Fixed Point Theory, 11(2010), No. 1, 29-36
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A KRASNOSELSKII TYPE RESULT FOR COINCIDENCES AND PERIODIC SOLUTIONS FOR FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we give a version of Krasnoselskii's fixed point theorem in cones for coincidences and applications to functional-differential equations.
Key Words and Phrases: Krasnoselskii type theorem for coincidences, periodic solution, functional-differential equations.
2000 Mathematics Subject Classification: 34K13, 54H25.

1. Introduction

The purpose of this article is to give a general existence principle of Krasnoselskii type for coincidences. We study the existence of positive solutions (in a cone) to the operator equation

$$Lx = T(x) \tag{1}$$

where L is a linear map and T is a nonlinear operator. Here $L, T : X \to Y$ where X is a Banach space and Y is a normed space.

There are many contributions on the existence of solutions for this problem when L is a linear Fredholm map of index zero, which use coincidence degree methods firstly introduced by Mawhin [2] or, theorems of Leray-Schauder type such as those introduced in [5](see also [3], [6]). In this setting, if L is a Fredholm map of index zero, i.e.

$$\operatorname{Im} L \text{ is closed and } \dim\ker L = co\dim\operatorname{Im} L < \infty,$$

and we take $X = X_1 \oplus X_2, Y = Y_1 \oplus Y_2$, where $X_1 = \ker L$ and $Y_2 = \operatorname{Im} L$ and $P : X \to X_1, Q : Y \to Y_1$ two continuous linear projectors and $J : X_1 \to Y_1$ a linear isomorphism, then $L + JP$ is a bijective linear map. Then the equation (1) is equivalent to the fixed point problem

$$x = (L + JP)^{-1}(T + JP)(x).$$

Our approach is different, since in our case L can be any linear map, so we will need another way to transform the coincidence equation (1) to a fixed point problem.

In what follows by a cone of X we mean a closed convex subset $K \subset X$ with $K \setminus \{0\} \neq \emptyset, \lambda K \subset K$ for every $\lambda \in \mathbb{R}_+$ and $K \cap (-K) = \{0\}$. Any cone K induces a partial order relation in X, denoted by $\preceq$. Hence $u \preceq v$ if and only if $v - u \in K$. We shall say that $u \prec v$ if $v - u \in K \setminus \{0\}$.

We use the following version of Krasnoselskii's theorem(see [1]).

Theorem 1. *Let $(X, ||\cdot||)$ be a normed linear space, $K \subset X$ a cone, $r, R \in \mathbb{R}_+$, $0 < r < R$, $K_{r,R} := \{u \in K : r \leq ||u|| \leq R\}$ and let $N : K_{r,R} \to K$ be a compact map. Assume that one of the following conditions is satisfied:*

(a) $N(u) \not\prec u$ if $||u|| = r$, and $N(u) \not\succ u$ if $||u|| = R$;

(b) $N(u) \not\succ u$ if $||u|| = r$, and $N(u) \not\prec u$ if $||u|| = R$.

Then N has a fixed point u in K with $r \leq ||u|| \leq R$.

2. The abstract coincidence result

Theorem 2. *Let $K \subset X$ a cone, $r, R \in \mathbb{R}_+$, $0 < r < R$, $T : K \to Y$ a completely continuous map and $J : X \to Y$ a linear map such that*

(a) $L + J : X \to Y$ is invertible

(b) $(T + J)(K) \subseteq (L + J)(K) := \widetilde{K}$.

Assume that one of the following conditions is satisfied:

$$(i) \quad \begin{cases} Lx - T(x) \notin \widetilde{K} \text{ for } ||x|| = r \\ T(x) - Lx \notin \widetilde{K} \text{ for } ||x|| = R \end{cases}$$

$$(ii) \quad \begin{cases} T(x) - Lx \notin \widetilde{K} \text{ for } ||x|| = r \\ Lx - T(x) \notin \widetilde{K} \text{ for } ||x|| = R \end{cases}$$

Then there exist $x \in K_{r,R}$ such that $Lx = T(x)$.

Proof. Since $L + J$ is invertible, the coincidence equation (1) is equivalent with the fixed point equation

$$x = (L + J)^{-1}(T + J)(x).$$

We define the operator $\widetilde{T} : K_{r,R} \to K$ by

$$\widetilde{T}(x) = (L + J)^{-1}(T + J)(x).$$

By condition (b), it is easy to see that $\widetilde{T}$ is well defined and completely continuous.

We will consider that condition (i) takes place, in the other case the proof can be made by a similar reasoning.

We shall prove that $\widetilde{T}$ satisfies the assumptions of Theorem 1.

First step. We will show that $\widetilde{T}(x) \not\prec x$ for $||x|| = r$.

Assume that there exists $x \in K$ with $||x|| = r$ such that $x - \widetilde{T}(x) \in K \setminus \{0\}$. This involves that there exists $y \in K \setminus \{0\}$ such that $x - \widetilde{T}(x) = y$. Then

$$x - (L + J)^{-1}(T + J)(x) = y$$

and by linearity of $L + J$ we obtain

$$(L + J)(x) - (T + J)(x) = (L + J)(y).$$

Therefore

$$(L - T)(x) = (L + J)(y)$$

and since y was from K we obtain that $(L-T)(x) \in \widetilde{K}$ for $||x|| = r$ which is a contradiction to the first relation from (i).

Second step. We will show that $\widetilde{T}(x) \not\succ x$ for $||x|| = R$.

In a similar way, if we assume that there exists $x \in K$ with $||x|| = R$ such that $\widetilde{T}(x) - x \in K \setminus \{0\}$ we obtain that there exists $y \in K \setminus \{0\}$ such that $\widetilde{T}(x) - x = y$ and then will obtain the contradiction $(T-L)(x) \in \widetilde{K}$ for $||x|| = R$, to the second relation from (i).

Thus we can apply Krasnoselskii's theorem and the proof is finished.

3. Application

In what follows we will apply the above result to give an existence result for positive periodic solutions of the problem (inspired by [4])

$$\begin{cases} x^{'}(t) = a(t)x(t) - F(x)(t) \\ x(0) = x(\tau) \end{cases} \tag{2}$$

Here $\tau > 0$ is a number, $a \in C_\tau(\mathbb{R}, \mathbb{R}_+) := \{x \in C(\mathbb{R}, \mathbb{R}_+) : x(t+\tau) = x(t)$ for any $t \in [0,\tau]\}$ and $F : C_\tau(\mathbb{R}, \mathbb{R}_+) \to C_\tau(\mathbb{R}, \mathbb{R}_+)$ is continuous.

We will say that F is nondecreasing if for any $u, v \in X$ with $u(t) \le v(t)$ we have that $F(u)(t) \le F(v)(t)$ for every $t \in [0,\tau]$. Similarly, F is nonincreasing if for any $u, v \in X$ with $u(t) \le v(t)$ we have that $F(u)(t) \ge F(v)(t)$ for every $t \in [0,\tau]$.

By integration we obtain

$$x(t) - x(0) = \int_0^t [a(s)x(s) - F(x)(s)]ds$$

If we take

$$\begin{aligned} &X = Y = C_\tau(\mathbb{R}, \mathbb{R}_+), \\ &Lx(t) = x(t) - x(0), \\ &J(x)(t) = -\int_0^t a(s)x(s)ds, \\ &T(x)(t) = \int_0^t [a(s)x(s) - F(x)(s)]ds; \end{aligned}$$

then problem (2) is equivalent with the equation

$$(L+J)(x) = (T+J)(x).$$

Remark 1. *It is a well know fact that problem (2) is also equivalent with the equation*

$$x(t) = \int_t^{t+\tau} G(t,s)F(x)(s)ds$$

where G is the Green's kernel given by

$$G(t,s) = e^{-\int_t^s a(\theta)d\theta} \cdot (1 - e^{-\int_0^\tau a(\theta)d\theta})^{-1}.$$

If we denote by $\delta = e^{-\int_0^\tau a(\theta)d\theta} < 1$ *then it is easy to verify that Green's kernel satisfy the property*

$$0 < \frac{\delta}{1-\delta} \le G(t,s) \le \frac{1}{1-\delta},\ \textit{for any } s \in [t, t+\tau].$$

The next lemma is concerning the invertibility of the operator $L+J$ in Theorem 2.

Lemma 3. *The operator $L+J$ build above is invertible and*

$$(L+J)^{-1}(y)(t) = -\int_t^{t+\tau} G(t,s)y^{'}(s)ds$$

Proof. Let $y = (L+J)(x)$. Then

$$y(t) = x(t) - x(0) - \int_0^t a(s)x(s)ds.$$

From here

$$y^{'}(t) = x^{'}(t) - a(t)x(t),$$

and using Remark 1 we have

$$x(t) = -\int_t^{t+\tau} G(t,s)y^{'}(s)ds =: (L+J)^{-1}(y)(t).$$

The second lemma is concerning with condition (b) in Theorem 2.

Lemma 4. $(T+J)(K) \subseteq (L+J)(K)$ *where* $K = \{x \in X : x(t) \ge \delta\,||x||$ *for any* $t \in [0,\tau]\}$.

Proof. To prove this, it is sufficient to show that

$$(L+J)^{-1}(T+J)(K) \subseteq K.$$

Let $t^* \in [0,\tau]$ a fixed number and $x \in K$. Using Green's kernel properties we have

$$\begin{aligned}
\delta\left|\left|(L+J)^{-1}(T+J)(x)\right|\right| &= \delta \sup_{t\in[0,\tau]} \int_t^{t+\tau} G(t,s)F(x)(s)ds \\
&\le \delta \sup_{t\in[0,\tau]} \int_t^{t+\tau} \frac{1}{1-\delta}F(x)(s)ds = \frac{\delta}{1-\delta}\int_{t^*}^{t^*+\tau} F(x)(s)ds \\
&\le \int_{t^*}^{t^*+\tau} G(t^*,s)F(x)(s)ds = (L+J)^{-1}(T+J)(x)(t^*)
\end{aligned}$$

and so $(L+J)^{-1}(T+J)(x) \in K$.

The following four lemmas are concerning with conditions (i) and (ii) from Theorem 2.

Lemma 5. *Assume that*

(1) *F is nondecreasing;*

(2) *there exists $r > 0$ such that* $\max_{t\in[0,\tau]} F(r)(t) < \frac{1-\delta}{\tau} r$.

Then $T(x) - Lx \notin (L+J)(K)$ for any $x \in K$ with $||x|| = r$.

Proof. Let $x \in K$ with $||x|| = r$ and suppose that there exists $\overline{y} \in K$ such that $T(x) - Lx = (L+J)(\overline{y})$. Then

$$\int_0^t [a(s)x(s) - F(x)(s)]ds - x(t) + x(0) = \overline{y}(t) - \overline{y}(0) - \int_0^t a(s)\overline{y}(s)ds.$$

From here we obtain that

$$x^{'}(t) + \overline{y}^{'}(t) = a(t)[x(t) + \overline{y}(t)] - F(x)(t),$$

and using Remark 1 we have

$$\overline{y}(t) = \int_t^{t+\tau} G(t,s)F(x)(s)ds - x(t).$$

Since $||x|| = r$ there exists $t^* \in [0,\tau]$ such that $x(t^*) = r$. Then, by Green's kernel properties we have

$$\overline{y}(t^*) = \int_{t^*}^{t^*+\tau} G(t^*,s)F(x)(s)ds - r \le \frac{1}{1-\delta}\int_{t^*}^{t^*+\tau} F(x)(s)ds - r$$
$$= \frac{1}{1-\delta}\int_0^{\tau} F(x)(s)ds - r.$$

Further, $x \in K$ so $x(t) \le ||x|| = r$ for any $t \in [0,\tau]$, and by monotonicity of F and condition (2) we have

$$\overline{y}(t^*) \le \frac{1}{1-\delta}\int_0^{\tau} F(r)(s)ds - r < \frac{1}{1-\delta}\int_0^{\tau} \frac{1-\delta}{\tau} r ds - r = 0.$$

This contradicts $\overline{y} \in K$ so the assumption we have made is false.

Lemma 6. *Assume that*

(1) *F is nondecreasing;*

(2) *there exists $r > 0$ such that* $\min_{t\in[0,\tau]} F(\delta r)(t) > \frac{1-\delta}{\delta\tau} r$.

Then $Lx - T(x) \notin (L+J)(K)$ for any $x \in K$ with $||x|| = r$.

Proof. Let $x \in K$ with $||x|| = r$ and suppose that there exists $\widetilde{y} \in K$ such that $Lx - T(x) = (L+J)(\widetilde{y})$. Then

$$x(t) - x(0) - \int_0^t [a(s)x(s) - F(x)(s)]ds = \widetilde{y}(t) - \widetilde{y}(0) - \int_0^t a(s)\widetilde{y}(s)ds.$$

From here we obtain that

$$x^{'}(t) - \widetilde{y}^{'}(t) = a(t)[x(t) - \widetilde{y}(t)] - F(x)(t),$$

and using Remark 1 we have

$$\widetilde{y}(t) = x(t) - \int_t^{t+\tau} G(t,s)F(x)(s)ds.$$

Since $||x|| = r$ there exists $t^* \in [0,\tau]$ such that $x(t^*) = r$. Then, by Green's kernel properties we have

$$\begin{aligned}\widetilde{y}(t^*) = r - \int_{t^*}^{t^*+\tau} G(t^*,s)F(x)(s)ds &\leq r - \frac{\delta}{1-\delta}\int_{t^*}^{t^*+\tau} F(x)(s)ds \\ &= r - \frac{\delta}{1-\delta}\int_0^{\tau} F(x)(s)ds.\end{aligned}$$

Further, $x \in K$ so $x(t) \geq \delta\, ||x|| = \delta r$ for any $t \in [0,\tau]$, and by monotonicity of F and condition (2) we have

$$\widetilde{y}(t^*) \leq r - \frac{\delta}{1-\delta}\int_0^{\tau} F(\delta r)(s)ds < r - \frac{\delta}{1-\delta}\int_0^{\tau} \frac{1-\delta}{\delta\tau} r ds = 0.$$

This contradicts $\widetilde{y} \in K$ and so the assumption we have made is false.

Lemma 7. *Assume that*

(1) *F is nonincreasing;*

(2) *there exists $r > 0$ such that $\max\limits_{t\in[0,\tau]} F(\delta r)(t) < \frac{1-\delta}{\tau} r$.*

Then $T(x) - Lx \notin (L+J)(K)$ for any $x \in K$ with $||x|| = r$.

Proof. Let $x \in K$ with $||x|| = r$ and suppose that there exists $\overline{y} \in K$ such that $T(x) - Lx = (L+J)(\overline{y})$. Then, as in proof of Lemma we obtain

$$\overline{y}(t) = \int_t^{t+\tau} G(t,s)F(x)(s)ds - x(t).$$

Since $||x|| = r$ there exists $t^* \in [0, \tau]$ such that $x(t^*) = r$. Then, by Green's kernel properties we have

$$\overline{y}(t^*) = \int_{t^*}^{t^*+\tau} G(t^*, s)F(x)(s)ds - r \leq \frac{1}{1-\delta} \int_{t^*}^{t^*+\tau} F(x)(s)ds - r$$
$$= \frac{1}{1-\delta} \int_0^\tau F(x)(s)ds - r.$$

Further, $x \in K$ so $x(t) \geq \delta\, ||x|| = \delta r$ for any $t \in [0, \tau]$, and by monotonicity of F and condition (2) we have

$$\overline{y}(t^*) \leq \frac{1}{1-\delta} \int_0^\tau F(\delta r)(s)ds - r < \frac{1}{1-\delta} \int_0^\tau \frac{1-\delta}{\tau} r ds - r = 0.$$

This contradicts $\overline{y} \in K$ so the assumption we have made is false.

Lemma 8. *Assume that*
(1) *F is nonincreasing;*
(2) *there exists $r > 0$ such that* $\min_{t\in[0,\tau]} F(r)(t) > \frac{1-\delta}{\delta\tau} r.$

Then $Lx - T(x) \notin (L + J)(K)$ for any $x \in K$ with $||x|| = r$.

Proof. Let $x \in K$ with $||x|| = r$ and suppose that there exists $\widetilde{y} \in K$ such that $Lx - T(x) = (L + J)(\widetilde{y})$. Then, as in proof of Lemma we obtain

$$\widetilde{y}(t) = x(t) - \int_t^{t+\tau} G(t, s)F(x)(s)ds.$$

Since $||x|| = r$ there exists $t^* \in [0, \tau]$ such that $x(t^*) = r$. Then, by Green's kernel properties we have

$$\widetilde{y}(t^*) = r - \int_{t^*}^{t^*+\tau} G(t^*, s)F(x)(s)ds \leq r - \frac{\delta}{1-\delta} \int_{t^*}^{t^*+\tau} F(x)(s)ds$$
$$= r - \frac{\delta}{1-\delta} \int_0^\tau F(x)(s)ds.$$

Further, $x \in K$ so $x(t) \leq ||x|| = r$ for any $t \in [0, \tau]$, and by monotonicity of F and condition (2) we have

$$\widetilde{y}(t^*) \leq r - \frac{\delta}{1-\delta} \int_0^\tau F(r)(s)ds < r - \frac{\delta}{1-\delta} \int_0^\tau \frac{1-\delta}{\delta\tau} r ds = 0.$$

This contradicts $\widetilde{y} \in K$ so the assumption we have made is false.

We are now ready to state the main existence result for this application.

Theorem 9. *Assume that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:*

$$i)\begin{cases} F \text{ is nondecreasing} \\ \max_{t\in[0,\tau]} F(r)(t) < \dfrac{1-\delta}{\tau} r \\ \min_{t\in[0,\tau]} F(\delta R)(t) > \dfrac{1-\delta}{\delta\tau} R \end{cases}$$

$$ii)\begin{cases} F \text{ is nondecreasing} \\ \min_{t\in[0,\tau]} F(\delta r)(t) > \dfrac{1-\delta}{\delta\tau} r \\ \max_{t\in[0,\tau]} F(R)(t) < \dfrac{1-\delta}{\tau} R \end{cases}$$

$$iii)\begin{cases} F \text{ is nonincreasing} \\ \max_{t\in[0,\tau]} F(\delta r)(t) < \dfrac{1-\delta}{\tau} r \\ \min_{t\in[0,\tau]} F(R)(t) > \dfrac{1-\delta}{\delta\tau} R \end{cases}$$

$$iv)\begin{cases} F \text{ is nonincreasing} \\ \min_{t\in[0,\tau]} F(r)(t) > \dfrac{1-\delta}{\delta\tau} r \\ \max_{t\in[0,\tau]} F(\delta R)(t) < \dfrac{1-\delta}{\tau} R \end{cases}$$

Then there exists a positive periodic solution x of equation (2) such that $r \leq ||x|| \leq R$.

Proof. The proof is a simple consequence of Theorem 2 and Lemmas 3 - 8.

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Received: 21.09.2009; Accepted: 13.12.2009.