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SEMIFIXED SETS OF MAPS AND RÅDSTRÖM EMBEDDING IN LOCALLY CONVEX TOPOLOGICAL LINEAR SPACES

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Abstract. In the space of the compact convex subsets of a locally convex topological linear space some results concerning the existence of fixed and semifixed sets for singlevalued and multivalued maps are established. The approach is based on a suitable version of the classical embedding construction of Radstrom.

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1. Introduction

In the present paper we study the existence of semifixed sets for maps in hyperspaces (i.e. spaces of sets) in the case where the underlying space $\mathbb{E}$ is a Hausdorff locally convex topological linear space. When $\mathbb{E}$ is Banach, semifixed sets of maps have been considered in [3], where also an application to set differential equations can be found.

Let $\mathfrak{X}$ be the hyperspace of all nonempty compact convex subsets of $\mathbb{E}$, equipped with the Hausdorff topology, and let $\mathcal{C}(\mathfrak{X})$ be the family of all nonempty compact convex subsets of $\mathfrak{X}$.

A *semifixed set* of a singlevalued map $\varphi : \mathcal{A} \to \mathfrak{X}$, where $\phi \neq \mathcal{A} \subset \mathfrak{X}$, is a set $A \in \mathcal{A}$ satisfying one of the relations:

$$A \cap \varphi(A) \neq \phi\ , \quad A \subset \varphi(A)\ , \quad A \supset \varphi(A)\ .$$

More generally, a *semifixed set* of a multivalued map $\Phi : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$ is a set $A \in \mathcal{A}$ such that, for some $F \in \Phi(A)$, one of the relations is valid:

$$A \cap F \neq \phi\ , \quad A \subset F\ , \quad A \supset F\ .$$

If $A = \varphi(A)$ or $A \in \Phi(A)$, then A is a *fixed set* of φ or Φ.

The theory of fixed points in topological linear spaces has a long history, starting with the fundamental contributions of Tychonoff [20] and Ky Fan [6]. For developments, applications and more recent results, see Górniewicz [8], Bugajewski [1], Cobzaş [2], Park [16], [17], where also additional bibliography can be found.

We shall prove the following hyperspace versions of the classical fixed point theorems of Tychonoff [21] and Ky Fan [6].

Denote by $\mathcal{A}$ a nonempty compact convex subset of $\mathfrak{X}$. Then:

I. Any continuous singlevalued map $\varphi : \mathcal{A} \to \mathcal{A}$ has a fixed set.

II. Any upper semicontinuous multivalued map $\Phi : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$, with values contained in $\mathcal{A}$, has a fixed set.

The latter result will be used to prove the existence of semifixed sets of maps, under Tychonoff or Ky Fan type assumptions, namely:

I'. Let $\varphi : \mathcal{A} \to \mathfrak{X}$ be a continuous singlevalued map with the property that for each $X \in \mathcal{A}$ there is $Z \in \mathcal{A}$ such that $Z \cap \varphi(X) \neq \phi$ (resp. $Z \subset \varphi(X)$, $Z \supset \varphi(X)$). Then there exists a set $A \in \mathcal{A}$ such that $A \cap \varphi(A) \neq \phi$ (resp. $A \subset \varphi(A)$, $A \supset \varphi(A)$).

II'. Let $\Phi : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$ be an upper semicontinuous multivalued map satisfying the following condition: for each $X \in \mathcal{A}$ there are sets $Z \in \mathcal{A}$ and $F \in \Phi(X)$ such that $Z \cap F \neq \phi$ (resp. $Z \subset F$, $Z \supset F$). Then there exists a set $A \in \mathcal{A}$ such that $A \cap F$ (resp. $A \subset F$, $A \supset F$) for some $F \in \Phi(A)$.

In this context an important role is played by an embedding result of the space $\mathfrak{X}$ into a positively-semilinear subspace of a Hausdorff locally convex topological linear space. This is essentially due to Hörmander [9], who established it by a functional analysis approach (though for a different class of convex sets not closed under the addition operation). Here an alternative topological construction is presented which is based, as in Rådström [20], on the well known technique of embedding an Abelian semigroup with cancellation law into an Abelian group. This technique with its refinements was previously used by Urbański [22] in order to extend the Rådström embedding theorem to arbitrary topological linear spaces. Yet, for the sake of completeness, we shall review the Rådström construction in Hausdorff locally convex topological linear spaces, a setting with additional structure which is appropriate in the investigation of the semifixed sets of maps we discuss here.

It is worth noting that fixed and semifixed sets of maps can occur in the theory of set differential and fuzzy differential equations [3], [4], [13], [14], [18], [19].

The paper consists of five sections, the first of which is the introduction. Section 2 contains notation and preliminary results. In Section 3, the Rådström embedding is discussed. Fixed and semifixed set for singlevalued and multivalued maps are presented in Section 4 and Section 5, respectively.

2. Notation and preliminary results

Throughout this paper $\mathbb{E}$ is a Hausdorff locally convex topological linear real space, $2^{\mathbb{E}}$ is the family of all nonempty subsets of $\mathbb{E}$, and $\mathfrak{X}$ is the space defined by

$$\mathfrak{X} = \{X \in 2^{\mathbb{E}} | X \text{ is compact convex}\} .$$

For $X, Y \in \mathfrak{X}$ and $\lambda \in \mathbb{R}^+ = [0, +\infty)$, set $X + Y = \{x + y \in \mathbb{E} | x \in X \text{ and } y \in Y\}$, $\lambda X = \{\lambda x \in \mathbb{E} | x \in X\}$. $\mathfrak{X}$ is closed under the above operations of *addition* and *multiplication* by nonnegative scalars. Moreover $\mathfrak{X}$ is a *positively-semilinear space*, by which we mean that for X, Y, $Z \in \mathfrak{X}$ and $\lambda, \mu \in \mathbb{R}^+$ the following properties are satisfied: (i) $X+\{0\}=X$ (0 is the zero of $\mathbb{E}$); (ii) $X+Y=Y+X$; (iii) $(X+Y)+Z=X+(Y+Z)$; (iv) $1X=X$; (v) $\lambda(\mu X)=(\lambda\mu)X$; (vi) $\lambda(X+Y)=\lambda X+\lambda Y$; (vii) $(\lambda+\mu)X=\lambda X+\mu X$.

Remark 2.1. Properties (i)-(vi) remain valid for X, Y, $Z \in 2^{\mathbb{E}}$ and $\lambda, \mu \in \mathbb{R}^+$, while (vii) holds provided $X \in 2^{\mathbb{E}}$ is convex and $\lambda, \mu \in \mathbb{R}^+$. Moreover, the definition of λX is meaningful when $X \in 2^{\mathbb{E}}$ and $\lambda \in \mathbb{R}$, in which case (iv)-(vi) are valid, while (vii) holds in the weaker form (vii)′ $(\lambda+\mu)X \subset \lambda X+\mu X$.

The closure of a set A contained in a topological space is denoted by $\overline{A}$.

For a set $A \subset \mathbb{E}$ let us recall the following definitions: A is *balanced* if $a \in A$ and $\lambda \in [-1,1]$ imply $\lambda a \in A$; A is *bounded* if for each neighborhood U of 0 there exists a number $\lambda > 0$ such that $A \subset \lambda U$; A is *absorbent* if for each $x \in \mathbb{E}$ there exists $r > 0$ such that $x \in \lambda A$ for all λ with $|\lambda| \geq r$. When A is convex and balanced, then A is absorbent if and only if for each $x \in \mathbb{E}$ there exists $\lambda > 0$ such that $x \in \lambda A$.

By a *net* $\{S_i\}_{i\in I}$ we mean any map $S : I \to \mathcal{Z}$, where I is a directed set. A net $\{T_j\}_{j\in J}$ is a *subnet* of $\{S_i\}_{i\in I}$ if there exists a function $N : J \to I$ such that

(a) $T_j = S_{N(j)}$ for each $j \in J$;

(b) for each $i_0 \in I$ there exists $j_0 \in J$ such that $j \geq j_0$ implies $N(j) \geq i_0$.

We refer to Kelley [12] for properties of nets that we shall use in the sequel.

The proof of the following cancellation law lemma can be found in Hu and Papageorgiou [10], p. 64. For further results in this direction, see Pallaschke and Urbański [15].

Lemma 2.2. *Let A, B, $C \subset \mathbb{E}$ be nonempty sets, with B closed and convex and C bounded. Then $A+C \subset B+C$ implies $A \subset B$. If A and B are both closed and convex, then $A+C=B+C$ implies $A=B$.*

Lemma 2.3. *Let $K \subset \mathbb{E}$ be a nonempty compact set such that $K \subset A+U$, where $A \subset \mathbb{E}$ is nonempty and U is an open neighborhood of 0. Then there exists θ, with $0 < \theta < 1$, such that $K \subset A+\theta U$. Moreover if H, $K \subset \mathbb{E}$ are nonempty compact, $A \subset \mathbb{E}$ is nonempty convex, and U is an open convex and balanced neighborhood of 0, then*

$$K+H \subset A+U+H \quad \textit{implies} \quad K \subset A+U\ .$$

Proof. If the first statement is not true there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$ such that $x_n \notin A+\theta_n U$ for every $n \in \mathbb{N}$, where $\theta_n = n/(n+1)$. As K is compact, $\{x_n\}_{n\in\mathbb{N}}$ has a subnet $\{x_{N(i)}\}_{i\in I}$ which converges to some $x \in K$. Since $x \in A+U$, there exist $a \in A$ and $u \in U$ such that $x = a+u$. Moreover $\lambda \mapsto \lambda u$ is continuous and $u \in U$, an open set, hence there is $0 < \sigma < 1$ such that $u \in \sigma U$. Hence $x \in a+\sigma U \subset A+\sigma U$ which shows that $A+\sigma U$ is an open neighborhood of x. Fix $n_0 \in \mathbb{N}$ so that $\theta_{n_0} > \sigma$. By definition of subnet there is $i_1 \in I$ such that if $i \geq i_1$ one has $N(i) \geq n_0$, which implies $\theta_{N(i)} \geq \theta_{n_0} > \sigma$. Thus $x \in A+\sigma U \subset A+\theta_{N(i)}U$ for all $i \geq i_1$. On the other hand $x \in A+\sigma U$, an open set, hence there is $i_2 \in I$ such that $x_{N(i)} \in A+\sigma U$, for all $i \geq i_2$. Then by taking $i \in I$ with $i \geq i_1$, $i \geq i_2$ one has $x_{N(i)} \in A + \theta_{N(i)}U$ and so, from the contradiction, the first statement follows.

Consider the second statement. As $K + H$ is compact, there exists $0 < \theta < 1$ such that $K + H \subset A + H + \theta U \subset \overline{A + \theta U} + H$. By [11], Theorem 4, p. 66, H is totally bounded and a fortiori bounded, hence Lemma 2.2 implies $K \subset \overline{A + \theta U}$. Moreover,

$$\overline{A + \theta U} \subset A + U \ . \tag{2.1}$$

In fact each $z \in \overline{A + \theta U}$ satisfies $(z + (1-\theta)U) \cap (A + \theta U) \neq \phi$ and thus, for some $a \in A$ and u, $v \in U$, one has $z + (1-\theta)u = a + \theta v$. Since U is balanced and convex, it follows that $z = a + \theta v + (1-\theta)(-u) \in A + U$, proving (2.1). Therefore $K \subset \overline{A + \theta U} \subset A + U$, completing the proof.

We now introduce in $\mathfrak{X}$ the Hausdorff topology (similar to the topology induced by the Hausdorff distance in a metric space) and we review some of its properties.

Set

$$\mathcal{F} = \{\mathcal{N} \subset 2^{\mathbb{E}} | \mathcal{N} \quad \text{is a base of neighborhoods of } 0 \in \mathbb{E}\} \ .$$

For $\mathcal{N} \in \mathcal{F}$, $A \in \mathfrak{X}$ and $U \in \mathcal{N}$, set

$$\mathcal{V}_{\mathcal{N}}(A, U) = \{X \in \mathfrak{X} | X \subset A + U \text{ and } A \subset X + U\} \ ,$$

and let

$$\mathcal{B}_{\mathcal{N}}(A) = \{\mathcal{V}_{\mathcal{N}}(A, U)\}_{U \in \mathcal{N}} \ .$$

Proposition 2.4. *Let $\mathcal{N} \in \mathcal{F}$. For each $A \in \mathfrak{X}$ the family $\mathcal{B}_{\mathcal{N}}(A)$ has the following properties:*

(i): *if $\mathcal{V}_{\mathcal{N}}(A, U) \in \mathcal{B}_{\mathcal{N}}(A)$ then $A \in \mathcal{V}_{\mathcal{N}}(A, U)$;*

(ii): *if $\mathcal{V}_{\mathcal{N}}(A, U_i) \in \mathcal{B}_{\mathcal{N}}(A)$, $i = 1, 2$, then there exists $\mathcal{V}_{\mathcal{N}}(A, U) \in \mathcal{B}_{\mathcal{N}}(A)$ such that $\mathcal{V}_{\mathcal{N}}(A, U) \subset \mathcal{V}_{\mathcal{N}}(A, U_1) \cap \mathcal{V}_{\mathcal{N}}(A, U_2)$;*

(iii): *for each $\mathcal{V}_{\mathcal{N}}(A, U) \in \mathcal{B}_{\mathcal{N}}(A)$ there exists $\mathcal{V}_{\mathcal{N}}(A, U_1) \in \mathcal{B}_{\mathcal{N}}(A)$ such that $\mathcal{V}_{\mathcal{N}}(A, U_1) \subset \mathcal{V}_{\mathcal{N}}(A, U)$ and such that, for every $B \in \mathcal{V}_{\mathcal{N}}(A, U_1)$ there exists $\mathcal{V}_{\mathcal{N}}(B, U_2) \in \mathcal{B}_{\mathcal{N}}(B)$ with $\mathcal{V}_{\mathcal{N}}(B, U_2) \subset \mathcal{V}_{\mathcal{N}}(A, U)$.*

Proof. (i) is obvious, while (ii) is valid if one takes $U \in \mathcal{N}$ such that $U \subset U_1 \cap U_2$. To show (iii), let V be a convex neighborhood of 0 contained in U, and take $U_1 = U_2 = W$ with $W \in \mathcal{N}$ satisfying $W \subset V/2$. For any $X \in \mathcal{V}_{\mathcal{N}}(B, U_2)$, where $B \in \mathcal{V}_{\mathcal{N}}(A, U_1)$, one has $X \subset B + U_2$, $B \subset X + U_2$, $B \subset A + U_1$, $A \subset B + U_1$, which imply $X \subset A + 2W \subset A + U$, $A \subset X + 2W \subset X + U$. Hence $X \in \mathcal{V}_{\mathcal{N}}(A, U)$, showing that $\mathcal{V}_{\mathcal{N}}(B, U_2) \subset \mathcal{V}_{\mathcal{N}}(A, U)$. As the inclusion $\mathcal{V}_{\mathcal{N}}(A, U_1) \subset \mathcal{V}_{\mathcal{N}}(A, U)$ is obvious, also (iii) holds, completing the proof.

For fixed $\mathcal{N} \in \mathcal{F}$ consider the family of sets $\{\mathcal{B}_{\mathcal{N}}(A)\}_{A \in \mathfrak{X}}$. A set $\mathcal{A} \in \mathfrak{X}$ is said to be *open* of for every $A \in \mathcal{A}$ there exists a set $\mathcal{V}_{\mathcal{N}}(A, U) \in \mathcal{B}_{\mathcal{N}}(A)$ such that $\mathcal{V}_{\mathcal{N}}(A, U) \subset \mathcal{A}$.

By virtue of [11], Theorem 1, p. 7, and Proposition 2.4 one has:

Proposition 2.5. *The system $\tau_{\mathcal{N}}$ of all open sets of $\mathfrak{X}$ is a topology for $\mathfrak{X}$. Moreover, for each $A \in \mathfrak{X}$, the family $\mathcal{B}_{\mathcal{N}}(A)$ is a base of neighborhoods of A in the resulting topological space $(\mathfrak{X}, \tau_{\mathcal{N}})$.*

Proposition 2.6. *If $\mathcal{N}'$, $\mathcal{N}'' \in \mathcal{F}$ are arbitrary bases of neighborhoods of* 0*, then the resulting topologies $\tau_{\mathcal{N}'}$ and $\tau_{\mathcal{N}''}$ for $\mathfrak{X}$ coincide.*

Proof. Let $A \in \mathfrak{X}$ and let $\mathcal{V}_{\mathcal{N}'}(A, U') \in \mathcal{B}_{\mathcal{N}'}(A)$. As $U' \in \mathcal{N}'$ and $\mathcal{N}''$ is a base of neighborhoods of 0, there exists $U'' \in \mathcal{N}''$ such that $U'' \subset U'$. Hence $\mathcal{V}_{\mathcal{N}''}(A, U'') \in \mathcal{B}_{\mathcal{N}''}(A)$ satisfies $\mathcal{V}_{\mathcal{N}''}(A, U'') \subset \mathcal{V}_{\mathcal{N}'}(A, U')$. This and the analogous relation obtained by interchanging the roles of U', $\mathcal{N}'$ and U'', $\mathcal{N}''$ yield the result.

Remark 2.7. By Proposition 2.6, the resulting topology $\tau_{\mathcal{N}}$ for $\mathfrak{X}$ is independent of $\mathcal{N} \in \mathcal{F}$ and thus it will be denoted by τ.

In the sequel the space $\mathfrak{X}$ is understood to be endowed with the topology τ and the resulting topological space will be denoted by $(\mathfrak{X}, \tau)$.

Set

$$\mathcal{N}_0 = \{U \in 2^{\mathbb{E}} | U \text{ is an open convex balanced neighborhood of } 0 \in \mathbb{E}\} \ , \tag{2.2}$$

and observe that $\tau_{\mathcal{N}_0} = \tau$, for $\mathcal{N}_0 \in \mathcal{F}$.

Proposition 2.8. *For each $A \in \mathfrak{X}$ the base of neighborhoods $\mathcal{B}_{\mathcal{N}_0}(A)$ of A consists of open and convex sets.*

Proof. Each $\mathcal{V}_{\mathcal{N}_0}(A, U) \in \mathcal{B}_{\mathcal{N}_0}(A)$ is open. Let $X \in \mathcal{V}_{\mathcal{N}_0}(A, U)$, thus $X \subset A + U$, $A \subset X + U$. By Lemma 2.3, there is $0 < \theta < 1$ such that $X \subset A + \theta U$, $A \subset X + \theta U$. Each $Y \in \mathcal{V}_{\mathcal{N}_0}(X, (1-\theta)U)$ satisfies $Y \subset X + (1-\theta)U$, $X \subset Y + (1-\theta)U$ and hence

$$Y \subset A + \theta U + (1-\theta)U = A + U \ , \qquad A \subset Y + (1-\theta)U + \theta U = Y + U \ ,$$

for U is convex. Therefore $Y \in \mathcal{V}_{\mathcal{N}_0}(A, U)$, and thus $\mathcal{V}_{\mathcal{N}_0}(X, (1-\theta)U) \subset \mathcal{V}_{\mathcal{N}_0}(A, U)$. Hence $\mathcal{V}_{\mathcal{N}_0}(A, V)$ is open. Furthermore $\mathcal{V}_{\mathcal{N}_0}(A, U)$ is convex for, if $X_i \in \mathcal{V}_{\mathcal{N}_0}(A, U)$, $i = 1, 2$ and $\lambda \in [0, 1]$, one has $X_i \subset A + U$, $A \subset X_i + U$, $i = 1, 2$, which imply $\lambda X_1 + (1-\lambda)X_2 \subset A + U$, $A \subset \lambda X_1 + (1-\lambda)X_2 + U$. Therefore $\lambda X_1 + (1-\lambda)X_2 \in \mathcal{V}_{\mathcal{N}_0}(A, U)$, completing the proof.

Proposition 2.9. *The topological space $(\mathfrak{X}, \tau)$ is Hausdorff.*

Proof. Let A, $B \in \mathfrak{X}$, $A \neq B$. Let $a \in A \backslash B$ (if $a \in B \backslash A$, the argument is similar). By the Hahn-Banach theorem [11], Theorem 6, p. 73, there exist a continuous linear functional $f : \mathbb{E} \to \mathbb{R}$ and an $\varepsilon > 0$ such that

$$\sup\{f(x) | x \in B\} + \varepsilon < f(a) \ . \tag{2.3}$$

Taking any $U \in \mathcal{N}_0$, with $U \subset \{x \in \mathbb{E} | \ |f(x)| < \varepsilon/2\}$, one has

$$\mathcal{V}_{\mathcal{N}_0}(A, U) \cap \mathcal{V}_{\mathcal{N}_0}(B, U) = \phi \ . \tag{2.4}$$

In the contrary case, there exists $X \in \mathfrak{X}$ such that $X \subset A + U$, $A \subset X + U$, $X \subset B + U$, $B \subset X + U$, which imply $A \subset B + 2U$, $B \subset A + 2U$. Hence $a = b + 2u$ for some $b \in B$ and $u \in U$, and so $f(a) = f(b) + 2f(u) < \sup\{f(x) | x \in B\} + \varepsilon$, contradicting (2.3). Therefore (2.4) holds, completing the proof.

Proposition 2.10. *Let $\mathfrak{X}$ be endowed with the topology τ, and $\mathfrak{X} \times \mathfrak{X}$ and $\mathbb{R}^+ \times \mathfrak{X}$ with the respective product topologies. Then*

(i): *the map $(X, Y) \mapsto X + Y$ is continuous from $\mathfrak{X} \times \mathfrak{X}$ to $\mathfrak{X}$;*

(ii): *the map* $(\lambda, X) \mapsto \lambda X$ *is continuous from* $\mathbb{R}^+ \times \mathfrak{X}$ *to* $\mathfrak{X}$.

Proof. (i) Let $(A, B) \in \mathfrak{X} \times \mathfrak{X}$ and let $\mathcal{V}_{\mathcal{N}_0}(A + B, U) \in \mathcal{B}_{\mathcal{N}_0}(A + B)$, where $\mathcal{N}_0$ is given by (2.2). Then each $(X, Y) \in \mathcal{V}_{\mathcal{N}_0}(A, (1/2)U) \times \mathcal{V}_{\mathcal{N}_0}(B, (1/2)U)$ satisfies $X \subset A + (1/2)U$, $A \subset X + (1/2)U$, $Y \subset B + (1/2)U$, $B \subset Y + (1/2)U$, and thus $X + Y \subset A + B + U$, $A + B \subset X + Y + U$. Hence $X + Y \in \mathcal{V}_{\mathcal{N}_0}(A + B, U)$, and (i) holds.

(ii) Let $(\lambda_0, A) \in \mathbb{R}^+ \times \mathfrak{X}$ and $\mathcal{V}_{\mathcal{N}_0}(\lambda_0 A, U) \in \mathcal{B}_{\mathcal{N}_0}(\lambda_0 A)$ be given. Fix $\mu > 0$ so that $A \subset \mu U$ and let $0 < \sigma < 1/2(\lambda_0 + 1)$.

Let $(\lambda, X) \in [\lambda_0, \lambda_0 + 1/2(\mu + 1)] \times \mathcal{V}_{\mathcal{N}_0}(A, \sigma U)$. Clearly

$$X \subset A + \sigma U \qquad A \subset X + \sigma U \ , \tag{2.5}$$

and thus

$$\begin{aligned} \lambda X \subset \lambda(A + \sigma U) &= \lambda_0 A + (\lambda - \lambda_0)A + \lambda_0 \sigma U + (\lambda - \lambda_0)\sigma U \\ &\subset \lambda_0 A + (\lambda - \lambda_0)\mu U + \lambda_0 \sigma U + \sigma U \ , \end{aligned}$$

for $(\lambda - \lambda_0)\sigma \le (1/2(\mu + 1))\sigma < \sigma$. Since $(\lambda - \lambda_0)\mu + (\lambda_0 + 1)\sigma < 1$, it follows

$$\lambda X \subset \lambda_0 A + U \ . \tag{2.6}$$

On the other hand $X \subset A + \sigma U \subset \mu U + \sigma U \subset (\mu + 1)U$, and hence $(\lambda - \lambda_0)X \subset (\lambda - \lambda_0)(\mu + 1)U \subset (1/2)U$. As the latter set is a balanced neighborhood of 0, one has $0 \in (\lambda - \lambda_0)X + (1/2)U$ and thus, by adding $\lambda_0 X$ to each side,

$$\lambda_0 X \subset \lambda X + (1/2)U \ . \tag{2.7}$$

By virtue of (2.5) and (2.7) one has

$$\lambda_0 A \subset \lambda_0 X + \lambda_0 \sigma U \subset \lambda X + (1/2)U + \lambda_0 \sigma U \subset \lambda X + U$$

which, combined with (2.6), implies $\lambda X \in \mathcal{V}_{\mathcal{N}_0}(\lambda_0 A, U)$.

Let $(\lambda, X) \in [\lambda_0 - 1/2(\mu+1), \lambda_0] \cap \mathbb{R}^+$. Clearly $0 \in (\lambda_0 - \lambda)A + (\lambda_0 - \lambda)(-A) \subset (\lambda_0 - \lambda)A + (\lambda_0 - \lambda)\mu U$, as $A \subset \mu U$, a balanced set. Then from (2.5), as $(\lambda_0 - \lambda)\mu + \lambda\sigma < 1$, one has

$$\lambda X \subset \lambda A + \lambda \sigma U \subset \lambda A + (\lambda_0 - \lambda)A + (\lambda_0 - \lambda)\mu U + \lambda \sigma U \subset \lambda_0 A + U \ . \tag{2.8}$$

On the other hand from (2.5) it follows

$$\begin{aligned} \lambda_0 A \subset \lambda_0 X + \lambda_0 \sigma U &= \lambda X + (\lambda_0 - \lambda)X + \lambda_0 \sigma U \\ &\subset \lambda X + (\lambda_0 - \lambda)A + (\lambda_0 - \lambda)\sigma U + \lambda_0 \sigma U \\ &\subset \lambda X + (\lambda_0 - \lambda)\mu U + (\lambda_0 - \lambda)\sigma U + \lambda_0 \sigma U \subset \lambda X + U, \end{aligned} \tag{2.9}$$

for $A \subset \mu U$ and $(\lambda_0 - \lambda)\mu + (\lambda_0 - \lambda)\sigma + \lambda_0 \sigma < 1$. Combining (2.8) and (2.9) gives $\lambda X \in \mathcal{V}_{\mathcal{N}_0}(\lambda_0 A, U)$, hence (ii) holds, completing the proof.

The following proposition shows that τ is a natural topology for $\mathfrak{X}$.

Proposition 2.11. *Let* $\mathfrak{X}_0$ *be the subspace of* $(\mathfrak{X}, \tau)$ *consisting of all singleton subsets of* $\mathbb{E}$, *with the induced topology. Then the map* $J : \mathfrak{X}_0 \to \mathbb{E}$ *defined by* $J(\{x\}) = x$ *is a positively-semilinear homeomorphism from* $\mathfrak{X}_0$ *onto* $\mathbb{E}$.

Proof. J is bijective and positively-semilinear, i.e. $J(\lambda\{x\}+\mu\{x'\}) = \lambda x+\mu x'$ for all $\{x\}$, $\{x'\} \in \mathfrak{X}_0$ and λ, $\mu \geq 0$. Moreover, J is a homemorphism. In fact, for $\{a\} \in \mathfrak{X}$ and $U \in \mathcal{N}_0$, one has

$$J(\mathfrak{X}_0 \cap \mathcal{V}_{\mathcal{N}_0}(\{a\}, U)) = \{J(\{x\})|\{x\} \subset \{a\} + U \ , \ \{a\} \subset \{x\} + U\} = a + U \ ,$$

where the last equality holds, as U is balanced. Since, by Proposition 2.8, $\mathfrak{X}_0 \cap \mathcal{V}_{\mathcal{N}_0}(\{a\}, U)$ is an open neighborhood of $\{a\}$, it follows that J is continuous. The proof that J^{-1} is continuous is similar and so it is omitted. This completes the proof.

3. Rådström embedding

In this section we shall construct a topological linear space $\mathfrak{Y}$ in which $\mathfrak{X}$ will be embedded as a positively-semilinear subspace. This will be done by using the classical method of embedding an Abelian semigroup with cancellation rule into an Abelian group (see [7] and, for spaces of sets, [20], [22]).

Introduce in $\mathfrak{X} \times \mathfrak{X}$ the relation $\sim$ defined by

$$(X', Y') \sim (X'', Y'') \text{ if and only if } X' + Y'' = X'' + Y' \ .$$

By using Lemma 2.2, one can easily see that $\sim$ is an equivalence relation. For $(X, Y) \in \mathfrak{X} \times \mathfrak{X}$, denote by $[X, Y]$ the equivalence class containing (X, Y), and set $\mathfrak{Y} = (\mathfrak{X} \times \mathfrak{X})/\sim$, that is

$$\mathfrak{Y} = \{\Gamma \subset \mathfrak{X} \times \mathfrak{X}|\Gamma = [X, Y] \text{ for some } (X, Y) \in \mathfrak{X} \times \mathfrak{X}\} \ .$$

Thus $\mathfrak{Y}$ is the set of all disjoint equivalence classes of $\mathfrak{X} \times \mathfrak{X}$.

For $[X, Y]$, $[X', Y'] \in \mathfrak{Y}$ and $\lambda \in \mathbb{R}$ define the *sum* $[X, Y]+[X', Y']$ and the *product* by a scalar $\lambda[X, Y]$, by setting

$$[X, Y] + [X', Y'] = [X + X', Y + Y']$$

$$\lambda[X, Y] = \begin{cases} [\lambda X, \lambda Y] & \text{if} \quad \lambda \geq 0 \\ [|\lambda|Y, |\lambda|X] & \text{if} \quad \lambda < 0 \end{cases} \ .$$

Remark 3.1. The above definitions are meaningful. In fact, if $(A, B) \sim (X, Y)$, $(A', B') \sim (X', Y')$ and $\lambda \in \mathbb{R}$, then

$$[A + A', B + B'] = [X + X', Y + Y'] \qquad \lambda[A, B] = \lambda[X, Y] \ .$$

Moreover, it is easy to prove the following

Proposition 3.2. *The space $\mathfrak{Y}$ endowed with the above operations of addition, and multiplication by scalars $\lambda \in \mathbb{R}$ is a linear real space with zero $[Z, Z] \equiv [0, 0]$.*

Set

$$\mathcal{K} = \{\Gamma \in \mathfrak{Y}|\Gamma = [X + Z, Z] \quad \text{for some} \quad X, Z \in \mathfrak{X}\} \ . \tag{3.1}$$

Remark 3.3. $\mathcal{K}$ is a positively-semilinear subspace of $\mathfrak{Y}$, i.e. $\mathcal{K}$ is closed under the operations of *addition* and *multiplication* by scalars $\lambda \in \mathbb{R}^+$. In fact, for $[X' + Z', Z']$, $[X'' + Z'', Z''] \in \mathcal{K}$ and $\lambda, \mu \in \mathbb{R}^+$, setting $X = \lambda X' + \mu X''$, $Z = \lambda Z' + \mu Z''$, one has $\lambda[X' + Z', Z'] + \mu[X'' + Z'', Z''] = [X + Z, Z] \in \mathcal{K}$.

Now let $J : \mathcal{K} \to \mathfrak{X}$ be the map given by

$$J([X+Z,Z]) = X \quad \text{for each} \quad [X+Z,Z] \in \mathcal{K} \ .$$

This definition is not ambiguous for, if $(X'+Z',Z') \sim (X+Z,Z)$, then by Lemma 2.2 one has $J([X'+Z',Z']) = J([X+Z,Z])$.

The following proposition is immediate.

Proposition 3.4. *The map $J : \mathcal{K} \to \mathfrak{X}$ is a bijection from $\mathcal{K}$ onto $\mathfrak{X}$. Moreover, for $[X+Z,Z]$, $[X'+Z',Z'] \in \mathcal{K}$ and $\lambda, \mu \in \mathbb{R}^+$ one has*

$$J(\lambda[X+Z,Z] + \mu[X'+Z',Z']) = \lambda J([X+Z,Z]) + \mu J([X'+Z',Z']) \ ,$$

i.e. J is a positively-semilinear isomorphism of $\mathcal{K}$ onto $\mathfrak{X}$.

Let $\mathcal{N}_0$ be given by (2.2). For $[A,B] \in \mathfrak{Y}$ and $U \in \mathcal{N}_0$ set

$$\mathcal{W}([A,B],U) = \{[X,Y] \in \mathfrak{Y} | X+B \subset A+Y+U \text{ and } A+Y \subset X+B+U\} \ ,$$

and let

$$\mathcal{B}([A,B]) = \{\mathcal{W}([A,B],U)\}_{U \in \mathcal{N}_0} \ .$$

Proposition 3.5. *The definition of $\mathcal{W}([A,B],U)$ is meaningful, that is it is independent of the representatives in the equivalence classes $[A,B]$, $[X,Y]$.*

Proof. Let $(A',B') \sim (A,B)$ and $(X',Y') \sim (X,Y)$, thus

$$A'+B = A+B' \qquad X'+Y = X+Y' \ . \tag{3.2}$$

It suffices to show that

$$\begin{cases} X'+B' \subset A'+Y'+U \\ A'+Y' \subset X'+B'+U \end{cases} \quad \text{if and only if} \quad \begin{cases} X+B \subset A+Y+U \\ A+Y \subset X+B+U \end{cases} \ . \tag{3.3}$$

Suppose that both inclusions on the left hand side of (3.3) are satisfied. Then, adding $X+B$ to each side and taking into account (3.2), one has

$$X+B+X'+B' \subset X+B+A'+Y'+U = A+Y+X'+B'+U$$

$$A+Y+X'+B' = A'+B+X+Y' \subset X+B+X'+B'+U \ .$$

Hence, by Lemma 2.3, the inclusions on the right hand side of (3.3) follow at once, proving the "only if" part. The proof of the "if" part is analogous, and thus it is omitted. This completes the proof.

Proposition 3.6. *For each $[A,B] \in \mathfrak{Y}$ the family $\mathcal{B}([A,B])$ has the following properties:*

- (i): *if $\mathcal{W}([A,B],U) \in \mathcal{B}([A,B])$ then $[A,B] \in \mathcal{W}([A,B],U)$;*
- (ii): *if $\mathcal{W}([A,B],U_1) \in \mathcal{B}([A,B])$, $i = 1,2$, then there exists $\mathcal{W}([A,B],U_2) \in \mathcal{B}([A,B])$ such that $\mathcal{W}([A,B],U) \subset \mathcal{W}([A,B],U_1) \cap \mathcal{W}([A,B]),U_2)$;*
- (iii): *for each $\mathcal{W}([A,B],U) \in \mathcal{B}([A,B])$ there exists $\mathcal{W}([A,B],V) \in \mathcal{B}([A,B])$ such that $\mathcal{W}([A,B],V) \subset \mathcal{W}([A,B],U)$ and such that for every $[A',B'] \in \mathcal{W}([A,B],V)$ there exists $\mathcal{W}([A',B'],U') \in \mathcal{B}([A',B'])$ with $\mathcal{W}([A',B'],U') \subset \mathcal{W}([A,B],U)$.*

Proof. (i) is obvious, while (ii) holds if one takes $U = U_1 \cap U_2$, a set in $\mathcal{N}_0$. It will be shown that (iii) holds with $V = U' = (1/2)U$. Let $[X, Y] \in \mathcal{W}([A', B'], U')$, where $[A', B'] \in \mathcal{W}([A, B], V)$, hence

$$X + B' \subset A' + Y + U' \qquad A' + Y \subset X + B' + U'$$
$$A' + B \subset A + B' + V \qquad A + B' \subset A' + B + V \ .$$

It follows

$$X + B' + B \subset A' + Y + B + U' \subset A + B' + Y + U' + V = A + Y + B' + U$$
$$A + Y + A' \subset A + X + B' + U' \subset A' + B + X + U' + V = X + B + A' + U \ ,$$

and so, by Lemma 2.3,

$$X + B \subset A + Y + U \qquad A + Y \subset X + B + U \ .$$

Therefore $\mathcal{W}([A', B'], U') \subset \mathcal{W}([A, B], U)$. As the inclusion $\mathcal{W}([A, B], V) \subset \mathcal{W}([A, B], U)$ is evident, then also (iii) holds, completing the proof.

Now consider the family $\{\mathcal{B}([A, B])\}_{[A,B] \in \mathfrak{Y}}$. A set $\mathcal{A} \subset \mathfrak{Y}$ is said to be *open* if for every $[A, B] \in \mathcal{A}$ there exists a set $\mathcal{W}([A, B], U) \in \mathcal{B}([A, B])$ such that $\mathcal{W}([A, B], U) \subset \mathcal{A}$.

By virtue of [11], Theorem 1, p. 7, and Proposition 3.6 one has:

Proposition 3.7. *The system σ of all open subsets of $\mathfrak{Y}$ is a topology for $\mathfrak{Y}$. Moreover, for each $[A, B] \in \mathfrak{Y}$, the family $\mathcal{B}([A, B])$ is a base of neighborhoods of $[A, B]$ in the resulting topological space $(\mathfrak{Y}, \sigma)$.*

In the sequel the space $\mathfrak{Y}$ will be supposed endowed with the topology σ.

Proposition 3.8. *For each $[A, B] \in \mathfrak{Y}$ the base of neighborhoods $\mathcal{B}([A, B])$ of $[A, B]$ consists of open and convex sets.*

Proof. Each $\mathcal{W}([A, B], U) \in \mathcal{B}([A, B])$ is open. To see this, take any $[A', B'] \in \mathcal{W}([A, B], U)$. Hence $A' + B \subset A + B' + U$, $A + B' \subset A' + B + U$ and thus, by Lemma 2.3, there exists $0 < \theta < 1$ such that

$$A' + B \subset A + B' + \theta U \qquad A + B' \subset A' + B + \theta U \ . \tag{3.4}$$

Then, $\mathcal{W}([A', B']), (1 - \theta)U) \subset \mathcal{W}([A, B], U)$. In fact any $[X, Y] \in \mathcal{W}([A', B'], (1 - \theta)U)$ satisfies $X + B' \subset A' + Y + (1 - \theta)U$, $A' + Y \subset X + B' + (1 - \theta)U$ and hence, by (3.4),

$$X + B' + A' + B \subset A' + Y + (1 - \theta)U + A + B' + \theta U = Y + A + A' + B' + U$$
$$A' + Y + A + B' \subset X + B' + (1 - \theta)U + A' + B + \theta U = X + B + A' + B' + U \ .$$

By Lemma 2.3 it follows $X + B \subset A + Y + U$, $A + Y \subset X + B + U$, i.e. $[X, Y] \in \mathcal{W}([A, B], U)$, and thus $\mathcal{W}([A, B]), U)$ is open.

$\mathcal{W}([A, B], U)$ is convex. In fact, if $[X_i, Y_i] \in \mathcal{W}([A, B], U)$, $i = 1, 2$, and $\lambda \in [0, 1]$, one has $X_i + B \subset A + Y_i + U$, $A + Y_i \subset X_i + B + U$, $i = 1, 2$, and so $\lambda X_1 + (1 - \lambda)X_2 + B \subset A + \lambda Y_1 + (1 - \lambda)Y_2 + U$, $A + \lambda Y_1 + (1 - \lambda)Y_2 \subset \lambda X_1 + (1 - \lambda)X_2 + B + U$. Hence $\lambda[X_1, Y_1] + (1 - \lambda)[X_2, Y_2] \in \mathcal{W}([A, B], U)$, i.e. the latter set is convex, completing the proof.

Proposition 3.9. *Let $\mathfrak{Y}$ be endowed with the topology σ, and let $\mathfrak{Y} \times \mathfrak{Y}$ and $\mathbb{R} \times \mathfrak{Y}$ be endowed with the respective product topologies. Then,*

(i): *the map $([X,Y],[X',Y']) \mapsto [X,Y]+[X',Y']$ is continuous from $\mathfrak{Y} \times \mathfrak{Y}$ to $\mathfrak{Y}$;*

(ii): *the map $(\lambda,[X,Y]) \mapsto \lambda[X,Y]$ is continuous from $\mathbb{R} \times \mathfrak{Y}$ to $\mathfrak{Y}$.*

Proof. (i) Let $([A,B],\ [A',B']) \in \mathfrak{Y} \times \mathfrak{Y}$ and let $\mathcal{W}([A+A',B+B'],U) \in \mathcal{B}([A+A',B+B'])$. Then each $([X,Y],[X',Y']) \in \mathcal{W}([A,B],(1/2)U) \times \mathcal{W}([A',B'],(1/2)U)$ satisfies

$$X+B \subset A+Y+(1/2)U \qquad A+Y \subset X+B+(1/2)U$$
$$X'+B' \subset A'+Y'+(1/2)U \qquad A'+Y' \subset X'+B'+(1/2)U\ ,$$

which imply

$$X+X'+B+B' \subset A+A'+Y+Y'+U \qquad A+A'+Y+Y' \subset X+X'+B+B'+U\ .$$

Therefore $[X,Y]+[X',Y'] \in \mathcal{W}([A+A',B+B'],U)$, and so (i) holds.

(ii) Let $(\lambda_0,[A,B]) \in \mathbb{R} \times \mathfrak{Y}$ and $\mathcal{W}(\lambda_0[A,B],U) \in \mathcal{B}(\lambda_0[A,B])$ be arbitrary. Fix $\mu > 0$ so that $A, B \subset \mu U$ and take σ satisfying $0 < \sigma < 1/2(\lambda_0+1)$. Set $\nu = 1/4(\mu+1)$. We first consider the case $\lambda_0 \geq 0$.

Case 1. Let $\lambda_0 \geq 0$. Then

$$\lambda \in [\lambda_0,\lambda_0+\nu] \quad \text{and} \quad [X,Y] \in \mathcal{W}([A,B],\sigma U) \text{ imply } \lambda[X,Y] \in \mathcal{W}(\lambda_0[A,B],U)\ , \tag{3.5}$$

$$\lambda \in [\lambda_0-\nu,\lambda_0] \cap \mathbb{R}^+ \text{ and } [X,Y] \in \mathcal{W}([A,B],\sigma U) \text{ imply } \lambda[X,Y] \in \mathcal{W}(\lambda_0[A,B],U)\ . \tag{3.6}$$

Since the proofs of (3.5) and (3.6) are similar, we shall prove only (3.5). We have

$$\begin{aligned}
\lambda X + \lambda_0 B &= \lambda X + (\lambda + (\lambda_0 - \lambda))B \\
&\subset \lambda X + \lambda B + (\lambda - \lambda_0)(-B)\ , \qquad \text{by Remark 2.1} \\
&= \lambda(X+B) + (\lambda-\lambda_0)\mu U\ , \qquad \text{as } B \subset \mu U, \text{ a balanced set} \\
&\subset \lambda(A+Y+\sigma U) + (\lambda-\lambda_0)\mu U\ , \quad \text{for} \quad [X,Y] \in \mathcal{W}([A,B],\sigma U) \\
&= \lambda_0 A + (\lambda-\lambda_0)A + \lambda Y + \lambda\sigma U + (\lambda-\lambda_0)\mu U\ , \\
&\subset \lambda_0 A + \lambda Y + 2(\lambda-\lambda_0)\mu U + \lambda\sigma U\ , \quad \text{for} \quad A \subset \mu U \\
&\subset \lambda_0 A + \lambda Y + U\ ,
\end{aligned}$$

because $2(\lambda-\lambda_0)\mu \leq \mu/2(\mu+1) < 1/2$ and $\lambda\sigma < (\lambda_0+1)\sigma < 1/2$. Similarly one can show that $\lambda_0 A + \lambda Y \subset \lambda X + \lambda_0 B + U$. Hence $\lambda[X,Y] \in \mathcal{W}(\lambda_0[A,B],U)$, and so (3.5) is proved.

Case 2. $\lambda_0 \leq 0$. Then,

$$\lambda \in [\lambda_0-\nu,\lambda_0] \quad \text{and} \quad [X,Y] \in \mathcal{W}([A,B],\sigma U) \text{ imply } \lambda[X,Y] \in \mathcal{W}(\lambda_0[A,B],U)\ , \tag{3.7}$$

$$\lambda \in [\lambda_0,\lambda_0+\nu] \cap \mathbb{R}^- \text{ and } [X,Y] \in \mathcal{W}([A,B],\sigma U) \text{ imply } \lambda[X,Y] \in \mathcal{W}(\lambda_0[A,B],U)\ , \tag{3.8}$$

where $\mathbb{R}^- = (-\infty,0]$.

Consider (3.7). Clearly $\lambda_0[A, B] = |\lambda_0|[B, A]$, $\lambda[X, Y] = |\lambda|[Y, X]$, for $\lambda, \lambda_0 \leq 0$. Since $|\lambda| \in [|\lambda_0|,\ |\lambda_0| + \nu]$ and $[Y, X] \in \mathcal{W}([B, A], \sigma U)$, (3.5) implies $|\lambda|[Y, X] \in \mathcal{W}(|\lambda_0|[B, A], U)$, and hence $\lambda[X, Y] \in \mathcal{W}(\lambda_0[A, B], U)$. The proof of (3.6) is similar, and so it is omitted.

In conclusion, the map $(\lambda, [X, Y]) \mapsto \lambda[X, Y]$ is continuous at each $\lambda_0 \neq 0$, by Case 1 and Case 2, while it is continuous at $\lambda_0 = 0$ by (3.5) and (3.7). This completes the proof.

Proposition 3.10. *The space $(\mathfrak{Y}, \sigma)$ is a Hausdorff locally convex topological linear real space.*

Proof. By virtue of Propositions 3.2, 3.7, 3.8 and 3.9, $(\mathfrak{Y}, \sigma)$ is a locally convex topological linear real space. To prove that $(\mathfrak{Y}, \sigma)$ is Hausdorff, fix two arbitrary distinct points $[A, B]$ $[A', B'] \in \mathfrak{Y}$, hence $A + B' \neq A' + B$. It suffices to show that there is $U \in \mathcal{N}_0$ such that

$$\mathcal{W}([A, B], U) \cap \mathcal{W}([A', B'], U) = \phi\ . \tag{3.9}$$

Let $c \in (A + B')\backslash(A' + B)$ (if $c \in (A' + B)\backslash(A + B')$ the argument is similar). By the Hahn-Banach theorem there exist a continuous linear functional $f : \mathbb{E} \to \mathbb{R}$ and an $\varepsilon > 0$ such that

$$\sup\{f(x)|x \in A' + B\} + \varepsilon < f(c)\ . \tag{3.10}$$

Now if one takes $U \in \mathcal{N}_0$, with $U \subset \{x \in \mathbb{E}|\ |f(x)| < \varepsilon/2\}$, then (3.9) is satisfied. In the contrary case, there is $[X, Y] \in \mathfrak{Y}$ such that

$$X + B \subset A + Y + U\ , \quad A + Y \subset X + B + U\ ,$$

$$X + B' \subset A' + Y + U\ , \quad A' + Y \subset X + B' + U\ .$$

By virtue of the third and the second of the above relations, one has

$$X + B' + A \subset A + Y + A' + U \subset X + A' + B + 2U$$

and hence, by Lemma 2.3, $A + B' \subset A' + B + 2U$. It follows that $c = d + 2u$ for some $d \in A' + B$ and $u \in U$, and thus $f(c) = f(d) + 2f(u) < \sup\{f(x)|x \in A' + B\} + \varepsilon$, a contradiction to (3.10). Therefore (3.9) is valid, completing the proof.

The following embedding theorem concludes the foregoing elementary construction. It was established, for different spaces of convex sets, by Rådström [20] in normed spaces, by Hörmander [9] in locally convex topological linear spaces, and by Urbański [22] in topological linear spaces.

Theorem 3.11. *Let $\mathcal{K}$ be the positively-semilinear subspace of $\mathfrak{Y}$ given by (3.1) equipped with the relative $\sigma_{\mathcal{K}}$ topology, induced by the σ topology of $\mathfrak{Y}$. Let $\mathfrak{X}$ be endowed with the topology τ. Then the map $J : \mathcal{K} \to \mathfrak{X}$, given by $J([X + Z, Z]) = X$ has the following properties:*

(i): *J is a positively-semilinear isomorphism from $\mathcal{K}$ onto $\mathfrak{X}$;*
(ii): *J is a homeomorphism from $(\mathcal{K}, \sigma_{\mathcal{K}})$ onto $(\mathfrak{X}, \tau)$.*

Proof. (i) follows from Proposition 3.4.

(ii) J is continuous. In fact, given $[A+B,B] \in \mathcal{K}$ and $\mathcal{V}_{\mathcal{N}_0}(A,U) \in \mathcal{B}_{\mathcal{N}_0}(A)$, take any $[X+Y,Y] \in \mathcal{W}([A+B,B],U) \cap \mathcal{K}$. Hence

$$X+Y+B \subset A+B+Y+U \ , \qquad A+B+Y \subset X+Y+B+U \tag{3.11}$$

and thus, by Lemma 2.3, $X \subset A+U$, $A \subset X+U$, that is $J([X+Y,Y]) = X \in \mathcal{V}_{\mathcal{N}_0}(A,U)$. Therefore J is continuous from $(\mathcal{K}, \sigma_{\mathcal{K}})$ to $(\mathfrak{X}, \tau)$.

J^{-1} is continuous. In fact, fix arbitrary $A \in \mathfrak{X}$ and $\mathcal{W}([A+B,B] \cap \mathcal{K}$, where $\mathcal{W}([A+B,B],U) \in \mathcal{B}([A+B,B])$. Clearly, each $X \in \mathcal{V}_{\mathcal{N}_0}(A,U)$ satisfies $X \subset A+U$, $A \subset X+U$. From these, adding to each side $B+Y$, with $Y \in \mathfrak{X}$, one obtains (3.11) and hence $J^{-1}(X) = [X+Y,Y] \in \mathcal{W}([A+B,B],U) \cap \mathcal{K}$. Therefore also J^{-1} is continuous, and thus J is a homeorphism from $\mathcal{K}$ to $\mathfrak{X}$, completing the proof.

4. Semifixed sets of singlevalued maps

In this section we prove some fixed and semifixed set results for singlevalued maps under Tychonoff type assumptions.

Theorem 4.1. *Let $\mathcal{A}$ be a nonempty compact convex subset of $(\mathfrak{X}, \tau)$. Then any continuous singlevalued map $\varphi : \mathcal{A} \to \mathcal{A}$ has a fixed set, i.e. there exists at least one set $A \in \mathcal{A}$ such that $A = \varphi(A)$.*

Proof. By Theorem 3.11, $J^{-1} : \mathfrak{X} \to \mathcal{K}$ is a positively-semilinear isomorphism and, moreover, J^{-1} is continuous from $(\mathfrak{X}, \tau)$ onto $(\mathcal{K}, \sigma_{\mathcal{K}})$. It follows that the set $\Xi = J^{-1}(\mathcal{A}) \subset \mathcal{K}$ is compact and convex. Define $\psi : \Xi \to \Xi$ by

$$\psi(\xi) = (J^{-1} \cdot \varphi \cdot J)(\xi) \qquad \text{for every} \qquad \xi \in \Xi \ .$$

Furthermore Ξ is contained in $(\mathfrak{Y}, \sigma)$, which is a Hausdorff locally convex topological linear space, by Proposition 3.10. Hence, by Tychonoff's fixed point theorem (see [21], or [5] p. 414), there exists $\xi \in \Xi$ such that $\xi = \psi(\xi)$. Hence $J(\xi) = \varphi(J(\xi))$, and thus the set $A = J(\xi) \in \mathcal{A}$ satisfies $A = \varphi(A)$, completing the proof.

Let $\mathcal{Z}$ and $\mathcal{Y}$ be topological spaces. A multifunction Φ, defined on $\mathcal{Z}$, whose values are compact nonempty subsets of $\mathcal{Y}$ is said to be *upper semicontinuous* if, for each $z_0 \in \mathcal{Z}$ and each open set $U \subset \mathcal{Y}$ such that $\Phi(z_0) \subset U$, there exists a neighborhood W of z_0 such that $\Phi(z_0) \subset U$ for every $z \in W$.

Denote by $\mathcal{C}(\mathfrak{X})$ (resp. $\mathcal{C}(\mathfrak{Y})$) the family of all nonempty compact convex subsets of $\mathfrak{X}$ (resp. $\mathfrak{Y}$).

Theorem 4.2. *Let $\mathcal{A}$ be a nonempty compact convex subset of $(\mathfrak{X}, \tau)$. Then any upper semicontinuous multifunction $\Phi : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$, with values $\Phi(X) \subset \mathcal{A}$ for all $X \in \mathcal{A}$, has a fixed set, i.e. there exists at least one set $A \in \mathcal{A}$ such that $A \in \Phi(A)$.*

Proof. By Theorem 3.11, $\Xi = J^{-1}(\mathcal{A})$ is a compact convex subset of $\mathcal{K}$ homeomorphic to $\mathcal{A}$, where Ξ and $\mathcal{A}$ are equipped with the induced topologies of $\mathcal{K}$ and $\mathfrak{X}$.

For $\xi \in \Xi$ put

$$\Psi(\xi) = (J^{-1} \cdot \Phi \cdot J)(\xi) \ , \tag{4.1}$$

and observe that (4.1) defines a multifunction $\Psi : \Xi \to \mathcal{C}(\mathfrak{Y})$ with values $\Psi(\xi) \subset \Xi$ for all $\xi \in \Xi$.

Ψ is upper semicontinuous. Let $\xi_0 \in \Xi$ and let $U \subset \Xi$ be an open set such that $\Psi(\xi_0) \subset U$. As Φ is upper semicontinuous at $A_0 = J(\xi_0) \in \mathcal{A}$ and $\Phi(J(\xi_0)) \subset J(U)$, where $J(U)$ is open in $\mathcal{A}$, there exists an open neighborhood $W_{A_0} \subset \mathcal{A}$ of A_0 such that

$$\Phi(X) \subset J(U) \qquad \text{for every} \quad X \in W_{A_0} \ . \tag{4.2}$$

Let ξ be in $V_{\xi_0} = J^{-1}(W_{A_0})$, an open neighborhood of ξ_0. As $J(\xi) \in W_{A_0}$, (4.2) implies $\Phi(J(\xi)) \subset J(U)$, and thus $\Psi(\xi) = J^{-1}(\Phi(J(\xi))) \subset U$, proving that Ψ is upper semicontinuous.

Furthermore, Ξ is contained in $(\mathfrak{Y}, \sigma)$ which, by Proposition 3.10, is a Hausdorff locally convex topological linear space. Hence, by Ky Fan's fixed point theorem [6], there exists $\xi \in \Xi$ such that $\xi \in \Psi(\xi)$. Then the set $A = J(\xi) \in \mathcal{A}$ satisfies $A \in \Phi(A)$, completing the proof.

Theorem 4.3. *Let $\mathcal{A}$ be a nonempty compact convex subset of $(\mathfrak{X}, \tau)$. Let $\varphi : \mathcal{A} \to \mathfrak{X}$ be a continuous function satisfying the following condition: (γ) for each $X \in \mathcal{A}$ there is a set $Z \in \mathcal{A}$ such that*

$$Z \cap \varphi(X) \neq \phi \qquad (\textit{resp.}\ Z \subset \varphi(X)\ ,\quad Z \supset \varphi(X))\ .$$

Then φ has a semifixed set, i.e. there exists at least one set $A \in \mathcal{A}$ such that

$$A \cap \varphi(A) \neq \phi \qquad (\textit{resp.}\ A \subset \varphi(A)\ ,\quad A \supset \varphi(A))\ .$$

Proof. Consider first the intersection case. For each $X \in \mathcal{A}$ set

$$\Omega(X) = \{Z \in \mathcal{A} | Z \cap \varphi(X) \neq \phi\}\ . \tag{4.3}$$

It is easy to verify that $\Omega(X) \subset \mathcal{A}$ is nonempty and convex. Furthermore, $\Omega(X)$ is compact. For this it suffices to show that $\Omega(X)$ is closed in $\mathcal{A}$. Let $Z \in \overline{\Omega(X)}$ and let $\{Z_i\}_{i \in I}$ be a net in $\Omega(X)$ converging to $Z \in \mathcal{A}$, whence

$$Z_i \cap \varphi(X) \neq \phi \quad \text{for each } i \in I\ . \tag{4.4}$$

We have $Z \cap \varphi(X) \neq \phi$. In the contrary case, by the Hahn-Banach theorem, there exist a continuous linear functional $f : \mathbb{E} \to \mathbb{R}$ and constants $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\sup\{f(x) | x \in Z\} \leq \lambda - \varepsilon < \lambda + \varepsilon \leq \inf\{f(x) | x \in \varphi(X)\}\ .$$

From the latter, taking $U \in \mathcal{N}_0$ so that $U \subset \{x \in \mathbb{E} |\ |f(x)| < \varepsilon\}$, one can easily show that

$$Z' \cap \varphi(X) = \phi \quad \text{for all} \quad Z' \in \mathcal{V}_{\mathcal{N}_0}(Z, U)\ . \tag{4.5}$$

Since $Z_i \underset{I}{\longrightarrow} Z$, there is $i \in I$ such that $Z_i \in \mathcal{V}_{\mathcal{N}_0}(Z, U)$ and hence, by (4.5), one has $Z_i \cap \varphi(X) = \phi$, a contradiction to (4.4). Therefore $Z \cap \varphi(X) \neq \phi$, that is $Z \in \Omega(X)$, which shows that $\Omega(X)$ is closed.

Thus (4.3) defines a multifunction $\Omega : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$, with values $\Omega(X) \subset \mathcal{A}$, for each $X \in \mathcal{A}$.

Ω is upper semicontinuous. In the contrary case there is $A \in \mathcal{A}$ and an open set $\mathcal{W} \subset \mathfrak{X}$, with $\Omega(A) \subset \mathcal{W}$, such that the following property holds: (δ) for each neighborhood $\mathcal{V}$ of A, where $\mathcal{V} \in \mathcal{B}$ and $\mathcal{B} = \mathcal{B}_{\mathcal{N}_0}(A)$, there exist $X_{\mathcal{V}} \in \mathcal{V}$ and $Z_{\mathcal{V}} \in \mathcal{A}$ such that:

$$Z_{\mathcal{V}} \cap \varphi(X_{\mathcal{V}}) \neq \phi \quad \text{and} \quad Z_{\mathcal{V}} \notin \mathcal{W}, \qquad \text{for every} \quad \mathcal{V} \in \mathcal{B}\ . \tag{4.6}$$

We now make $\mathcal{B}$ a directed set with partial order $\geq$ induced by the reverse inclusion relation in $\mathcal{B}$. By construction, $\{X_{\mathcal{V}}\}_{\mathcal{V}\in\mathcal{B}}$ is a net in $\mathcal{A}$ converging to A. On the other hand $\{Z_{\mathcal{V}}\}_{\mathcal{V}\in\mathcal{B}}$ is a net in $\mathcal{A}$, a compact space, and thus by Kelley [12], p. 136, it has a subnet $\{Z'_{\mathcal{U}}\}_{\mathcal{U}\in\mathcal{C}}$, where $Z'_{\mathcal{U}} = Z_{N(\mathcal{U})}$, which converges to some $B \in \mathcal{A}$. Moreover $\{X'_{\mathcal{U}}\}_{\mathcal{U}\in\mathcal{C}}$ where $X'_{\mathcal{U}} = X_{N(\mathcal{U})}$, is a subnet of $\{X\}_{\mathcal{V}\in\mathcal{B}}$ and thus it converges to A. Therefore,

$$X'_{\mathcal{U}} \underset{\mathcal{C}}{\longrightarrow} A\ , \quad Z'_{\mathcal{U}} \underset{\mathcal{C}}{\longrightarrow} B\ , \quad \varphi(X'_{\mathcal{U}}) \underset{\mathcal{C}}{\longrightarrow} \varphi(A)\ , \tag{4.7}$$

where the latter is valid by the continuity of φ.

We have

$$B \cap \varphi(A) \neq \phi\ . \tag{4.8}$$

In the contrary case there exist a continuous linear functional $f : \mathbb{E} \to \mathbb{R}$ and constants $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ such that $\sup\{f(x)|x \in B\} \leq \lambda - \varepsilon < \lambda + \varepsilon \leq \inf\{f(x)|x \in \varphi(A)\}$. Then, fixing $U \in \mathcal{N}_0$ such that $U \subset \{x \in \mathbb{E}||f(x)| < \varepsilon\}$, one can show as before that the following property holds:

$$Z \cap Y = \phi \quad \text{for all} \quad Z \in \mathcal{V}_{\mathcal{N}_0}(B, U)\ , \quad Y \in \mathcal{V}_{\mathcal{N}_0}(\varphi(A), U)\ . \tag{4.9}$$

By virtue of (4.7) for some $\mathcal{U}_0 \in \mathcal{C}$ one has

$$Z_{N(\mathcal{U}_0)} \in \mathcal{V}_{\mathcal{N}_0}(B, U)\ , \quad \varphi(X_{N(\mathcal{U}_0)}) \in \mathcal{V}_{\mathcal{N}_0}(\varphi(A), U)$$

and hence, by (4.9),

$$Z_{N(\mathcal{U}_0)} \cap \varphi(X_{N(\mathcal{U}_0)}) = \phi\ .$$

But the latter contradicts (4.6), for $\mathcal{N}(\mathcal{U}_0) \in \mathcal{B}$, and thus (4.8) is valid.

By (4.8) it follows that $B \in \Omega(A) \subset \mathcal{W}$. As $\mathcal{W}$ is open, by (4.7) there exists $\mathcal{U}' \in \mathcal{C}$ such that $\mathcal{U} \geq \mathcal{U}'$ implies $Z_{N(\mathcal{U})} \in \mathcal{W}$, a contradiction to (4.6), as $N(\mathcal{U}) \in \mathcal{B}$.

It has been proved that $\Omega : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$ is a upper semicontinuous multifunction with nonempty compact convex values $\Omega(X) \subset \mathcal{A}$, for all $X \in \mathcal{A}$. Then, by Theorem 4.2 there exists $A \in \mathcal{A}$ such that $A \in \Omega(A)$. From the definition of Ω it follows that $A \cap \varphi(A) \neq \phi$. In the inclusion cases the argument is analogous and so it is omitted. This completes the proof.

5. Semifixed sets of multivalued maps

In this section we prove a semifixed set result for multivalued maps under Ky Fan type assumptions.

For $\mathcal{D}$ a nonempty subset of $\mathfrak{X}$ and $V \in \mathcal{N}_0$, set

$$\mathcal{V}(\mathcal{D}, V) = \bigcup_{D\in\mathcal{D}} \mathcal{V}_{\mathcal{N}_0}(D, V)\ .$$

Clearly $\mathcal{V}(\mathcal{D}, V)$ is an open subset of $\mathfrak{X}$.

Lemma 5.1. *Let $\mathcal{D}$ be a nonempty compact subset of $\mathfrak{X}$. Let $\{Y_i\}_{i\in I}$ be a net in $\mathfrak{X}$ satisfying the following condition: (δ) for every $V \in \mathcal{N}_0$ there exists $i_0 \in I$ such that*

$$Y_i \in \mathcal{V}_{\mathcal{N}_0}(\mathcal{D}, V) \qquad \textit{for all} \quad i \geq i_0\ . \tag{5.1}$$

Then $\{Y_i\}_{i\in I}$ has a subnet $\{Y'_j\}_{j\in J}$ which converges to some set $D \in \mathcal{D}$.

Proof. For (i,V), $(i',V') \in \mathcal{W} = I \times \mathcal{N}_0$ define $(i',V') \geq (i,V)$ iff $i' \geq i$ and $V' \subset V$, and observe that $\mathcal{W}$ equipped with the relation $\geq$ is a directed set.

For $(i,V) \in \mathcal{W}$ put

$$\Gamma(i,V) = \{j \in I | j \geq i \quad \text{and} \quad Y_j \in \mathcal{V}(\mathcal{D},V)\} \ . \tag{5.2}$$

$\Gamma(i,V)$ is nonempty. In fact let $i_0 \in I$ correspond to V according to property (δ), and take $j \in I$ satisfying $j \geq i$ and $j \geq i_0$. By the latter and (5.1) one has $Y_j \in \mathcal{V}_{\mathcal{N}_0}(\mathcal{D},V)$, and thus $\Gamma(i,V) \neq \phi$. Furthermore,

$$(i',V') \geq (i,V) \quad \text{implies} \quad \Gamma(i',V') \subset \Gamma(i,V) \ , \tag{5.3}$$

because if $j \in \Gamma(i',V')$ one has $j \geq i' \geq i$ and $Y_j \in \mathcal{V}(\mathcal{D},V') \subset \mathcal{V}(\mathcal{D},V)$, and thus $j \in \Gamma(i,V)$.

Now fix a map $N : \mathcal{W} \to I$ satisfying

$$N(i,V) \in \Gamma(i,V) \quad \text{for every} \quad (i,V) \in \mathcal{W} \ . \tag{5.4}$$

N has the following property: (ε) for each $j_0 \in I$ there exists $(i_0,V_0) \in \mathcal{W}$ such that

$$(i,V) \geq (i_0,V_0) \quad \text{implies} \quad N(i,V) \geq j_0 \ . \tag{5.5}$$

In fact, set $(i_0,V_0) = (j_0,V_0)$, with $V_0 \in \mathcal{N}_0$, and let $(i,V) \geq (i_0,V_0)$. By (5.4) and (5.3) one has: $N(i,V) \in \Gamma(i,V) \subset \Gamma(i_0,V_0)$. Hence, by (5.2), $N(i,V) \geq i_0 = j_0$, and (5.5) is valid.

As $(\mathcal{W},\geq)$ is a directed set and $N : \mathcal{W} \to I$ satisfies property (ε), it follows that $\{Y_{N(i,V)}\}_{(i,V)\in\mathcal{W}}$ is a subnet of $\{Y_i\}_{i\in I}$.

For every $(i,V) \in \mathcal{W}$ there is $F_{N(i,V)} \in \mathcal{D}$ such that

$$Y_{N(i,V)} \in \mathcal{V}_{\mathcal{N}_0}(F_{N(i,V)},V) \ . \tag{5.6}$$

In fact, combining (5.4) and (5.2) gives $Y_{N(i,V)} \in \mathcal{V}(\mathcal{D},V)$, from which (5.6) follows for some set, say $F_{N(i,V)}$, in $\mathcal{D}$.

By construction $\{F_{N(i,V)}\}_{(i,V)\in\mathcal{W}}$ is a net in $\mathcal{D}$, a compact space, and thus it has a subnet $\{F'_{M(Z)}\}_{Z\in\mathcal{Z}}$ which converges to some set $D \in \mathcal{D}$. Here M is a map from a directed set $\mathcal{Z}$ to $\mathcal{W}$, i.e.

$$M(Z) = (i(Z),V(Z)) \in \mathcal{W} \quad \text{for every} \quad Z \in \mathcal{Z} \ ,$$

which satisfies the following condition: (η) for each $(i_0,V_0) \in \mathcal{W}$ there exists $Z_0 \in \mathcal{Z}$ such that

$$Z \geq Z_0 \quad \text{implies} \quad (i(Z),V(Z)) \geq (i_0,V_0) \ .$$

It will be shown that the subnet $\{Y'_{M(Z)}\}_{Z\in\mathcal{Z}}$ of $\{Y_{N(i,V)}\}_{(i,V)\in\mathcal{W}}$, which for convenience we write $\{Y_{N(i(Z),V(Z))}\}_{Z\in\mathcal{Z}}$, converges to D. In the sequel $\{F_{N(i(Z),V(Z))}\}_{Z\in\mathcal{Z}}$ will be used to denote the subnet $\{F'_{M(Z)}\}_{Z\in\mathcal{Z}}$ of $\{F_{N(i,V)}\}_{(i,V)\in\mathcal{W}}$.

Let $U \in \mathcal{N}_0$ be arbitrary. As $F_{N(i(Z),V(Z))} \xrightarrow[\mathcal{Z}]{} D$, there exists $Z_1 \in \mathcal{Z}$ such that

$$F_{N(i(Z),V(Z))} \in \mathcal{V}_{\mathcal{N}_0}(D,U/2) \quad \text{for all} \quad Z \geq Z_1 \ . \tag{5.7}$$

On the other hand, by (η), in correspondence of $(i(Z_1),U/2)$ there exists $Z_2 \in \mathcal{Z}$ such that

$$(i(Z),V(Z)) \geq (i(Z_1),U/2) \quad \text{for all} \quad Z \geq Z_2 \ . \tag{5.8}$$

Fix $Z_0 \in \mathcal{Z}$, such that $Z_0 \geq Z_1$, $Z_0 \geq Z_2$. Then for all $Z \geq Z_0$ one has

$$\begin{aligned} Y_{N(i(Z),V(Z))} &\in \mathcal{V}_{\mathcal{N}_0}(F_{N(i(Z),V(Z))}, V(Z)) && \text{by (5.6)} \\ &\subset \mathcal{V}_{\mathcal{N}_0}(F_{N(i(Z),V(Z))}, U/2) && \text{by (5.8)} \\ &\subset \mathcal{V}_{\mathcal{N}_0}(D, U) && \text{by (5.7)} . \end{aligned}$$

Since $U \in \mathcal{N}_0$ is arbitrary, it follows that the subnet $\{Y'_{M(Z)}\}_{Z\in\mathcal{Z}}$ of $\{Y_{N(i,V)}\}_{(i,V)\in\mathcal{W}}$ converges to D. As the latter is a subnet of $\{Y_i\}_{i\in I}$, the proof is complete.

Lemma 5.2. *Let $\mathcal{A}$ be a nonempty compact subset of $\mathfrak{X}$ and let $\Phi : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$ be upper semicontinuous. Then the set $\Phi(\mathcal{A}) = \{Y \in \mathfrak{X} | Y \in \Phi(X)$ for some $X \in \mathcal{A}\}$ is compact.*

Proof. Let $\{Y_i\}_{i\in I}$ be an arbitrary net in $\Phi(\mathcal{A})$, and let $\{X_i\}_{i\in I}$ correspond, where $Y_i \in \Phi(X_i)$, for each $i \in I$. Since $\mathcal{A}$ is compact, $\{X_i\}_{i\in I}$ has a subnet, say $\{X'_j\}_{j\in J}$, which converges to some $A \in \mathcal{A}$. Let $\{Y'_j\}_{j\in J}$ be the corresponding subnet of $\{Y_i\}_{i\in I}$.

By hypothesis Φ is upper semicontinuous, and thus for each $V \in \mathcal{N}_0$ there exists a neighborhood $\mathcal{U}$ of A, relative to $\mathcal{A}$, such that $\Phi(X) \subset \mathcal{V}(\Phi(A), V)$ for all $X \in \mathcal{U}$. Moreover, as $X'_j \underset{J}{\longrightarrow} A$, there is $j_0 \in J$ such that $j \geq j_0$ implies $X'_j \in \mathcal{U}$. Therefore,

$$Y'_j \in \Phi(X'_j) \subset \mathcal{V}(\Phi(A), V) \quad \text{for all} \quad j \geq j_0 .$$

By Lemma 5.1, $\{Y'_j\}_{j\in J}$ has a subnet $\{Y''_l\}_{l\in L}$ which converges to some $F \in \Phi(A) \subset \Phi(\mathcal{A})$. Clearly $\{Y''_l\}_{l\in L}$ is a subnet of $\{Y_i\}_{i\in I}$ and so, by Kelley [12], p. 136, $\Phi(\mathcal{A})$ is compact. This completes the proof.

Theorem 5.3. *Let $\mathcal{A}$ be a nonempty compact convex subset of $(\mathfrak{X}, \tau)$. Let $\Phi : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$ be a upper semicontinuous multifunction satisfying the following condition: (θ) for every $X \in \mathcal{A}$ there exist sets $Z \in \mathcal{A}$ and $F \in \Phi(X)$ such that*

$$Z \cap F \neq \phi \qquad (\textit{resp. } Z \subset F \ , \quad Z \supset F) .$$

Then Φ has a semifixed set, i.e. there exists at least one set $A \in \mathcal{A}$ and an $F \in \Phi(A)$ such that

$$A \cap F \neq \phi \quad (\textit{resp. } A \subset F \ , \quad A \supset F) .$$

Proof. Consider first the intersection case. For every $X \in \mathcal{A}$ set

$$\Omega(X) = \{Z \in \mathcal{A} | \text{ there exists } F \in \Phi(X) \text{ such that } Z \cap F \neq \phi\} . \tag{5.9}$$

It is easy to verify that $\Omega(X) \subset \mathcal{A}$ is nonempty and convex. Moreover, to show that $\Omega(X)$ is compact it suffices to prove that it is closed in $\mathcal{A}$. To see that, let $Z \in \overline{\Omega(X)}$ and let $\{Z_i\}_{i\in I}$ be an arbitrary net in $\Omega(X)$ converging to $Z \in \mathcal{A}$. Further let $\{F_i\}_{i\in I}$ be a corresponding net in $\Phi(X)$ such that

$$F_i \in \Phi(X) \ , \qquad Z_i \cap F_i \neq \phi \quad \text{for every } i \in I . \tag{5.10}$$

Since $\{F_i\}_{i\in I} \subset \Phi(X)$, a compact set, there exists a subnet $\{F'_j\}_{j\in J}$, where $F'_j = F_{N(j)}$, which converges to some $F \in \Phi(X)$. Moreover $\{Z'_j\}_{j\in J}$, where $Z'_j = Z_{N(j)}$, is a subnet of $\{Z_i\}_{i\in I}$ and thus it converges to Z.

It will be shown that

$$Z \cap F \neq \phi . \tag{5.11}$$

Suppose, by contradiction, that $Z \cap F = \phi$. Then, by the Hahn-Banach theorem, there exist a continuous linear functional $f : \mathbb{E} \to \mathbb{R}$, and constants $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\sup\{f(x)|x \in Z\} \leq \lambda - \varepsilon < \lambda + \varepsilon \leq \inf\{f(x)|x \in F\} \ . \tag{5.12}$$

Now taking $U \in \mathcal{N}_0$ so that

$$U \subset \{x \in \mathbb{E}||f(x)| < \varepsilon\} \ , \tag{5.13}$$

one has

$$Z' \cap Y' = \phi \quad \text{for all} \quad Z' \in \mathcal{V}_{\mathcal{N}_0}(Z,U) \ , \quad Y' \in \mathcal{V}_{\mathcal{N}_0}(F,U) \ . \tag{5.14}$$

In fact, as $Z' \subset Z + U$ and $Y' \subset F + U$, in view of (5.12) and (5.13), it follows that

$$Z' \subset \{x \in \mathbb{E}|f(x) < \lambda\} \ , \quad Y' \subset \{x \in \mathbb{E}|f(x) > \lambda\}$$

and thus (5.14) is valid.

Since $Z_{N(j)} \underset{J}{\longrightarrow} Z$ and $F_{N(j)} \underset{J}{\longrightarrow} F$, for some $j \in J$ one has $Z_{N(j)} \in \mathcal{V}_{\mathcal{N}_0}(Z,U)$, $F_{N(j)} \in \mathcal{V}_{\mathcal{N}_0}(F,U)$. By (5.14) one has $Z_{N(j)} \cap F_{N(j)} = \phi$, a contradiction to (5.10), for $N(j) \in I$. Hence (5.11) is valid and, clearly, $Z \in \Omega(X)$, proving that $\Omega(X)$ is closed in $\mathcal{A}$ and hence also compact.

Therefore (5.9) defines a multifunction $\Omega : \mathcal{A} \to \mathcal{C}(\mathfrak{X})$ with nonempty compact convex values $\Omega(X) \subset \mathcal{A}$, for each $X \in \mathcal{A}$.

Ω is upper semicontinuous. In the contrary case there exist $A \in \mathcal{A}$ and an open set $\mathcal{H} \subset \mathfrak{X}$, with $\Omega(A) \subset \mathcal{H}$, such that the following property holds: for each $U \in \mathcal{N}_0$ there exist $X_U \in \mathcal{V}_{\mathcal{N}_0}(A,U)$, $Z_U \in \mathcal{A}$ and $F_U \in \Phi(X_U)$ such that

$$Z_U \cap F_U \neq \phi \ , \quad Z_U \notin \mathcal{H} \ , \qquad \text{for each} \quad U \in \mathcal{N}_0 \ . \tag{5.15}$$

We suppose that $\mathcal{N}_0$ is directed under the partial order $\geq$ induced by the reverse inclusion relation. By construction $\{X_U\}_{U \in \mathcal{N}_0}$ is a net in $\mathcal{A}$ converging to A; moreover $\{Z_U\}_{U \in \mathcal{N}_0}$ and $\{F_U\}_{U \in \mathcal{N}_0}$ are nets in $\mathcal{A}$ and $\Phi(\mathcal{A})$, both compact sets, the latter by Lemma 5.2. Hence there exist subnets, say $\{Z_{N(V)}\}_{V \in \mathcal{V}}$ and $\{F_{N(V)}\}_{V \in \mathcal{V}}$, which converge, respectively, to $Z \in \mathcal{A}$ and $F \in \Phi(\mathcal{A})$. Further, as $\{X_{N(V)}\}_{V \in \mathcal{V}}$ converges to A, $F_{N(V)} \in \Phi(X_V)$ for each $V \in \mathcal{V}$, and Φ is upper semicontinuous at A, Lemma 5.1 implies that $F \in \Phi(A)$. On the other hand as before one can show that $Z \cap F \neq \phi$. Therefore, from the definition of $\Omega(A)$, it follows that $Z \in \Omega(A)$.

Now $Z_{N(V)} \underset{\mathcal{V}}{\longrightarrow} Z$, where $Z \in \Omega(A) \subset \mathcal{H}$, and hence there exists $V_0 \in \mathcal{V}$ such that

$$Z_{N(V)} \in \mathcal{H} \qquad \text{for all} \quad V \geq V_0 \ ,$$

a contradiction to (5.15), for $N(V) \in \mathcal{N}_0$. Therefore Ω is upper semicontinuous.

By Theorem 4.2, there exists $A \in \mathcal{A}$ such that $A \in \Omega(A)$. Consequently for some $F \in \Phi(A)$ one has $A \cap F \neq \phi$. In the inclusion cases the argument is similar and so it is omitted. This completes the proof.

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