

## SCHAUDER-TYCHONOFF'S FIXED POINT THEOREM IN A VECTOR LATTICE

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*Dedicated to Wataru Takahashi on the occasion of his retirement*

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**Abstract.** In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice using the topology introduced by [2]. The purpose of this paper is to show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem.

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### 1. INTRODUCTION

There are many fixed point theorems in a topological vector space, for instance, Takahashi's fixed point theorem and Fan-Browder's fixed point theorem in a topological vector space, Tychonoff's fixed point theorem in a locally convex space, Schauder's fixed point theorem in a normed space, and so on; see for example [7].

In this paper, we consider fixed point theorems in a vector lattice. As known well every topological vector space has a linear topology. On the other hand, although every vector lattice does not have a topology, it has two lattice operators, which are the supremum  $\vee$  and the infimum  $\wedge$ , and also an order is introduced from these operators; see also [5, 8] about vector lattices. There are some methods how to introduce a topology to a vector lattice. One method is to assume that the vector lattice has a linear topology [1]. On the other hand, there is another method to make up a topology in a vector lattice, for instance, in [2] one method is introduced in case of the vector lattice with unit.

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In the previous paper [4] we show Takahashi's and Fan-Browder's fixed point theorems in a vector lattice using the topology introduced by [2]. The purpose of this paper is to show Schauder-Tychonoff's fixed point theorem using Fan-Browder's fixed point theorem.

## 2. TOPOLOGY IN A VECTOR LATTICE AND TAKAHASHI'S AND FAN-BROWDER'S FIXED POINT THEOREMS

First we introduce a topology in a vector lattice introduced by [2]; see also [4].

Let  $X$  be a vector lattice.  $e \in X$  is said to be an unit if  $e \wedge x > 0$  for any  $x \in X$  with  $x > 0$ . Let  $\mathcal{K}_X$  be the class of units of  $X$ . In case where  $X$  is the set of real numbers  $\mathbf{R}$ ,  $\mathcal{K}_{\mathbf{R}}$  is the set of positive real numbers. Let  $X$  be a vector lattice with unit and let  $Y$  be a subset of  $X$ .  $Y$  is said to be open if for any  $x \in Y$  and for any  $e \in \mathcal{K}_X$  there exists  $\varepsilon \in \mathcal{K}_{\mathbf{R}}$  such that  $[x - \varepsilon e, x + \varepsilon e] \subset Y$ . Let  $\mathcal{O}_X$  be the class of open subsets of  $X$ .  $Y$  is said to be closed if  $Y^C \in \mathcal{O}_X$ . For  $e \in \mathcal{K}_X$  and for an interval  $[a, b]$  we consider the following subset

$$[a, b]^e = \{x \mid \text{there exists some } \varepsilon \in \mathcal{K}_{\mathbf{R}} \text{ such that } x - a \geq \varepsilon e \text{ and } b - x \geq \varepsilon e\}.$$

By the definition of  $[a, b]^e$  it is easy to see that  $[a, b]^e \subset [a, b]$ . Every mapping from  $X \times \mathcal{K}_X$  into  $(0, \infty)$  is said to be a gauge. Let  $\Delta_X$  be the class of gauges in  $X$ . For  $x \in X$  and  $\delta \in \Delta_X$ ,  $O(x, \delta)$  is defined by

$$O(x, \delta) = \bigcup_{e \in \mathcal{K}_X} [x - \delta(x, e)e, x + \delta(x, e)e]^e.$$

$O(x, \delta)$  is said to be a  $\delta$ -neighborhood of  $x$ . Suppose that for any  $x \in X$  and for any  $\delta \in \Delta_X$  there exists  $U \in \mathcal{O}_X$  such that  $x \in U \subset O(x, \delta)$ .

For a subset  $Y$  of  $X$  we denote by  $cl(Y)$  and  $int(Y)$ , the closure and the interior of  $Y$ , respectively. Let  $X$  and  $Y$  be vector lattices with unit,  $x_0 \in Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ .  $f$  is said to be continuous in the sense of topology at  $x_0$  if for any  $V \in \mathcal{O}_Y$  with  $f(x_0) \in V$  there exists  $U \in \mathcal{O}_X$  with  $x_0 \in U$  such that  $f(U \cap Z) \subset V$ .

Let  $X$  be a vector lattice with unit.  $X$  is said to be Hausdorff if for any  $x_1, x_2 \in X$  with  $x_1 \neq x_2$  there exists  $O_1, O_2 \in \mathcal{O}_X$  such that  $x_1 \in O_1$ ,  $x_2 \in O_2$  and  $O_1 \cap O_2 = \emptyset$ . A subset  $Y$  of  $X$  is said to be compact if for any open covering of  $Y$  there exists a finite sub-covering. A subset  $Y$  of  $X$  is said to be normal if for any closed subsets  $F_1$  and  $F_2$  with  $F_1 \cap F_2 \cap Y = \emptyset$  there exists  $O_1, O_2 \in \mathcal{O}_X$  such that  $F_1 \subset O_1$ ,  $F_2 \subset O_2$  and  $O_1 \cap O_2 \cap Y = \emptyset$ .

A vector lattice is said to be Archimedean if it holds that  $x = 0$  whenever there exists  $y \in X$  with  $y \geq 0$  such that  $0 \leq rx \leq y$  for any  $r \in \mathcal{K}_{\mathbf{R}}$ .

We obtain the following Takahashi's fixed point theorem in a vector lattice; see [4].

**Theorem 2.1.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $Y$  a compact subset of  $X$  and  $Z$  a convex subset of  $Y$ . Suppose that a mapping  $f$  from  $Z$  into  $2^Y$  satisfies*

(0)  $f^{-1}(y)$  is convex for any  $y \in Y$ ,

and there exists a mapping  $g$  from  $Z$  into  $2^Y$  satisfying the following conditions:

- (1)  $g(z)$  is a subset of  $f(z)$  for any  $z \in Z$ ;
- (2)  $g^{-1}(y)$  is non-empty for any  $y \in Y$ ;
- (3)  $g(z)$  is an open subset of  $X$  for any  $z \in Z$ .

Then there exists  $z_0 \in Z$  such that  $z_0 \in f(z_0)$ .

In the above theorem, putting  $Z = Y$  and  $g = f$ , the following theorem is obtained. It is Fan-Browder's fixed point theorem in a vector lattice.

**Theorem 2.2.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $Y$  a compact convex subset of  $X$ . Suppose that a mapping  $f$  from  $Y$  into  $2^Y$  satisfies the following conditions:*

- (1)  $f^{-1}(y)$  is non-empty and convex for any  $y \in Y$ ;
- (2)  $f(y)$  is an open subset of  $X$  for any  $y \in Y$ .

Then there exists  $y_0 \in Y$  such that  $y_0 \in f(y_0)$ .

In the above theorem, changing from  $f$  to  $f^{-1}$ , the following theorem is obtained; see [7].

**Theorem 2.3.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $Y$  a compact convex subset of  $X$ . Suppose that a mapping  $f$  from  $Y$  into  $2^Y$  satisfies the following conditions:*

- (1)  $f^{-1}(y)$  is an open subset of  $X$  for any  $y \in Y$ ;
- (2)  $f(y)$  is non-empty and convex for any  $y \in Y$ .

Then there exists  $y_0 \in Y$  such that  $y_0 \in f(y_0)$ .

Moreover the following holds.

**Theorem 2.4.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $Y$  a compact convex subset of  $X$  and  $A \subset Y \times Y$ . Suppose that  $A$  satisfies the following conditions:*

- (1)  $\{x \mid (x, y) \in A\}$  is closed for any  $y \in Y$ ;
- (2)  $\{y \mid (x, y) \notin A\}$  is convex for any  $x \in Y$ ;
- (3)  $(x, x) \in A$  for any  $x \in Y$ .

Then there exists  $x_0 \in Y$  such that  $\{x_0\} \times Y \subset A$ .

### 3. SCHAUDER-TYCHONOFF'S FIXED POINT THEOREM

Let  $X$  be a vector lattice with unit and  $Y$  a vector lattice. Let  $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$  be the class of  $\{v_e \mid e \in \mathcal{K}_X\}$  satisfying the following conditions:

- (U1)  $v_e \in Y$  with  $v_e > 0$ ;
- (U2)<sup>d</sup>  $v_{e_1} \geq v_{e_2}$  if  $e_1 \geq e_2$ ;
- (U3)<sup>s</sup> For any  $e \in \mathcal{K}_X$  there exists  $\theta(e) \in \mathcal{K}_\mathbf{R}$  such that  $v_{\theta(e)e} \leq \frac{1}{2}v_e$ .

Let  $x_0 \in Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ .  $f$  is said to be continuous at  $x_0$  if there exists  $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$  such that for any  $e \in \mathcal{K}_X$  there exists  $\delta \in \mathcal{K}_\mathbf{R}$  such that for any  $x \in Z$  if  $|x - x_0| \leq \delta e$ , then  $|f(x) - f(x_0)| \leq v_e$ . In particular if  $Y$  has an unit, then we consider often  $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$  satisfying the following condition instead of (U1):

(U1)<sup>u</sup>  $v_e \in \mathcal{K}_Y$ .

*Example 3.1.* We consider a sufficient condition such that there exists  $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$  satisfying the condition (U1)<sup>u</sup>. Let  $X$  be an Archimedean vector lattice. Then there exists a positive homomorphism  $f$  from  $X$  into  $\mathbf{R}$ , that is,  $f$  satisfies the following conditions:

- (H1)  $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$  for any  $x, y \in X$  and for any  $\alpha, \beta \in \mathbf{R}$ ;  
(H2)  $f(x) \geq 0$  for any  $x \in X$  with  $x \geq 0$ .

Indeed it is shown as follows. By [8] (see Theorem IV.11.1) for any Archimedean vector lattice  $X$  there exists the completion  $\hat{X}$  of  $X$ . By [8] (see Theorem V.4.2) for the complete vector lattice  $\hat{X}$  there exists an extremally disconnected compact set  $\Omega$  and a vector sublattice  $Y$  of  $C_\infty(\Omega)$  such that  $\hat{X}$  is isomorphic to  $Y$ , where

$$C_\infty(\Omega) = \left\{ f \mid \begin{array}{l} f \text{ is continuous from } \Omega \text{ into } [-\infty, \infty] \text{ and} \\ f^{-1}(\{\pm\infty\}) \text{ is nowhere dense} \end{array} \right\}.$$

Therefore it may be assumed that  $X$  is a vector sublattice of  $C_\infty(\Omega)$ . Take  $\omega \in \Omega$  arbitrary and let  $f(x) = x(\omega)$  for any  $x \in X$ . Then  $f$  satisfies the conditions (H1) and (H2). Suppose that  $X$  satisfies that there exists a homomorphism  $f$  from  $X$  into  $\mathbf{R}$  satisfying the following condition instead of (H2):

- (H2)<sup>s</sup>  $f(x) > 0$  for any  $x \in X$  with  $x > 0$ .

Then for any  $e_Y \in \mathcal{K}_Y$   $\{f(e)e_Y\}$  satisfies the conditions (U1)<sup>u</sup>(U2)<sup>d</sup>(U3)<sup>s</sup> clearly. Therefore if  $X$  is Archimedean and there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup>, then it may be assumed that every  $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$  satisfies the condition (U1)<sup>u</sup>.

Let  $X$  and  $Y$  be vector lattices with unit,  $Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ . Suppose that there exists  $P \subset Y$  satisfying the following conditions:

- (P1)  $P$  is open and convex;  
(P2) If  $x \in P$  and  $x \leq y$ , then  $y \in P$ ;  
(P3)  $0 \notin P$ ;  
(P4)  $\{x \mid x > 0\} \subset P$ .

Let  $\mathcal{P}_Y$  be the class of the above  $P$ 's.  $f$  is said to be upper semi-continuous with respect to  $P \in \mathcal{P}_Y$  if  $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$  for any  $y \in Y$ .  $f$  is said to be lower semi-continuous with respect to  $P \in \mathcal{P}_Y$  if  $\{x \mid f(x) - y \in P\} \in \mathcal{O}_X \cap Z$  for any  $y \in Y$ .  $f$  is said to be semi-continuous with respect to  $P \in \mathcal{P}_Y$  if it is upper and lower semi-continuous with respect to  $P \in \mathcal{P}_Y$ .

*Example 3.2.* We consider of a sufficient condition to satisfy  $\mathcal{P}_X \neq \emptyset$ . Let  $X$  be an Archimedean vector lattice with unit. Suppose that there exists a homomorphism  $f$  from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup>. Let  $0 < \beta < 1$  and  $\delta(x, e) = \frac{\beta f(x)}{f(e)}$  for any  $x \in X$  with  $x > 0$  and for any  $e \in \mathcal{K}_X$ . Put  $P = \bigcup_{x \in X \text{ with } x > 0} \text{int}(O(x, \delta))$ . Then  $P$  is open and  $\{x \mid x > 0\} \subset P$ .

Note that by the condition (H2)<sup>s</sup> for any  $x_1, x_2 \in X$  with  $x_1, x_2 > 0$  and  $x_1 \neq x_2$ ,  $\frac{x_1}{f(x_1)}$  and  $\frac{x_2}{f(x_2)}$  are incomparable mutually. Therefore  $x - \delta(x, e)e \not\leq 0$  for any  $x \in X$  with  $x > 0$  and for any  $e \in \mathcal{K}_X$ . Assume that  $0 \in P$ . Then there exists  $x \in X$  with

$x > 0$  and  $e \in \mathcal{K}_X$  such that  $0 \in [x - \delta(x, e)e, x + \delta(x, e)e]^e$ . It is a contradiction. Therefore  $0 \notin P$ .

Note that  $x \in \text{int}(A)$  if and only if there exists  $\delta_x \in \Delta_X$  such that  $O(x, \delta_x) \subset A$ . Let  $x \in P$  and  $x \leq y$ . Then there exists  $z \in X$  with  $z > 0$  and  $\delta_x \in \Delta_X$  such that  $O(x, \delta_x) \subset O(z, \delta)$ . Let  $\delta_y(u, e) = \delta_x(u - y + x, e)$ . Since  $\delta(x_2, e) \leq \delta(x_1 + x_2, e)$  for any  $x_1, x_2 \in X$  with  $x_1, x_2 > 0$ , it holds that  $x_1 + O(x_2, \delta) \subset O(x_1 + x_2, \delta)$ . Therefore

$$O(y, \delta_y) = y - x + O(x, \delta_x) \subset y - x + O(z, \delta) \subset O(z + y - x, \delta),$$

that is,  $y \in \text{int}(O(z + y - x, \delta)) \subset P$ .

Let  $x_0, x_1 \in P$  and  $\alpha \in \mathbf{R}$  with  $0 \leq \alpha \leq 1$ . Then for  $i = 0, 1$  there exists  $y_i \in X$  with  $y_i > 0$  and  $\delta_i \in \Delta_X$  such that  $O(x_i, \delta_i) \subset O(y_i, \delta)$ . Let  $\delta_\alpha(z, e) = (1 - \alpha)\delta_0(x_0, e) + \alpha\delta_1(x_1, e)$ . Take  $z \in O((1 - \alpha)x_0 + \alpha x_1, \delta_\alpha)$  arbitrary. Then there exists  $e \in \mathcal{K}_X$  such that

$$\begin{aligned} z &\in [(1 - \alpha)x_0 + \alpha x_1 - \delta_\alpha((1 - \alpha)x_0 + \alpha x_1, e)e, \\ &\quad (1 - \alpha)x_0 + \alpha x_1 + \delta_\alpha((1 - \alpha)x_0 + \alpha x_1, e)e]^e \\ &= (1 - \alpha)[x_0 - \delta_0(x_0, e)e, x_0 + \delta_0(x_0, e)e]^e \\ &\quad + \alpha[x_1 - \delta_1(x_1, e)e, x_1 + \delta_1(x_1, e)e]^e. \end{aligned}$$

Since  $\delta(\alpha x, e) = \alpha\delta(x, e)$  for any  $x \in X$  with  $x > 0$  and for any  $\alpha \in \mathcal{K}_{\mathbf{R}}$ , it holds that  $O(\alpha x, \delta) = \alpha O(x, \delta)$ . Since

$$\begin{aligned} &\delta(z_0, e_0)e_0 + \delta(z_1, e_1)e_1 \\ &= \delta\left(z_0 + z_1, \frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1\right) \left(\frac{f(z_0)}{f(e_0)}e_0 + \frac{f(z_1)}{f(e_1)}e_1\right) \end{aligned}$$

for any  $z_0, z_1 \in X$  with  $z_0, z_1 > 0$ , it holds that  $O(z_0, \delta) + O(z_1, \delta) \subset O(z_0 + z_1, \delta)$ . Then

$$\begin{aligned} z &\in (1 - \alpha)O(x_0, \delta_0) + \alpha O(x_1, \delta_1) \\ &\subset (1 - \alpha)O(y_0, \delta) + \alpha O(y_1, \delta) = O((1 - \alpha)y_0, \delta) + O(\alpha y_1, \delta) \\ &\subset O((1 - \alpha)y_0 + \alpha y_1, \delta). \end{aligned}$$

Therefore  $O((1 - \alpha)x_0 + \alpha x_1, \delta_\alpha) \subset O((1 - \alpha)y_0 + \alpha y_1, \delta)$ , that is,  $(1 - \alpha)x_0 + \alpha x_1 \in \text{int}(O((1 - \alpha)y_0 + \alpha y_1, \delta)) \subset P$ .

*Example 3.3.* We consider of another simple sufficient condition to satisfy  $\mathcal{P}_X \neq \emptyset$ . Let  $X$  be a Hilbert lattice with unit, that is,  $X$  has an inner product  $\langle \cdot, \cdot \rangle$  and for any  $x, y \in X$  if  $|x| \leq |y|$ , then  $\langle x, x \rangle \leq \langle y, y \rangle$ . Then for any  $e \in \mathcal{K}_X$   $P = \{x \mid \langle x, e \rangle > 0\}$  satisfies the conditions (P1)-(P4). Actually it is possible to show as follows.

It is clear that  $P$  is convex and  $0 \notin P$ .

Note that  $\langle x, y \rangle \geq 0$  if  $x, y \geq 0$ . Actually since  $|x - y| \leq x + y$  and  $\langle |x - y|, |x - y| \rangle = \langle x - y, x - y \rangle$ , it holds that  $\langle x - y, x - y \rangle \leq \langle x + y, x + y \rangle$ . Therefore it holds that  $\langle x, y \rangle \geq 0$ . Let  $x \in P$  and  $x \leq y$ . Then  $\langle y, e \rangle \geq \langle x, e \rangle > 0$  proving that  $y \in P$ .

Assume that there exists  $x \in X$  with  $x > 0$  such that  $\langle x, e \rangle = 0$ . Then since  $\langle x + e, x + e \rangle = \langle x - e, x - e \rangle = \langle |x - e|, |x - e| \rangle$ ,

$$0 = \langle x + e + |x - e|, x + e - |x - e| \rangle = 4\langle x \vee e, x \wedge e \rangle \geq 4\langle x \wedge e, x \wedge e \rangle > 0.$$

It is a contradiction. Therefore  $\{x \mid x > 0\} \subset P$ .

For  $x \in P$  and  $e_1 \in \mathcal{K}_K$  putting  $\delta < \frac{\langle x, e \rangle}{\langle e_1, e \rangle}$ , then  $\langle x - \delta e_1, e \rangle > 0$ . Therefore  $P$  is open.

**Lemma 3.1.** *Let  $X$  be an Archimedean vector lattice with unit,  $Y$  a vector lattice with unit,  $Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup> and that  $\mathcal{P}_Y \neq \emptyset$ . Then  $f$  is semi-continuous with respect to any  $P \in \mathcal{P}_Y$  if it is continuous at any  $x \in Z$ .*

*Proof.* Take  $y \in Y$  and  $x_0 \in \{x \mid y - f(x) \in P\} \cap Z$  arbitrary. By the assumption there exists  $\{v_e\} \in \mathcal{U}_Z^s(\mathcal{K}_X, \geq)$  such that for any  $e \in \mathcal{K}_X$  there exists  $\delta(e) \in \mathcal{K}_\mathbf{R}$  such that for any  $x \in Z$  if  $|x - x_0| \leq \delta(e)e$ , then  $|f(x) - f(x_0)| \leq v_e$ . By the assumption it may be assumed that  $v_e \in \mathcal{K}_Y$  for any  $e \in \mathcal{K}_X$ . Since  $P$  is open, there exists a natural number  $n(e)$  such that  $[y - f(x_0) - 2^{-n(e)}v_e, y - f(x_0) + 2^{-n(e)}v_e] \subset P$ . If  $|x - x_0| \leq \delta(\theta(e, n(e))e)\theta(e, n(e))e$ , where  $\theta(e, n) = \underbrace{\theta(\theta(\dots\theta(\theta(e)e)\dots e))}_n$ , then  $|f(x) - f(x_0)| \leq$

$v_{\theta(e, n(e))e} \leq 2^{-n(e)}v_e$ . Therefore  $y - f(x) \in [y - f(x_0) - 2^{-n(e)}v_e, y - f(x_0) + 2^{-n(e)}v_e] \subset P$ , that is,  $[x_0 - \delta(\theta(e, n(e))e)\theta(e, n(e))e, x_0 + \delta(\theta(e, n(e))e)\theta(e, n(e))e] \subset \{x \mid y - f(x) \in P\} \cap Z$  proving that  $\{x \mid y - f(x) \in P\} \in \mathcal{O}_X \cap Z$ . Therefore  $f$  is upper semi-continuous with respect to  $P$ . Similarly it can be proved that  $f$  is lower semi-continuous with respect to  $P$ .  $\square$

**Lemma 3.2.** *Let  $X$  be an Archimedean vector lattice with unit,  $Y$  a vector lattice with unit,  $x_0 \in Z \subset X$  and  $f$  a mapping from  $Z$  into  $Y$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup>. Then  $f$  is continuous at  $x_0$  in the sense of topology if it is continuous at  $x_0$ .*

*Proof.* By the assumption there exists  $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$  such that for any  $e \in \mathcal{K}_X$  there exists  $\delta(e) \in \mathcal{K}_\mathbf{R}$  such that for any  $x \in Z$  if  $|x - x_0| \leq \delta(e)e$ , then  $|f(x) - f(x_0)| \leq v_e$ . By the assumption it may be assumed that  $v_e \in \mathcal{K}_Y$  for any  $e \in \mathcal{K}_X$ . Let  $\delta_Y$  be a gauge in  $Y$ . Take a natural number  $n(e)$  such that  $2^{-n(e)} < \delta_Y(f(x_0), v_e)$  and put  $\delta_X(x, e) = \theta(e, n(e))\delta(\theta(e, n(e))e)$ , where  $\theta(e, n) = \underbrace{\theta(\theta(\dots\theta(\theta(e)e)\dots e))}_n$ . Let

$x \in O(x_0, \delta_X)$ . There exists  $e \in \mathcal{K}_X$  such that  $x \in [x_0 - \delta_X(x_0, e)e, x_0 + \delta_X(x_0, e)e]^e$ . Then

$$|f(x) - f(x_0)| \leq v_{\theta(e, n(e))e} \leq 2^{-n(e)}v_e < \delta_Y(f(x_0), v_e)v_e.$$

Therefore

$$f(x) \in [f(x_0) - \delta_Y(f(x_0), v_e)v_e, f(x_0) + \delta_Y(f(x_0), v_e)v_e]^{v_e} \subset O(f(x_0), \delta_Y)$$

proving that  $f$  is continuous at  $x_0$  in the sense of topology.  $\square$

**Theorem 3.1.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit,  $Y$  a vector lattice with unit and  $Z$  a compact convex subset of  $X$ . Suppose that  $\mathcal{P}_Y \neq \emptyset$  and that a mapping  $f$  from  $Z \times Z$  into  $Y$  satisfies that there exists  $P \in \mathcal{P}_Y$  such that*

- (1)  $f(\cdot, x_2)$  is upper semi-continuous with respect to  $P$  for any  $x_2 \in Z$ ;
- (2)  $f(x_1, \cdot)$  is convex for any  $x_1 \in Z$ ;

(3) *There exists  $c \in Y$  such that  $c - f(x, x) \notin P$  for any  $x \in Z$ .*

*Then there exists  $x_0 \in Z$  such that  $c - f(x_0, x) \notin P$  for any  $x \in Z$ .*

*Proof.* Let  $A = \{(x_1, x_2) \mid c - f(x_1, x_2) \notin P\}$ . By (1)  $\{x_1 \mid (x_1, x_2) \in A\}$  is closed for any  $x_2 \in Z$ . By (3)  $(x, x) \in A$  for any  $x \in Z$ . Let  $z_1, z_2 \in \{x_2 \mid (x_1, x_2) \notin A\}$  and  $0 \leq \alpha \leq 1$ . By (2) and convexity of  $P$

$$c - f(x_1, (1 - \alpha)z_1 + \alpha z_2) \geq (1 - \alpha)(c - f(x_1, z_1)) + \alpha(c - f(x_1, z_2)) \in P.$$

By (P2)  $(1 - \alpha)z_1 + \alpha z_2 \in \{x_2 \mid (x_1, x_2) \notin A\}$ , that is,  $\{x_2 \mid (x_1, x_2) \notin A\}$  is convex for any  $x_1 \in Z$ . By Theorem 2.4 there exists  $x_0 \in Z$  such that  $\{x_0\} \times Z \subset A$ . Therefore  $c - f(x_0, x) \notin P$  for any  $x \in Z$ .  $\square$

**Theorem 3.2.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $Z$  a compact convex subset of  $X$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup> and that a mapping  $f$  from  $Z$  into  $X$  is continuous. Then it holds that (1) or (2).*

- (1) *There exists  $x_0 \in Z$  such that  $f(x_0) = x_0$ .*
- (2) *There exists  $x_0 \in Z$  such that  $f(x_0) \neq x_0$  and  $|x_0 - f(x_0)| - |x - f(x_0)| \notin P$  for any  $P \in \mathcal{P}_X$  and for any  $x \in Z$ .*

*Proof.* Suppose that (1) is not satisfied. Then  $f(x) \neq x$  for any  $x \in Z$ . Take  $g(x_1, x_2) = |x_2 - f(x_1)| - |x_1 - f(x_1)|$ . Then  $g(\cdot, x_2)$  is continuous for any  $x_2 \in Z$ ,  $g(x_1, \cdot)$  is convex for any  $x_1 \in Z$  and by (P3)  $-g(x, x) = 0 \notin P$ . By Lemma 3.1 and Theorem 3.1 there exists  $x_0 \in Z$  such that  $-g(x_0, x) = |x_0 - f(x_0)| - |x - f(x_0)| \notin P$  for any  $x \in Z$ .  $\square$

**Theorem 3.3.** *Let  $X$  be a Hausdorff Archimedean vector lattice with unit and  $Z$  a compact convex subset of  $X$ . Suppose that there exists a homomorphism from  $X$  into  $\mathbf{R}$  satisfying the condition (H2)<sup>s</sup> and that a mapping  $f$  from  $Z$  into  $X$  is continuous. Then there exists  $x_0 \in Z$  such that  $f(x_0) = x_0$ .*

*Proof.* Assume that (2) in Theorem 3.2 holds. Then there exists  $x_0 \in Z$  such that  $f(x_0) \neq x_0$  and  $|x_0 - f(x_0)| - |x - f(x_0)| \notin P$  for any  $x \in Z$ . Since  $f(x_0) \neq x_0$ , by (P4)  $|x_0 - f(x_0)| \in P$ . Take  $x = f(x_0)$ . Then  $|x_0 - f(x_0)| \notin P$ . It is a contradiction. Therefore there exists  $x_0 \in Z$  such that  $f(x_0) = x_0$ .  $\square$

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