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A GENERALIZATION OF THE CARISTI FIXED POINT THEOREM IN METRIC SPACES

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Abstract. We provide the existence either of fixed points and of contractive fixed points in not necessarily complete metric spaces. As a consequence, we state a Caristi type fixed point theorem and generalize a fixed point theorem of [7].
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1. Introduction

The aim of the present paper is to ensure in metric spaces the existence of fixed points for functions not forced to be continuous.

In order to obtain our objective we introduce the definition of function *with partially complete graph*, which is more general than the property of function with complete graph introduced by Leader in [7]. In that paper the author extends the Meir-Keeler type results presented in [4], [5], [8] and [9] by proving the existence either of fixed points and of contractive fixed points for functions with complete graph in not necessarily complete metric spaces.

We recall that the Meir-Keeler property assumes a significant relevance in the fixed point theory, as it is shown by the numerous publications where it appears. In the setting of complete metric spaces we recall the recent papers [6] and [10].

For functions verifying our new property we first establish two necessary and sufficient conditions on the existence of a fixed point and of a contractive fixed point respectively. These results allow us to remove the completeness on the metric space in the Caristi fixed point theorem (see [3]) and in its more recent generalizations (see [1]). Moreover, we also prove a viable version of our Caristi type theorem.

Then, again in the new class of functions with partially complete graph, we state two theorems which strictly contain Theorem 3 in [7] and, moreover, we provide a corollary which improves Corollary 4 in [7] (cf. Remark 3.1).

Furthermore, we present an example showing that in our results the assumption on f to be with partially complete graph cannot be omitted, not even if the metric space is complete.

Finally we wish to observe that all our propositions extend Corollary 3.1 in [2], which ensures the existence of a contractive fixed point for functions defined in a complete metric space.

2. A Caristi type theorem

Let (X, d) be a metric space and $f : X \to X$ be a given function.

For every $z \in X$ the sequence $(f^p(z))_{p\in I\!N}$, where $f^0(z) = z$ and $f^p(z) = f(f^{p-1}(z))$ for $p \geq 1$, is said to be an *orbit* of f at z.

A point $x \in X$ is a *contractive fixed point* for f if it is a fixed point for f and if every orbit of f converges to x. Of course, if a contractive fixed point exists, then it is unique.

Two sequences $(x_n)_{n\in I\!N}$ and $(y_n)_{n\in I\!N}$ in X such that $\lim_{n\to+\infty} d(x_n, y_n) = 0$ are said to be *equivalent*; two equivalent sequences which are also Cauchy sequences are said to be *Cauchy equivalent*.

We introduce a new definition, which will play a crucial rule throughout the paper.

Definition 2.1. A function $f : S \to X$ is said to be *with partially complete graph* if for every Cauchy sequence $(x_n)_{n\in I\!N}$, $\{x_n : n \in I\!N\} \subset f(S)$, satisfying the properties

α) $(f(x_n))_{n\in I\!N}$ is a Cauchy sequence;

$\alpha\alpha$) for every $n \in I\!N$, the sequence $(f^p(x_n))_{p\in I\!N}$ has not constant subsequences;

$\alpha\alpha\alpha$) for every $p \in I\!N$, the sequence $(f^p(x_n))_{n\in I\!N}$ has not constant subsequences,

there exists $x \in X$ such that $\lim\limits_{n\to+\infty} x_n = x$ and $\lim\limits_{n\to+\infty} f(x_n) = f(x)$.

Thanks to this notion, we are able to remove the assumption of completeness on the metric space in the Caristi fixed point theorem (see [3]) and in its more recent generalizations obtained in [1].

To this end, we first provide two necessary and sufficient conditions for the existence of a fixed point and of a contractive fixed point respectively.

Theorem 2.1. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph. The function f has a fixed point if and only if there exists $x_0 \in X$ such that the orbit of f at x_0 is a Cauchy sequence.

Proof. The necessary condition is trivial by putting x_0 equal to the fixed point of f.

Assume now that there exists $x_0 \in X$ such that the orbit $(f^n(x_0))_{n\in I\!N}$ is a Cauchy sequence.

Let us first suppose that $(f^n(x_0))_{n\in I\!N}$ has a constant subsequence, that is there exist $z \in X$ and $(f^{n_k}(x_0))_{k\in I\!N}$ such that

$$f^{n_k}(x_0) = z \quad \text{, for every } k \in I\!N \ . \tag{1}$$

Since if a Cauchy sequence has a converging subsequence then the whole sequence converges, we deduce that

$$\lim_{n\to+\infty} f^n(x_0) = z \ . \tag{2}$$

The element z is a fixed point for f. In fact, from (1) we have

$$0 \leq d(f(z), z) = d\left(f\left(f^{n_k}(x_0)\right), z\right) = d\left(f^{1+n_k}(x_0), z\right) \quad \text{, for every } k \in I\!N \ .$$

By (2) we conclude $d(f(z), z) = 0$, i.e. $z = f(z)$.

Suppose now that sequence $(f^n(x_0))_{n\in I\!N}$ has not constant subsequences. Then we can say either that, for every $n \in I\!N$, sequence $(f^p(f^n(x_0)))_{p\in I\!N}$ has not constant subsequences and that, for every $p \in I\!N$, sequence $(f^p(f^n(x_0)))_{n\in I\!N}$ has not constant subsequences. Hence, sequence $(f^n(x_0))_{n\in I\!N}$ satisfies $\alpha), \alpha\alpha)$ and $\alpha\alpha\alpha)$ of Definition 2.1.

So, being f a function with partially complete graph, we can conclude that there exists $z \in X$ such that $\lim_{n\to+\infty} f^n(x_0) = z$ and $\lim_{n\to+\infty} f(f^n(x_0)) = f(z)$. By the uniqueness of the limit algorithm also in this case f has a fixed point. □

From the proof of the previous theorem we deduce immediately the following

Corollary 2.2. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph. If there exists $x_0 \in X$ such that the orbit of f at x_0 is a Cauchy sequence, then this sequence converges to a fixed point for f.

Theorem 2.3. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph. The function f has a contractive fixed point if and only if, for any $x, y \in X$, the orbits of f at x and y are Cauchy equivalent.

Proof. First we prove the sufficient condition. Let $x \in X$ be fixed and consider the orbit $(f^n(x))_{n\in I\!N}$. Since it is a Cauchy sequence, Corollary 2.2 states that the sequence converges to a fixed point for f, say z.

For every $y \in X$, the orbit of f at y converges to z too. In fact, let $w \in X$ be the limit of $(f^n(y))_{n\in I\!N}$. As above, w is a fixed point for f. By the hypotheses, sequences $(f^n(x))_{n\in I\!N}$ and $(f^n(y))_{n\in I\!N}$ are equivalent, therefore

$$d(z, w) = \lim_{n\to+\infty} d\left(f^n(x), f^n(y)\right) = 0 .$$

This gives $w = z$. Hence f has a contractive fixed point.

Now we establish the necessary condition.

Let us denote z the contractive fixed point of f. Note that for every $x, y \in X$, both the orbits $(f^n(x))_{n\in I\!N}$ and $(f^n(y))_{n\in I\!N}$ converge to z and, as a consequence, they are Cauchy sequences.

Moreover, we have

$$0 \le d(f^n(x), f^n(y)) \le d(f^n(x), z) + d(z, f^n(y)) \quad \text{, for every } n \in I\!N .$$

Consequently, letting $n \to +\infty$, we can conclude that the orbits of f at x and y are also equivalent.

□

We are now in position to provide, as the main result of the paper, a Caristi type fixed point theorem in metric spaces.

Theorem 2.4. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph.

Suppose that there exists a function $\phi : X \to [0, +\infty[$ such that

$$d(x, f(x)) \le \phi(x) - \phi(f(x)) \text{ , for every } x \in X . \tag{3}$$

Then every orbit of f converges to a fixed point for f.

Proof. Fixed $x \in X$, we begin by proving that the orbit $(f^n(x))_{n\in I\!N}$ is a Cauchy sequence. First of all, by (3), for every $n \in I\!N$ we have

$$0 \leq d\left(f^n(x), f(f^n(x))\right) \leq \phi\left(f^n(x)\right) - \phi\left(f^{n+1}(x)\right) .$$

It follows that $(\phi(f^n(x)))_{n\in I\!N}$ is a decreasing (non negative) sequence and it clearly implies its convergence in $I\!R_0^+$. Thus $(\phi(f^n(x)))_{n\in I\!N}$ is a Cauchy sequence.

Therefore, fixed $\varepsilon > 0$ there exists $\bar{n} = \bar{n}(\varepsilon) \in I\!N$ such that for every $n, p \geq \bar{n}$ (w.l.o.g. we take $n < p$), by applying (3), we get

$$d\left(f^n(x), f^p(x)\right) \leq d\left(f^n(x), f^{n+1}(x)\right) + d\left(f^{n+1}(x), f^{n+2}(x)\right) + \cdots +$$

$$+ d\left(f^{n+p-1}(x), f^p(x)\right) \leq \phi\left(f^n(x)\right) - \phi\left(f^p(x)\right) \leq \varepsilon,$$

that is $(f^n(x))_{n\in I\!N}$ is a Cauchy sequence.

Now, by applying Theorem 2.1, we can conclude that the orbit $(f^n(x))_{n\in I\!N}$ converges to a fixed point for f. □

Remark 2.1. Let us note that it is not possible to state the existence of a contractive fixed point in the class of functions of Theorem 2.4, as it is proved by the function $f(x) = x$, $x \in I\!R$. Nevertheless the thesis of Theorem 2.4 gives more than just the existence of fixed points; namely it asserts that it is possible to single out every fixed point for f.

Now, let us restrict our attention to the functions with complete graph, introduced by Leader in [7].

We recall that a function $f : X \to X$ is said to be *with complete graph* if for every Cauchy sequence $(x_n)_{n\in I\!N}$ such that $(f(x_n))_{n\in I\!N}$ is also a Cauchy sequence there exists a point $x \in X$ such that $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} f(x_n) = f(x)$.

In this setting, from Theorem 2.4 we can deduce the following

Corollary 2.5. Let (X, d) be a metric space and $f : X \to X$ be a function with complete graph.

Suppose that there exists a function $\phi : X \to [0, +\infty[$ such that

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \text{ , for every } x \in X \text{ .}$$

Then every orbit of f converges to a fixed point for f.

Proof. The assert follows by observing that a function with complete graph is also with partially complete graph. □

Remark 2.2. We note that Corollary 2.5 extends the similar results presented in [1], [3].

Moreover, we emphasize that there exist functions satisfying all the assumptions of Theorem 2.4 but not having all properties required in Corollary 2.5, as the following example shows.

Example 2.1. Let $([0,2[,d)$ be the metric space where d is the Euclidean metric. Let $f:[0,2[\to[0,2[$ be defined as

$$f(x)=\begin{cases} 0 & , \ x\in[0,1] \\ 1 & , \ x\in]1,2[\ . \end{cases}$$

This function is not with complete graph: in fact, the Cauchy sequence $\left(1+\frac{1}{2n}\right)_{n\in\mathbb{N}}$ is such that also $\left(f\left(1+\frac{1}{2n}\right)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence and $\left(1+\frac{1}{2n}\right)_{n\in\mathbb{N}}$ converges to 1, but $\lim_{n\to+\infty} f\left(1+\dfrac{1}{2n}\right)=1\neq 0=f(1)$.

On the other hand, obviously f is with partially complete graph since there not exist sequences in $f([0,2[)$ verifying $\alpha),\alpha\alpha)$ and $\alpha\alpha\alpha)$ of Definition 2.1.

Moreover, if we consider the function $\phi:[0,2[\to[0,+\infty[$ defined by

$$\phi(x)=\begin{cases} d(x,0) & , \ x\in[0,1] \\ d(x,1)+d(1,0) & , \ x\in]1,2[\ , \end{cases}$$

we can say that f verifies also (3).

The next theorem is a viable version of our Caristi type result.

Theorem 2.6. Let (X,d) be a metric space and let $x_0\in X$. Suppose that $f:\overline{B}(x_0,r)\to X$ is a function with partially complete graph and that there exists a function $\phi:X\to[0,+\infty[$ such that

$$\phi(x_0)\leq r \tag{4}$$

and

$$d(x,f(x)) \leq \phi(x)-\phi(f(x)) \ , \text{ for every } x\in\overline{B}(x_0,r) \ . \tag{5}$$

Then the orbit of f at x_0 converges to a fixed point for f belonging to $\overline{B}(x_0,r)$.

Proof. Let us consider the orbit $(f^n(x_0))_{n\in\mathbb{N}}$. By using (5), the fact that ϕ is a non negative function and (4), we get

$$d\left(x_0,f^1(x_0)\right)=d\left(x_0,f(x_0)\right) \leq \phi(x_0)-\phi(f(x_0)) \leq \phi(x_0) \leq r \ ,$$

so $f^1(x_0)$ belongs to $\overline{B}(x_0,r)$. Then, by proceeding as above, we also have

$$\begin{aligned} d\left(x_0,f^2(x_0)\right) &\leq d\left(x_0,f^1(x_0)\right)+d\left(f^1(x_0),f^2(x_0)\right)\leq \\ &\leq \phi(x_0)-\phi(f^1(x_0))+\phi(f^1(x_0))-\phi(f^2(x_0)) \\ &=\phi(x_0)-\phi(f^2(x_0))\leq \\ &\leq \phi(x_0)\leq r, \end{aligned}$$

so $f^2(x_0)$ belongs to $\overline{B}(x_0,r)$.

By means of an iterative process, we conclude that

$$\{f^n(x_0) \,:\, n\in\mathbb{N}\}\subset\overline{B}(x_0,r) \ .$$

We are now in position to use (5) on the elements of the orbit of f at x_0 and then we can proceed analogously as in the proof of Theorem 2.4 in order to obtain that $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence.

If the sequence $(f^n(x_0))_{n\in I\!N}$ does not satisfy the whole set of properties of Definition 2.1 (i.e. there exists $\bar{n} \in I\!N$ such that $(f^{p+\bar{n}}(x_0))_{p\in I\!N}$ has a constant subsequence), by proceeding as in the first part of the proof of Theorem 2.1, we can say that there exists a fixed point for f and this case is proved.

Otherwise, by recalling that f is a function with partially complete graph, the sequence $(f^n(x_0))_{n\in I\!N}$ converges in X to a point $\bar{x}$. Now, set $Y = \{f^n(x_0) : n \in I\!N\} \cup \{\bar{x}\}$, we consider the metric space $(Y, d_{|Y\times Y})$. Of course the range of $f_{|Y}$ is the space Y and the function $f_{|Y} : Y \to Y$ is with partially complete graph. Moreover, $f_{|Y}$ is easily seen to satisfy property (3). Therefore $f_{|Y}$ verifies all the assumptions of Theorem 2.4 and so we can claim that the orbit $(f^n(x_0))_{n\in I\!N}$ converges in $Y \subset \overline{B}(x_0, r)$ to a fixed point for f.

Then, this case is proved too. □

3. Some generalizations of Leader fixed point theorems

In the new class of functions with partially complete graph we state the following propositions which improve some of the results obtained in [7].

Theorem 3.1. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph.

The function f has a fixed point if and only if there exists $\bar{x} \in X$ with the property

(C) for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, \bar{x}) > 0$ and a natural number $r = r(\varepsilon, \bar{x})$ such that

$$d\left(f^{p+r}(\bar{x}), f^{q+r}(\bar{x})\right) < \varepsilon \quad \text{, for every } p, q \in I\!N \text{ with } d\left(f^p(\bar{x}), f^q(\bar{x})\right) < \varepsilon + \delta \ .$$

Moreover, if there exists $\bar{x}$ such that (C) is satisfied and the sequence $(f^n(\bar{x}))_{n\in I\!N}$ converges to an element $w \in X$, then we have

$$d\left(f^{p+r}(\bar{x}), w\right) < \varepsilon \quad \text{, for every } p \in I\!N \text{ with } d\left(f^p(\bar{x}), f^{p+r}(\bar{x})\right) < \delta$$

where ε, δ, r are the numbers defined in (C).

Proof. From our Theorem 2.1 and by applying Corollary 1 in [7], we can say immediately that

f has a fixed point if and only if there exists $\bar{x} \in X$ such that $(f^n(\bar{x}))_{n\in I\!N}$ is a Cauchy sequence if and only if there exists $\bar{x} \in X$ with the property (C)

and so we have the first part of the thesis.

The second part is an obvious consequence of Theorem 2 in [7]. □

Theorem 3.2. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph.

The function f has a contractive fixed point if and only if for any $x, y \in X$ the following condition holds

(CC) for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon, x, y) > 0$ and a natural number $r = r(\varepsilon, x, y)$ such that

$$d\left(f^{p+r}(x), f^{q+r}(y)\right) < \varepsilon, \quad \text{for every } p, q \in I\!N \text{ with } d\left(f^p(x), f^q(y)\right) < \varepsilon + \delta \ .$$

Proof. From our Theorem 2.3 and by applying Theorem 1 in [7], we have

f has a contractive fixed point if and only if for any $x, y \in X$ the sequences $(f^n(x))_{n\in I\!N}$, $(f^n(y))_{n\in I\!N}$ are Cauchy equivalent if and only if for any $x, y \in X$ condition (CC) is satisfied

which proves the proposition. □

Corollary 3.3. Let (X, d) be a metric space and $f : X \to X$ be a function with partially complete graph which satisfies the property

(s-CC) for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $r = r(\varepsilon) \in I\!N$ such that

$$d\left(f^r(x), f^r(y)\right) < \varepsilon \text{ , for every } x, y \in X \text{ with } d(x, y) < \varepsilon + \delta \text{ .}$$

Then f has a contractive fixed point, say w.
Moreover,

$$d\left(f^r(x), w\right) \leq \varepsilon \quad \text{, for every } x \in X \text{ with } d\left(x, f^r(x)\right) \leq \delta$$

where ε, δ, r are the numbers defined in (s-CC).

Proof. The first part of the thesis is a trivial consequence of our Theorem 3.2.
In order to obtain the second part, we fix $x \in X$ and, since w is a contractive fixed point, we have that the orbit of f at x converges to w.
Taking into account of (s-CC), fixed $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $r = r(\varepsilon) \in I\!N$ such that

$$d\left(f^{r+p}(x), f^{r+q}(x)\right) < \varepsilon \text{ , for every } p, q \in I\!N \text{ with } d(f^p(x), f^q(x)) < \varepsilon + \delta \text{ .}$$

Hence, by using Theorem 2 in [7], we can conclude that

$$d\left(f^{r+p}(x), w\right) \leq \varepsilon \text{ , for every } p \in I\!N \text{ such that } d\left(f^p(x), f^{r+p}(x)\right) \leq \delta \text{ .}$$

□

Remark 3.1. Thanks to Example 2.1 we can say that our Theorems 3.1 and 3.2 strictly contain Theorem 3 in [7]. Moreover, we note that our Corollary 3.3 improves Corollary 4 in [7].

Remark 3.2. We claim that, even if the metric space X is complete, Theorems 2.1, 2.3, 3.1, 3.2 and Corollary 3.3 are not still true if we drop the assumption on f to be with partially complete graph. This is proved by the following example.

Example 3.1. Let $([0, 1], d)$ be the complete metric space where d is the Euclidean metric. Let $f : [0, 1] \to [0, 1]$ be defined as

$$f(x) = \begin{cases} \frac{1}{2} & , \ x \in \{0\} \cup \left]\frac{1}{2}, 1\right] \\ \\ \frac{1}{m+1} & , \ x \in \left]\frac{1}{m+1}, \frac{1}{m}\right] \text{ , } m = 2, 3, \ldots . \end{cases}$$

First we prove that f satisfies the assumption (s-CC) of Corollary 3.3. Of course, as a consequence, there will be an $\bar{x}$ with the property (C) and also, for every $x, y \in [0, 1]$, the condition (CC) will be verified.

Let us fix $\varepsilon > 0$. We put $\delta(\varepsilon) = \varepsilon$ and $r = r(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1$, where the symbol $\left[\frac{1}{\varepsilon}\right]$ denotes the integer part of the number $\frac{1}{\varepsilon}$.

Indeed, we prove that

$$d\left(f^r(x), f^r(y)\right) < \varepsilon \text{ , for every } x, y \in [0,1] \text{ .} \tag{6}$$

To this end we observe that if $x = 0$ or if $x \in \left]\frac{1}{2}, 1\right]$, then $f^r(x) = \frac{1}{r+1} < \varepsilon$. On the other hand, if $x \in \left]\frac{1}{m+1}, \frac{1}{m}\right]$, $m \geq 2$, we have $f^r(x) = \frac{1}{m+r} < \varepsilon$. Therefore, for every $x \in [0,1]$, we get $f^r(x) \in]0, \varepsilon[$. Hence, (6) is satisfied obviously.

Finally, taking into account of our Corollary 3.3 and by observing that f has not fixed points, we can conclude that f cannot be a function with partially complete graph.

Remark 3.3. In [2] the authors prove, in the setting of the complete metric spaces, the existence of contractive fixed points for functions $f : X \to X$ satisfying the following property (cf. [2], Corollary 3.1)

there exists $p \in I\!N$ such that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, p) > 0$ for which $d\left(f^p(x), f^p(y)\right) < \varepsilon$ for every $x, y \in X$ with $d(x,y) < \varepsilon + \delta$.

Thanks to Example 3.1 we also claim that the above property cannot be weakened by requiring (s-CC), not even to the aim of obtaining just the existence of at least one fixed point.

References

[1] R.P. Agarwal, D. O'Regan and N. Shahzad, *Fixed point theory for generalized contractive maps of Meir-Keeler type*, Math. Nachr., **276**(2004), 3-22.

[2] T. Cardinali and P. Rubbioni, *An extension to multifunctions of the Keeler-Meir's fixed point theorem*, Fixed Point Theory, **7(1)**(2006), 23-36.

[3] J. Caristi, *Fixed point theorems for mapping satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215**(1976), 241-251.

[4] M. Hegedüs and T. Szilágyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive mappings*, Math. Japon., **25**(1980), 147-157.

[5] S. Kasahara, *A fixed point theorem of Meir-Keeler type*, Math. Sem. Notes Kobe Univ., **8**(1980), 131-135.

[6] W.A. Kirk, *Fixed points of asymptotic contractions*, J. Math. Anal. Appl., **277**(2003), 645-650.

[7] S. Leader, *Equivalent Cauchy sequences and contractive fixed points in metric spaces*, Studia Math., **66**(1983), 63-67.

[8] J. Matkowski, *Integrable solutions of functional equations*, Diss. Math., **127**(1975).

[9] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28**(1969), 326-329.

[10] T. Suzuki, *Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces*, Nonlinear Anal., **64**(2006), 971-978.

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