shown in [9] that K has normal structure if and only if it contains no diametral sequence. This fact of course implies that all compact convex sets have normal structure. However the main result in [9] deals explicitly with common fixed points of isometries.

Assume now that K has normal structure and is compact in some topology in which spheres are closed (e.g., the weak or weak* topologies). Then the following construction yields a uniquely determined point, called the (Brodskii-Milman) *center* of K. Let K_ε be the intersection of all closed spheres of radius $d-\varepsilon$ centered at points of K; normal structure shows that there exists $\varepsilon > 0$ such that $K_\varepsilon \neq \emptyset$. Compactness shows that, if $\varepsilon' = \sup\{\varepsilon : K_\varepsilon \neq \emptyset\}$, then $K_{\varepsilon'} \neq \emptyset$. Denote $K_{\varepsilon'} = K^1$. Transfinite induction, using this step for nonlimit ordinals and intersections for limit ordinals, yields a transfinite sequence $K^1 \supset \cdots \supset K^\theta \supset \cdots$; normal structure assures that these inclusions are proper until a θ is reached for which K^θ is a singleton, the center of K. This name is justified by the fact that the center of a convex bounded K for which the above construction can be carried through is a fixed point of every isometry of K onto itself. It is also shown that if K is compact in the norm topology, the center is a fixed point of every nonexpansive mapping of K onto itself. This is of course a direct consequence of a classical result of Freudenthal and Hurewicz [18]. (See the next section.)

In his review [12] of the Brodskii-Milman paper, M. M. Day states that they prove that the center of K is fixed under every isometry of K **into** itself. However it seems likely that the mapping is meant to be surjective, especially in view of the proof outlined above and the fact that the corresponding result about nonexpansive mappings is clearly false if the mapping is not surjective.

Now let X be a Banach space and K a nonempty bounded subset of X. For $x \in X$, set

$$r_x(K) = \sup\{\|x-y\| : y \in K\}$$

and denote the diameter of K by *diam* $K := \sup\{\|y-z\| : y, z \in K\}$. As usual, $B(x;r)$ denotes the closed ball centered at x with radius $r \geq 0$. $\overline{co}(S)$ denotes the closure of the convex hull of S. The number $r_K := \inf\{r_x(K) : x \in K\}$ is called the *Chebyshev radius* of K and the set $\mathcal{C}(K) := \{x \in K : r_x(K) = r_K\}$ is called the *Chebyshev center* of K. It is well known that if K is weakly compact and convex then $\mathcal{C}(K)$ is a nonempty closed convex subset of K, and if *diam* $K > 0$ then K has normal structure if and only if $\mathcal{C}(K)$ is a **proper** subset of K. Indeed, the following is a direct consequence of the definition of normal structure.

Lemma 1. *Suppose K is a weakly compact convex subset of a Banach space with diam $K > 0$, and suppose K has normal structure. Then diam $\mathcal{C}(K) = r_K <$ diam K.*

Another fact is immediate:

Proposition 1. *If $T : K \to K$ is a surjective isometry, then $T(\mathcal{C}(K)) = \mathcal{C}(K)$.*

This brings us to another classical open question.

Question 3. *If T is not surjective in the previous proposition, does one still have $T(\mathcal{C}(K)) \subset \mathcal{C}(K)$?*

Notice that if the answer to this question is affirmative, then the Brodskii-Milman center of K is a fixed point of every isometry of K **into** itself.

T.-C. Lim, et al., take up Question 3 in [31] and show that the answer is affirmative if the space X is uniformly convex. In this case the Chebyshev center of K is a singleton and coincides with the Brodskii-Milman center of K. But the general question remains open.

2.4. **Commuting nonexpansive mappings.** This brings us to the third influential paper, another 1963 paper by Ralph DeMarr [13], and its opening paragraph is prophetic. We quote:

> Kakutani [25] and Markov [32] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some nonempty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous (not necessarily linear) transformations. We shall show that it is true in a rather special, but non-trivial case, thus giving hope that further investigation of the general question will yield positive results.

DeMarr then proceeded to state his result: *Every commutative family of nonexpansive mappings which map a compact convex subset of a Banach space into itself has a common fixed point.* It is interesting to note that Lemma 1 of DeMarr's paper states precisely that if a nonempty compact convex subset of a Banach space has positive diameter, then it must contain a nondiametral point, a fact already apparent from the results of Brodskii and Milman.

DeMarr's paper motivated the Belluce-Kirk paper [4] in which it is proved that every finite family of commuting nonexpansive mappings defined on a weakly compact convex set K has a common fixed point if K has normal structure. Since DeMarr's result holds for infinite families this raised the obvious question of whether the result of [4] could be extended to infinite families, a question subsequently settled in the affirmative by T.-C. Lim [30], and also by R. E. Bruck in his definitive study [10] of nonexpansive retracts.

2.5. **Surjective isometries.** Another metric question may be of interest. The result of Freudenthal and Hurewicz mentioned Section 2.3 states that a surjective nonexpansive self-mapping of a compact metric space is necessarily an isometry. It is interesting to note that there are *noncompact* spaces for which this assertion also holds.

Example 1. Let $\{e_n\}$ be the standard unit basis in ℓ_2, and for each $n \geq 1$ let $L_n = \left\{te_n : 0 \leq t \leq 1 - \frac{1}{n}\right\}$. Take $X = \cup L_n$. Then the only surjective nonexpansive mapping of X onto X is the identity.

Example 2. Consider $\mathbb{R}^2$ with the radial metric ρ. Let $\{x_n\}$ be a sequence of distinct points on the unit sphere of $\mathbb{R}^2$. Let $y_n = x_n$ for $1 \leq n \leq 10$, and for $n > 10$ let y_n be the point on the segment $[0, x_n]$ such that $\rho(0, y_n) = 1 - \frac{1}{n}$. Now let $L_n = [0, y_n]$ and take $X = \cup L_n$. In this case every surjective nonexpansive mapping of X onto X is an isometry, but there exist nontrivial surjective nonexpansive mappings.

Question 4. *Is it possible to classify metric spaces for which surjective nonexpansive self-mappings are always isometries? Do such spaces share any additional properties with compactness?*

3. New Directions

There has been a trend over the years to seek applications of metric fixed point theory in settings where the underlying algebraic linear structure of a Banach space is not present, at least in any explicit sense. While this trend has been somewhat sporadic, several developments stand out.

First there was the era of the early to mid-eighties. This brought the discovery that ideas of nonlinear analysis could be applied to hyperbolic geometry. The pioneering paper [21] by Goebel, Sekowski, and Stachura appeared in 1980. Notable in this development is the 1984 book *Uniform convexity, hyperbolic geometry, and nonexpansive mappings* by Goebel and Reich [20]. At about the same time there was the realization that there is a beautiful synergism between hyperconvex metric spaces and nonexpansive mappings. This came to fruition with a 1986 paper of J.-B. Baillion [2]. The early eighties also brought the realization that certain iteration processes (Krasnoselskii, Ishikawa) could be carried out in a so-called hyperbolic setting (see below) ([26], [19]). Later came the discovery that complete $\mathbb{R}$-trees (often called metric trees) provided prime examples of hyperconvex metric spaces with unique topological properties (Kirk [27]). Also it became clear that many of the standard ideas of nonlinear analysis could be extended to the class of so-called CAT(0) spaces [28]. We discuss some of these developments in more detail below. First, however, we fix some terminology.

HYPERBOLIC SPACES. The meaning of the term 'hyperbolic metric' has been inconsistent. Here we adopt the terminology of Kohlenbach [29]. A *hyperbolic space* is a triple (X, ρ, W), where (X, ρ) is a metric space and $W : X \times X \times [0, 1] \to X$ satisfies

(W1) $\rho(z, W(x, y, \lambda)) \leq (1 - \lambda)\rho(z, x) + \lambda\rho(z, y)$,

(W2) $\rho\left(W(x, y, \lambda), W\left(x, y, \bar{\lambda}\right)\right) = \left|\lambda - \bar{\lambda}\right| \rho(x, y)$,

(W3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,

(W4) $\rho(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)\rho(x, y) + \lambda\rho(z, w)$.

If only axiom (W1) is assumed this structure is a convex metric space in the sense of Takahashi [43]. If (W1)-(W3) are assumed the notion is equivalent to spaces called of *hyperbolic type* in [19]. Axiom (W4) is used for example in [38]. However Kohlenbach's definition is less restrictive than that given in [38] in that it does not require the existence of metric lines. Hence it includes all CAT(0) spaces, whereas the definition in [38] includes only those CAT(0) space which have the unique geodesic extension property.

3.1. **CAT(0) spaces and graph theory.**

(Much of the material in this section is taken from the author's article [28].)

Theorem 7 (Kirk, 1981; cf. Penot, 1979)**.** *Let* (X, d) *be a nonempty bounded metric space and suppose the admissible sets* (*ball intersections*) *in* X *are countably compact and normal. Then every nonexpansive mapping* $T : X \to X$ *has a fixed point.*

The following statement is found in M. van de Vel's book "Theory of Convex Structures" [44].

> [Problem 6.23.3] Observe that an EP function $G \to G$ is nonexpansive. If G is a median graph, then the diametric convex sets are precisely the graphic cubes. However, the preceding discussion shows that the median convexity of G will rarely include metric disks. Nevertheless, the similarities between the Invariant Cube Theorem and [Theorem 7] are too detailed to be pure coincidences. Is there a common generalization of both results?

A few years later Victor Chepoi responded to the above question in [11].

A *Euclidean cell* is a convex polytope in some Euclidean space. By a *piecewise Euclidean* (PE) *cell complex* is meant a space $\mathfrak{H}$ formed by gluing together Euclidean cells via isometries of their faces, together with the subdivision of $\mathfrak{H}$ into cells. It is assumed that the intersection of two cells is either empty or a single face of each of the cells. The following can be found in the book by Bridson and Haefliger.

Proposition 2. *Let* $\mathfrak{H}$ *be a simply connected PE complex with only finitely many isometry types of cells and endowed with the intrinsic metric. Then* $\mathfrak{H}$ *satisfies* $CAT(0)$ *if and only if given any two geodesic segments* α *and* β *in* $\mathfrak{H}$, *then the function* $f : [0, 1] \to H$ *given by* $f(t) = d(\alpha(t), \beta(t))$ *is convex.*

Every median graph G gives rise to an abstract cubical complex $K(G)$ consisting of all cubes of G, i.e., subgraphs of G isomorphic to cubes of any dimension. The geometric realization $|K(G)|$ is called a *median complex.* In his paper Chepoi proves that CAT(0) cube complexes coincide with the cubical cell complexes arising from median graphs. In [11] Chepoi goes on to state:

> It seems that the CAT(0) property of median complexes gives a partial explanation to [van de Vel's problem]. Namely, assume additionally that $f : G \to G$ is a cell-to-cell map. Then f induces a continuous map $f : |K(G)| \to |K(G)|$ by extending f affinely over the geometric cubes. As a result, we obtain a nonexpansive map of a CAT(0) space $|K(G)|$. From [Proposition 2] and the result of Kirk [Theorem 7], f has a fixed point p. Consider the smallest cube C of G such that p belongs to the relative interior of $|C|$. Clearly, C is an invariant cube of the original map f. This remark can be extended to obtain fixed simplexes of edge-preserving maps of underlying graphs of CAT(0) simplicial complexes, but does not extend to arbitrary edge-preserving maps of CAT(0) polysimplicial complexes (cell complexes whose cells are products of simplexes).

These two quotes raised a number of questions, among them: What are edge preserving maps? What are median graphs. What does Invariant Cube Theorem say? What are CAT(0) spaces?

3.2. **Some answers.** A *graph* is an ordered pair (V, E) where V is a set and E is a binary relation on V $(E \subseteq V \times V)$. Elements of E are called *edges.* We are concerned here with (undirected) graphs that have a "loop" at every vertex (i.e., $(a, a) \in E$ for each $a \in V$) and no "multiple" edges. Such graphs are called *reflexive.* In this case $E \subseteq V \times V$ corresponds to a reflexive (and symmetric) binary relation on V.

Given a graph $G = (V, E)$, a path of G is a sequence $a_0, a_1, \cdots, a_{n-1}, \cdots$ with $(a_{i+1}, a_i) \in E$ for each $i = 0, 1, 2, \cdots$. A *cycle* is a finite path $(a_0, a_1 \cdots, a_{n-1})$ with $(a_0, a_{n-1}) \in E$. A graph is *connected* if there is a finite path joining any two of its vertices. A finite path $(a_0, a_1, \cdots, a_{n-1})$ is said to have *length* n. Finally, a *tree* is a connected graph with no cycles.

For a graph $G = (V, E)$ a map $f : V \to V$ is *edge-preserving* if $(a, b) \in E \Rightarrow (f(a), f(b)) \in E$. For such a mapping we simply write $f : G \to G$. There is a standard way of *metrizing* connected graphs; let each edge have length one and take distance $d(a, b)$ between two vertices a and b to be the length of the shortest path joining them. With this metric edge preserving mappings become precisely the *non-expansive* mappings. Keep in mind that in a reflexive graph an edge-preserving map may collapse edges between distinct points since loops are allowed.

The classical Fixed Edge Theorem in graph theory due to Nowakowski and Rival [34] asserts that an edge preserving mapping defined on a connected graph that has no cycles or infinite paths always leaves some edge of the graph fixed.

The Invariant Cube Theorem (due to H.-J. Bandelt and M. van de Vel [3]) appeared in 1987 . It involves median graphs. A *median graph* is a connected graph G (with standard metric d) that has the following property: For each triple of points $a_1, a_2, a_3 \in G$ there is a **unique** point $x \in G$ such that

$$d(a_i, x) + d(x, a_j) = d(a_i, a_j) \text{ for all } i \neq j \in \{1, 2, 3\}.$$

In this case x is called the *median* of a_1, a_2, a_3 and is written $x = m(a_1, a_2, a_3)$. Thus each three points of G have a unique common between point. In particular a connected graph which has no cycles (to which the Fixed Edge Theorem applies) is a median graph.

The ternary operation $m : G^3 \to G$ has the following properties, of which the first two are obvious; the third less so:

$(M1)$ $\quad m(a, b, c) = m(b, a, c) = m(c, b, a)$,
$(M2)$ $\quad m(a, a, b) = a$,
$(M3)$ $\quad m(m(a, b, c), u, v) = m(a, m(b, u, v), m(c, u, v))$.

Theorem 8 (Invariant Cube Theorem)**.** *Let G be a finite median graph and let $f : G \to G$ be an edge preserving function. Then there is a graphic cube of G that is mapped isomorphically onto itself by f. If the number of vertices of G is odd and f is an automorphism, then f has a fixed point.*

3.3. **The approximate fixed point property.** A subset K of a metric space is said to have the *approximate fixed point property* (for nonexpansive mappings) if given any nonexpansive $f : K \to K$, $\inf \{d(x, f(x)) : x \in K\} = 0$. To characterize this concept we need some more definitions.

Definition 3. A geodesic metric space X is said to have the *geodesic extension property* if for every local geodesic $c : [a, b] \to X$, with $a \neq b$, there exists $\varepsilon > 0$ and a local geodesic $c' : [a, b + \varepsilon] \to X$ such that $c' \mid_{[a,b]} = c$.

Definition 4 (Shafrir, 1990)**.** Let X be a metric space. A curve $\gamma : X \to [0, \infty)$ is said to be *directional (* with constant b) if there is $b \geq 0$ such that

$$t - s - b \leq d(\gamma(s), \gamma(t)) \leq t - s$$

for all $t \geq s \geq 0$. A subset of X is said to be *directionally bounded* if it does not contain a directional curve.

Lemma 2. *If X is a $CAT(0)$ space, then X has the geodesic extension property if and only if every non-constant geodesic $c : [a, b] \to X$ can be extended to a line $c : \mathbb{R} \to X$.*

If a complete CAT(0) space is homeomorphic to a finite dimensional manifold, then it always has the geodesic extension property ([8]).

In [41] it is proved that a closed convex subset of a complete hyperbolic metric space with the geodesic extension property has the approximate fixed point property for nonexpansive mappings if and only if it is directionally bounded. As an immediate corollary, a closed convex subset of a complete CAT(0) space with the geodesic extension property has the approximate fixed point property if and only if it is directionally bounded. However in this case a stronger assertion holds.

Theorem 9 ([28])**.** *If a complete $CAT(0)$ space is geodesically bounded then it is directionally bounded.*

In view of Shafrir's result we now have the following.

Corollary 1. *A closed convex subset of a complete $CAT(0)$ space with the geodesic extension property has the approximate fixed point property for nonexpansive mappings if and only if it does not contain a geodesic ray.*

Proof of Theorem 9. ([28])Suppose K is a closed convex set in X and suppose K contains a directional curve γ. We show that this implies K contains a geodesic ray.

Let $x_n = \gamma(n)$, $n = 0, 1, 2, \cdots$, and fix an arbitrary $\rho > b$, where b is the directional constant associated with γ. For each $n \geq \rho$, let y_n be the point of geodesic segment $[x_0, x_n]$ with distance ρ from x_0. Now suppose $m > n \geq \rho$, and let α be the comparison angle $\angle_{\bar{x}_0}(\bar{x}_n, \bar{x}_m)$in $\mathbb{R}^2$. By the law of cosines

$$\cos(\alpha) = \frac{d(x_0, x_n)^2 + d(x_0, x_m)^2 - d(x_n, x_m)^2}{2d(x_0, x_n)\, d(x_0, x_m)}.$$

Using the inequalities

$$n - b \leq d(x_0, x_n) \leq n;$$
$$m - b \leq d(x_0, x_m) \leq m;$$
$$m - n \geq d(x_n, x_m),$$

we have

$$\begin{aligned}\cos(\alpha) &\geq \frac{(n-b)^2+(m-b)^2-(m-n)^2}{2nm}\\ &= \frac{n}{2m}\left[\frac{(n-b)^2}{n^2}\right]+\frac{m}{2n}\left[\frac{(m-b)^2}{m^2}\right]-\frac{(m-n)^2}{2nm}\\ &= \frac{n}{2m}+\frac{m}{2n}-\frac{(m-n)^2}{2nm}-\frac{b}{n}-\frac{b}{m}+\frac{b^2}{nm}\\ &= 1-b\left(\frac{1}{n}+\frac{1}{m}-\frac{b}{nm}\right).\end{aligned}$$

Thus $\cos(\alpha) \to 1$ as $m, n \to \infty$; hence $\alpha \to 0$. If $\bar{y}_n, \bar{y}_m$ are the respective points of the comparison triangle $\Delta(\bar{x}_0, \bar{x}_n, \bar{x}_m)$ corresponding to y_n, y_m, then by the CAT(0) inequality $d(y_n, y_m) \leq d(\bar{y}_n, \bar{y}_m)$. The fact that $\alpha \to 0$ as $m, n \to \infty$ implies that $\{\bar{y}_n\}$, hence $\{y_n\}$, is a Cauchy sequence. Since $\rho > b$ is arbitrary it now follows that the sequence $\{[x_0, x_n]\}$ of geodesic segments converges to a geodesic ray issuing from x_0. □

Question 5. *Is the geodesic extension property needed for the characterization of Corollary 2.1?*

Other questions remain open as well.

Question 6. *Does Corollary 2.1 extend to hyperconvex metric spaces?*

Question 7. *Is it the case that a closed convex subset of a complete CAT(0) space has the fixed point property if and only if it is bounded (as in the Hilbert space case, cf. Ray,* [37]*)?*

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Received: 31.12.2009; Accepted:10.02.2010.