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# A NEW FIXED POINT THEOREM AND ITS APPLICATIONS IN EQUILIBRIUM THEORY

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**Abstract.** We give a new fixed-point theorem for lower semicontinuous correspondences and introduce the notion of $Q^{'}$-majorized correspondences. As applications we obtain some new equilibrium theorems which improve the results of X. Wu in [8], referred to abstract economies with lower semicontinuous correspondences, respectively X. Liu and H.Cai in [4], referred to abstract economies with Q-majorized correspondences.
**Key Words and Phrases**: Fixed point, Q-majorized correspondences, $Q'$-majorized correspondences, abstract economy, equilibrium.
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## 1. Introduction

Generally, the proofs of equilibrium theorems in game theory are based on fixed-point theorems. In this paper we obtain a fixed-point theorem for lower semicontinuous correspondences in Hausdorff local convex spaces and we use it to prove equilibrium existence theorems. X. Wu (in [8]) stated that, even if in recent years, many mathematicians have studied fixed point problems of l.s.c. correspondences, until that time, there had been no ideal result. X. Wu gave a new fixed point theorem for l.s.c correspondences in Hausdorff local convex spaces, which can almost compare with Himmelberg's fixed point theorem. Wu's Theorem was used by X. Liu and H.Cai (in [4]) to prove their results in equilibrium existence. To prove his fixed-point theorem, X. Wu used a Michael's selection theorem, which states that there exists an upper semicontinuous correspondence such that its values are included in the values of the l.s.c. correspondence. For the selection theorem one needs a metrizable set. X. Wu left an open problem: is the conclusion tenable if the metrizable condition is cancelled?

We get our result by using Yannelis and Prabhakar's selection theorem (see [9]) and an approximation method.The domain of the correspondence is not metrizable. We use this result in equilibrium theorems.

There is another fixed point theorem in a Hausdorff topological space, formulated by W. Kim in [2], but the l.s.c. correspondence must verify an additional topological condition.

Most of existence theorems of equilibrium deal with preference correspondences which have lower open sections or are majorized by correspondences with lower open sections. In the last years, some existence results were obtained for lower semicontinuous and upper semicontinuous correspondences. X. Z. Yuan and E. Taradfar (in [11]) proved existence theorems of equilibrium for economies with u-majorized correspondences (majorized by u.s.c., irreflexive correspondences with closed convex values) and X. Liu and H. Cai (in [4]) proved existence theorems of equilibrium for economies with Q-majorized correspondences (majorized by irreflexive, l.s.c. correspondences with open, convex values).

We propose the notion of $Q'$-majorized correspondences, notion which is stronger than the notion of Q-majorized correspondences. Finally we obtain existence theorems of equilibrium which drop the condition of set's being metrizable in the theorem of X. Liu and H. Cai (see [4]).

I am using the term correspondence instead of the well-known ones of set-valued function or multifunction. This term is used in specialized books (for example in [3]), espacially in economic mathematized literature.

The paper is organized in the following way: Section 2 contains preliminaries and notation. The fixed-point theorem is presented in Section 3. Section 4 contains the description of the model of an abstract economy (in Subsection 4.1), Preliminaries (Subsection 4.2) and the equilibrium theorems are stated in Subsection 4.3.

## 2. Preliminaries and notation

Throughout this paper, we shall use the following notation and definitions. Let $A$ be a subset of a topological space $X$.

1. $\mathcal{F}(A)$ denotes the family of all non-empty finite subset of A.
2. $2^A$ denotes the family of all subsets of $A$.
3. cl $A$ denotes the closure of $A$ in $X$.
4. If $A$ is a subset of a vector space, co$A$ denotes the convex hull of $A$.
5. If $F$, $T : A \to 2^X$ are correspondences, then co$T$, cl $T$, $T \cap F : A \to 2^X$ are correspondences defined by (co$T$)$(x)$ =co$T(x)$, (cl$T$)$(x)$ =cl$T(x)$ and $(T \cap F)(x) = T(x) \cap F(x)$ for each $x \in A$, respectively.
6. The graph of $T : X \to 2^Y$ is the set $\text{Gr}(T) = \{(x, y) \in X \times Y \mid y \in T(x)\}$
7. The correspondence $\overline{T}$ is defined by

$$\overline{T}(x) = \{y \in Y : (x, y) \in cl_{X\times Y}Gr(T)\}$$

(the set $cl_{X\times Y}Gr(T)$ is called the adherence of the graph of T).

It is easy to see that cl$T(x) \subset \overline{T}(x)$ for each $x \in X$.

**Lemma 2.1** [10]. *Let $X$ be a topological space, $Y$ be a non-empty subset of a topological vector space $E$, ß be a base of the zero neighborhoods in $E$ and $A : X \to 2^Y$. For each $V \in$ß, let $A_V : X \to 2^Y$ be defined by $A_V(x) = (A(x) + V) \cap Y$ for each $x \in X$. If $\widehat{x} \in X$ and $\widehat{y} \in Y$ are such that $\widehat{y} \in \cap_{V\in ß}\overline{A_V}(\widehat{x})$, then $\widehat{y} \in \overline{A}(\widehat{x})$.*

**Definition 2.1**. Let $X$, $Y$ be topological spaces and $T : X \to 2^Y$ be a correspondence

1. $T$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(x) \subset V$ for each $y \in U$.
2. $T$ is said to be *lower semicontinuous* if for each x$\in X$ and each open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \cap V \neq \emptyset$ for each $y \in U$.

The following three lemmas refere to the lower semicontinuous correspondences.

**Lemma 2.2** [10]. *Let $X$ and $Y$ be two topological spaces and let $A$ be a closed subset of $X$. Suppose $F_1 : X \to 2^Y$ , $F_2 : X \to 2^Y$ are lower semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the correspondence $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A, \\ F_2(x), & \text{if } x \in A \end{cases}$$

*is also lower semicontinuous.*

**Lemma 2.3.**[9]. *Let $X$ and $Y$ be two topological spaces and let $S : X \to 2^Y$ and $T : X \to 2^Y$ be correspondences such that*

*(i) $S$ is l.s.c. and has open upper sections*

*(ii) $T$ is l.s.c.*

*(iii) for all $x \in X$, $S(x) \cap T(x) \neq \emptyset$.*

*Then the correspondence $F : X \to 2^Y$ defined by $F(x) = S(x) \cap T(x)$ is l.s.c.*

**Lemma 2.4.** [10]. *Let $X$ be a topological space, $E$ be a topological vector space and $Y$ be a non-empty subset of $E$. Suppose $T : X \to 2^Y$ is a lower semicontinuous correspondence and $V$ is any non-empty open subset of $E$. Then the correspondence $H : X \to 2^Y$ defined by $H\ (x) = (T(x) + V) \cap Y$ for each $x \in X$ has an open graph in $X \times Y$.*

N. C. Yannelis and N. D. Prabhakar proved in [9] a continuous selection lemma.

**Lemma 2.5** [9]. *Let $X$ be a paracompact Hausdorff space and $Y$ be a linear topological space. Suppose $T : X \to 2^Y$ is a correspondence such that*

*(a) for each $x \in X$, $T(x)$ is non-empty and convex, and*

*(b) for each $y \in Y$, $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in $X$.*

*Then there exists a continuous function $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$.*

## 3. A fixed-point theorem

The purpose of this section is to give a fixed-point theorem for lower semicontinuous correspondences in a Hausdorff local convex space.

Our theorem improves Theorem 1 of X. Wu in [8], since for each $i \in I$, $X_i$ is not a convex set and $D_i$ is not metrizable, but we impose the stronger condition that $\overline{\mathrm{co}S_i}(x) \subset T_i(x)$ for each $x \in X$ instead of $\mathrm{clco}S_i(x) \subset T_i(x)$ for each $x \in X$. To prove his fixed-point theorem, X. Wu used a Michael's selection thorem, which states that there exists an upper semicontinuous selector of a l.s.c. correspondence. We get our result by using Yannelis and Prabhakar's selection theorem in [9] and an approximation method.

We present first Wu's Theorem:

**Theorem 3.1.** [8] *Let $I$ be an index set. For each $i \in I$, let $X_i$ be a nonempty convex subset of a paracompact set in a Hausdorff locally convex topological vector space $E_i$, $D_i$ a non-empty compact metrizable subset of $X_i$ and $S_i, T_i : X \to 2^{D_i}$ two correspondences with the following conditions:*

(1) *for each $x \in X$,* $\mathrm{clco}S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$.

*(2) $S_i$ is lower semi-continuous.*

*Then there exists an equilibrium point $x^* \in D := \prod_{i\in I} D_i$ such that $x_i^* \in$* $T_i(x^*)$ for each $i \in I$.

Our main result is the following.

**Theorem 3.2.** *Let $I$ be an index set. For each $i \in I$, let $X_i$ be a paracompact set in a Hausdorff local convex space $E_i$. Let $S_i, T_i : X = \prod_{i\in I} X_i \to 2^{E_i}$ be correspondences such that:*

(1) *$S_i(x)$ is non-empty, closed and $\overline{\mathrm{co}S_i}(x) \subset T_i(x)$ for each $x \in X$.*

(2) *$S_i$ is lower semi-continuous and there exists a compact convex set $D_i \subset X_i$ such that $S_i(x) \cap D_i \neq \emptyset$ for each $x \in X$.*

*Then there exists an equilibrium point $x^* \in D := \prod_{i\in I} D_i$ such that $x_i^* \in T_i(x^*)$ for each $i \in I$.*

**Proof.** Let ß$_i$ denote the family of all open convex neighborhoods of zero in $E_i$ and $\beta = \prod_{i\in I}$ ß$_i$. Let $V = \prod_{i\in I} V_i \in$ß be given.

By Lemma 2.4, the correspondence $H_i : X \to 2^{X_i}$ , defined by $H_i(x) = (\mathrm{co}S_i(x) + V_i) \cap X$ for each $x \in X$ has an open graph in $X \times X_i$, then it has open lower sections. The correspondence $G_i : X \to 2^{X_i}$ , defined by $G_i(x) = D_i$ for each $x \in X$ has open lower sections.

Since the intersection of two correspondences with open lower sections is a correspondence with open lower sections, it follows that $K_i : X \to 2^{X_i}$ , defined by $K_i(x) = H_i(x) \cap D_i$ for each $x \in X$ has open lower sections.

Since co$S_i$ has non-empty convex values and $D_i$ is convex, it follows that $K_i(x) = (\text{co}S_i(x) + V_i) \cap X_i \cap D_i = (\text{co}S_i(x) + V_i) \cap D_i$ is non-empty, convex. Then by Lemma 2.5 we have that there exists a continuous selection $f_i : X \to D_i$, $f_i(x) \in (\text{co}S_i(x) + V_i) \cap D_i$. Let $f : X \to D$ be defined by $f(x) = \prod_{i\in I} f_i(x)$ for each $x \in X$.

We apply the Brouwer-Schauder's Theorem to the restriction $f_{|D} : D \to D$ and we obtain a point $x_V^* = f(x_V^*)$ and it follows that $(x_V^*)_i \in (\text{co}S_i(x_V^*) + V_i) \cap D_i$ for each $i \in I$.

For each $V \in$ß we define the set $Q_V = \cap_{i\in I}\{x \in D : x_i \in (\text{co}S_i(x)+V_i)\cap D_i\}$.

$Q_V$ is non-empty because $x_V^* \in Q_V$, then cl$Q_V$ is non-empty. We show that the family $\{\text{cl}Q_V : V \in$ß$\}$ has the finite intersection property.

Let $\{V^1, V^2, ...V^n\}$ be any finite set of ß and let $V = \bigcap_{k=1}^{n} V^k$, then $V \in$ ß; Clearly $Q_V \subset \bigcap_{k=1}^{n} Q_{V^k}$ so that $\bigcap_{k=1}^{n} Q_{V^k} \neq \emptyset$. It follows that $\bigcap_{k=1}^{n} \text{cl}Q_{V^k} \neq \emptyset$.

Since $D$ is compact and the family $\{\text{cl}Q_V : V \in \prod_{i\in I}$ß$_i\}$ has the finite intersection property, we have that $\cap\{\text{cl}Q_V : V \in$ ß$\} \neq \emptyset$. Take any $x^* \in \cap\{\text{cl}Q_V : V \in$ß$\}$, then for each $V \in$ ß, $x^* \in \text{cl}\{x^* \in D : x^* \in \prod_{i\in I}(\text{co}S_i(x^*) + V_i) \cap D_i\}$.

Then $(x^*, x^*) \in \text{clGr}\prod_{i\in I}(\text{co}S_i(x^*) + V_i) \cap D_i$ and $x_i^* \in \overline{(\text{co}S_i(x^*) + V_i) \cap D_i}$ for every $V \in$ß and for each $i \in I$.

By Lemma 2.1 we have that for each $i \in I$, $x_i^* \in \overline{\text{co}S_i}(x^*) \subset T_i(x^*)$ and $x^* \in D$. □

The following results improve Wu's Corollaries in [8], since $X$ is not a convex set and $D$ is not metrizable.

**Corollary 3.1.** *Let $X$ be a non-empty subset of a Hausdorff locally convex space $E$, $D$ a non-empty compact convex subset of $X$ and $S, T : X \to 2^E$ be two correspondences with the following conditions:*

*(i) for each $x \in X, \overline{\text{co}S}(x) \subset T(x)$ and $S(x) \neq \emptyset$,*

*(ii) $S$ is l.s.c. on $X$ and*

*(iii) $S(x) \cap D \neq \emptyset$ for each $x \in X$.*

*Then there exists $x^* \in D$ and $x^* \in T(x^*)$.*

**Corollary 3.2.** *Let $X$ be a paracompact set in a Hausdorff local convex space $E$. Let $T : X \to 2^E$ be a correspondence l.s.c. such that $T(x)$ is non-empty, convex for each $x \in X$.*

*We also assume that there exists a compact convex set $D \subset X$ such that $T(x) \cap D \neq \emptyset$ for each $x \in X$.*

*Then there exists $x^* \in D$ such that $x^* \in \overline{T}(x^*)$.*

## 4. Applications in the equilibrium theory

**4.1. The model of an abstract economy.** Let $I$ be a nonempty set (the set of agents). For each $i \in I$, let $X_i$ be a non-empty topological vector space representing the set of actions and define $X := \prod_{i \in I} X_i$; let $A_i$, $B_i : X \to 2^{X_i}$ be the constraint correspondences and $P_i$ the preference correspondence.

**Definition 4.1.** [10]. An *abstract economy* $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ is defined as a family of ordered quadruples $(X_i, A_i, P_i, B_i)$.

**Definition 4.2** [10]. An *equilibrium* for $\Gamma$ is defined as a point $x^* \in X$ such that for each $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

**Remark 4.1.** When, for each $i \in I$, $A_i(x) = B_i(x)$ for all $x \in X$, this abstract economy model coincides with the classical one introduced by Borglin and Keiding in [1]. If in addition, $\overline{B}_i(x) = \mathrm{cl}_{X_i} B_i(x)$ for each $x \in X$, which is the case if $B_i$ has a closed graph in $X \times X_i$, the definition of equilibrium coincides with that one used by Yannelis and Prabhakar in [9].

**4.2. Preliminaries**. X. Liu and H. Cai defined in [4] the correspondences of class $Q$ and $Q$-majorized.

**Definition 4.3.** [4] Let $X$ be a topological space and $Y$ be a non-empty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

(1) $T$ is said to be *of class* $Q_\theta$ (or $Q$) if

(a) for each $x \in X$, $\theta(x) \notin \mathrm{cl} T(x)$ and

(b) $T$ is lower semicontinuous with open and convex values in $Y$;

(2) A correspondence $T_x : X \to 2^Y$ is said to be a *$Q_\theta$-majorant of $T$ at $x$* if there exists an open neighborhood $N(x)$ of $x$ such that

(a) For each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin \mathrm{cl} T_x(z)$

(b) $T_x$ is l.s.c. with open and convex values;

(3) $T$ is said to be *$Q_\theta$-majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists a $Q_\theta$-majorant $T_x$ of $T$ at $x$.

We introduce the following definitions.

**Definition 4.4.** Let $X$ be a topological space and $Y$ be a non-empty subset of a vector space E, $\theta : X \to E$ be a mapping and $T : X \to 2^Y$ be a correspondence.

(1) $T$ is said to be *of class* $Q'_\theta$ (or $Q'$) if

(a) for each $x \in X$, $\theta(x) \notin \overline{T(x)}$ and

(b) $T$ is lower semicontinuous with open and convex values in $Y$;

(2) A correspondence $T_x : X \to 2^Y$ is said to be a $Q'_\theta$*-majorant of* $T$ *at* $x$ if there exists an open neighborhood $N(x)$ of $x$ such that

(a) For each $z \in N(x)$, $T(z) \subset T_x(z)$ and $\theta(z) \notin \overline{T_x(z)}$

(b) $T_x$ is l.s.c. with open and convex values;

(3) $T$ is said to be $Q'_\theta$*-majorized* if for each $x \in X$ with $T(x) \neq \emptyset$ there exists a $Q'_\theta$-majorant $T_x$ of $T$ at $x$.

We need the following Lemma to prove the existence theorems in the next section.

**Lemma 4.1.** *Let $X$ be a paracompact topological space and $Y$ be a non-empty subset of a vector space $E$. Let $\theta : X \to E$ be a single-valued function and $P : X \to 2^Y \backslash \{\emptyset\}$ be* Q$'$-majorized. *Then there exists a correspondence $S : X \to 2^Y$ of class $Q'$ such that $P(x) \subset S(x)$ for each $x \in X$.*

The proof follows the same line as the proof of Theorem 1 in [4].

**4.3. Equilibrium theorems.** In this section, we state some new equilibrium existence theorems for abstract economies with any (countable or uncountable) set of players in locally convex spaces.

We present first the result obtained by X. Wu [8] and X. Liu and H. Cai [4].

**Theorem 4.1.** (X. Wu [8]). *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, such that for each $i \in I$, the following conditions are fulfilled:*

*(1) $X_i$ is a non-empty convex subset of a Hausdorff locally convex topological space $E_i$ and $D_i$ is a non-empty compact metrizable subset of $X_i$;*

*(2) for each $x \in X := \prod_{i \in I} X_i$, $P_i(x) \subset D_i(x)$, $A_i(x) \subset B_i(x) \subset D_i$ and $B_i(x)$ is nonempty convex;*

*(3) the set $W_i := \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is closed in $X$;*

*(4) the correspondences $A_{i|W_i}$ and $P_{i|W_i} : W_i \to 2^{D_i}$ are lower semicontinuous and either $A_i$ or $P_i : X \to 2^{D_i}$ has open sections, $B_i : X \to 2^{D_i}$ is lower semicontinuous;*

*(5) for each $x \in X$, $x_i \notin clco(A_i(x) \cap P_i(x))$.*

*Then there exists an equilibrium point $x^* \in D$ for $\Gamma$, i.e., for each $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.*

**Theorem 4.2.** (X. Liu, H. Cai [4]). *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$ :*

(1) *$X_i$ is a non-empty convex subset of Hausdorff locally topological vector space $E_i$, $X := \prod_{i\in I} X_i$ is paracompact and $D_i$ is a non-empty compact metrizable subset of $X_i$;*

(2) *$A_i$, $B_i$, $P_i$ are corespondences : $X \to 2^{D_i}$, for each $x \in X$, $A_i(x)$ is non-empty, $B_i$ is l.s.c. and convex closed valued; and* $\mathrm{cl}B_i(x) \subset D_i$;

(3) *the set $W_i : = \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is closed in $X$;*

(4) $A_i \cap P_i : X \to 2^{D_i}$ is *Q-majorized.*

*Then there exists an equilibrium point $x^* \in X$ for $\Gamma$, i.e., for each $i \in I$, $x_i^* \in \mathrm{cl}B_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.*

Now we present our equilibrium theorems.

Theorem 4.3 is an existence theorem of equilibria in which constraint and preference correspondences are lower semi-continuous. This theorem improves the theorem of X. Wu (see [8]) in the following ways: the sets $D_i$ are not metrizable, $X_i$ is not convex for each $i \in I$. We impose the stronger condition that for each $i \in I$, $x_i \notin \overline{(\mathrm{co}A_i \cap \mathrm{co}P_i)}(x)$ for each $x \in X$ instead of $x_i \notin \mathrm{clco}(A_i(x) \cap P_i(x))$ for each $x \in X$.

**Theorem 4.3.** *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i\in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$ :*

(1) *$X_i$ is a non-empty convex set in a Hausdorff locally convex space $E_i$, $X := \prod_{i\in I} X_i$ is paracompact and $D_i$ is a non-empty, convex, compact subset of $X_i$;*

(2) *$B_i$ is lower semicontinuous with non-empty convex values and for each $x \in X$, $A_i(x) \neq \emptyset$, $A_i(x) \subset B_i(x)$ and $\mathrm{cl}B_i(x) \cap D_i \neq \emptyset$;*

(3) *the set $W_i : = \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is closed in $X$;*

(4) *$A_{i|W_i}$ and $P_{i|W_i} : W_i \to 2^{D_i}$ are lower semi-continuous and either $A_i$ or $P_i : X \to 2^{D_i}$ has open sections;*

(5) *for each $x \in X$, $x_i \notin \overline{(\mathrm{co}A_i \cap \mathrm{co}P_i)}(x)$.*

*Then there exists an equilibrium point* $x^* \in D := \prod_{i \in I} D_i$ *for* $\Gamma$, *i.e., for each* $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ *and* $A_i(x^*) \cap P_i(x^*) = \emptyset$.

**Proof.** For each $i \in I$ and $x \in X$, let

$$F_i(x) = \begin{cases} \mathrm{co}A_i(x) \cap \mathrm{co}P_i(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \in X \setminus W_i; \end{cases}$$

$\mathrm{co}A_i$ and $\mathrm{co}B_i$ are l.s.c. with non-empty open convex values, then $\mathrm{co}A_i \cap \mathrm{co}P_i$ is l.s.c. by Lemma 2.3 and has nonempty convex values.

By Lemma 2.2, $F_i$ is l.s.c. and has non-empty and convex values.

Define $F : X \to 2^D$ by $F(x) = \prod_{i \in I} F_i(x)$. The correspondence $F$ is l.s.c with non-empty closed convex values and there exists a set $D$ such that $G(x) \cap D \neq \emptyset$ for each $x \in X$.

By Corollary 3.2, it follows that exists $x^* \in D$ such that $x^* \in \overline{F}(x^*)$, i.e., for each $i \in I$, $x_i^* \in \overline{F}_i(x^*)$.

If there exists some $i \in I_0$ such that $x^* \in W_i$, $x_i^* \in \overline{F}_i(x^*)$, by the definition of $F_i$, then $x_i^* \in \overline{(\mathrm{co}A_i \cap \mathrm{co}P_i)}(x^*)$, which contradicts assumption 5. Therefore, $x^* \notin W_i$ for all $i \in I_0$, i.e. $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. By the definition of $F_i$, we must have that $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. $\square$

**Theorem 4.4.** *Let* $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ *be an abstract economy, where* $I$ *is a (possibly uncountable) set of agents such that for each* $i \in I$ :

(1) $X_i$ *is a non-empty convex set in a Hausdorff locally convex space* $E_i$, $X := \prod_{i \in I} X_i$ *is paracompact and* $D_i$ is a non-empty, convex, compact subset of $X_i$*;*

(2) $B_i$ *is lower semicontinuous with non-empty convex values and for each* $x \in X$, $A_i(x) \neq \emptyset$, $A_i(x) \subset B_i(x)$ *and* $\mathrm{cl}B_i(x) \cap D_i \neq \emptyset$*;*

(3) *the set* $W_i := \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ *is closed in* $X$*;*

(4) *the correspondence* $A_i \cap P_i : X \to 2^{D_i}$ *is lower semi-continuous;*

(5) *for each* $x \in X$, $x_i \notin \overline{(\mathrm{co}A_i \cap \mathrm{co}P_i)}(x)$.

*Then there exists an equilibrium point* $x^* \in D := \prod_{i \in I} D_i$ *for* $\Gamma$, *i.e., for each* $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ *and* $A_i(x^*) \cap P_i(x^*) = \emptyset$.

**Proof.** For each $i \in I$ and $x \in X$, let

$$F_i(x) = \begin{cases} \mathrm{co}(A_i \cap P_i)(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \in X \setminus W_i; \end{cases}$$

We have that $co(A_i \cap P_i)$ is l.s.c. and has nonempty convex values.

By Lemma 2.2, $F_i$ is l.s.c. and has non-empty and convex values.

Define $F : X \to 2^D$ by $F(x) = \prod\limits_{i \in I} F_i(x)$. The correspondence $F$ is l.s.c with non-empty closed convex values and there exists a set $D$ such that $G(x) \cap D \neq \emptyset$ for each $x \in X$.

By Corollary 3.2, it follows that exists $x^* \in D$ such that $x^* \in \overline{F}(x^*)$, i.e., for each $i \in I$, $x_i^* \in \overline{F}_i(x^*)$.

If there exists some $i \in I_0$ such that $x^* \in W_i$, $x_i^* \in \overline{F}_i(x^*)$, by the definition of $F_i$, then $x_i^* \in \overline{(\text{co}A_i \cap \text{co}P_i)}(x^*)$, which contradicts assumption 5. Therefore, $x^* \notin W_i$ for all $i \in I_0$, i.e. $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. By the definition of $F_i$, we must have that $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. $\square$

Theorem 4.5 is an existence theorem of equilibria of a generalized game in which the intersection of constraint and preference correspondences is $Q^{'}-$majorized and with any (countable or uncountable) set of players in locally convex spaces. This theorem improves the theorem of X. Liu and H. Cai [4] in the following ways: the sets $D_i$ is not metrizable, $X_i$ is not convex for each $i \in I$. The correspondences $A_i \cap P_i$ is Q$^{'}$-majorized instead of being Q-majorized.

**Theorem 4.5.** *Let $\Gamma = (X_i, A_i, P_i, B_i)_{i \in I}$ be an abstract economy, where $I$ is a (possibly uncountable) set of agents such that for each $i \in I$ :*

(1) *$X_i$ is a non-empty convex set in a Hausdorff locally convex space $E_i$, $X := \prod\limits_{i \in I} X_i$ is paracompact and $D_i$ is a non-empty, convex, compact subset of $X_i$;*

(2) *$B_i$ is lower semicontinuous with non-empty convex values and for each $x \in X$, $A_i(x) \neq \emptyset$, $A_i(x) \subset B_i(x)$ and $\text{cl}B_i(x) \cap D_i \neq \emptyset$;*

(3) *the set $W_i := \{x \in X \ / \ (A_i \cap P_i)(x) \neq \emptyset\}$ is closed in $X$;*

(4) $A_i \cap P_i$ is *$Q^{'}$-majorized and $(A_i \cap P_i)(x) \cap D_i \neq \emptyset$ for each $x \in X$.*

*Then there exists an equilibrium point $x^* \in D$ for $\Gamma$, i.e., for each $i \in I$, $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.*

**Proof.** If $W_i = \emptyset$ for all $i \in I$, $B_i$ is l.s.c. and by Theorem 3.2, it follows that there exists an equilibrium point for $\Gamma$.

Let $I_0 = \{i \in I, W_i \neq \emptyset\}$, without loss of generality, we may assume that $I_0 \neq \emptyset$.

**Case 1.** For each $i \in I_0$, by 4) and Lemma 4.1, there exists a correspondence $S_i : X \to 2^{D_i}$ which is l.s.c. with open and convex values such that $(A_i \cap P_i)(x) \subset S_i(x)$ for each $x \in X$.

Define the correspondence $F_i : X \to 2^{D_i}$, by

$$F_i(x) = \begin{cases} S_i(x) \cap B_i(x), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \in X \setminus W_i; \end{cases}$$

$S_i$ is l.s.c. with open convex values, $B_i$ is l.s.c. with open convex values, then $S_i \cap B_i$ is l.s.c. by Lemma 2.3 and has nonempty convex values..

$F_i$ is l.s.c. with non-empty and convex values by Lemma 2.2.

**Case 2.** For $i \in I \setminus I_0$, we define $F_i : X \to 2^{D_i}$, by $F_i(x) = B_i(x)$ for each $x \in X$.

For $i \in I$, $G_i : X \to 2^{D_i}$, defined by $G_i(x) = clF_i(x)$ for each $x \in X$, is l.s.c with non-empty closed convex values.

Define $G : X \to 2^D$ by $G(x) = \prod\limits_{i \in I} G_i(x)$. The correspondence $G$ is l.s.c with non-empty closed convex values and there exists a set $X$ such that $G(x) \cap D \neq \emptyset$ for each $x \in X$.

By Corollary 3.2, it follows that exists $x^* \in D$ such that $x^* \in \overline{G}(x^*)$, i.e., for each $i \in I$, $x_i^* \in \overline{F}_i(x^*)$.

If there exists some $i \in I_0$ such that $x^* \in W_i$, $x_i^* \in \overline{F}_i(x^*)$, by the definition of $F_i$, then $x_i^* \in \overline{S_i(x^*) \cap B_i(x^*)} \subset \overline{S}_i(x^*)$, which contradicts that $S_i$ is of class $Q'$. Therefore, $x^* \notin W_i$ for all $i \in I_0$, i.e. $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. By the definition of $F_i$, we must have that $x_i^* \in \overline{B}_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ for all $i \in I_0$. □

## References


[1] A. Borglin and H. Keiding, *Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorem,* J. Math. Econom., **3**(1976), 313-316.

[2] W.K. Kim, *A fixed point theorem in a Hausdorff topological vector space,* Comment. Math. Univ. Carolinae, **36**(1995), 33-38.

[3] E. Klein and A.C. Thompson, *Theory of Correspondences,* Canadian Math. Soc. Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1984.

[4] X. Liu, H. Cai, *Maximal Elements and Equilibrium of Abstract Economy,* Appl. Math. Mech., **22**(2001), 1225-1230.

[5] W. Shafer and H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences,* J. Math. Econom., **2**(1975), 345-348.

[6] K.K. Tan and X.Z. Yuan, *Lower semicontinuity of multivalued mappings and equilibrium points,* World Congress of Nonlinear Analysis '92 (Berlin), vol I-IV, Walter de Gruyter, 1996, 1849-1860.

[7] G. Tian, *Fixed points theorems for mappings with non-compact and non-convex domains,* J. Math. Anal. Appl., **158**(1991), 161-167.

[8] X. Wu, *A new fixed point theorem and its applications,* Proc. Amer. Math. Soc., **125**(1997), 1779-1783.

[9] N.C. Yannelis and N.D. Prabhakar, *Existence of maximal elements and equilibrium in linear topological spaces,* J. Math. Econom., **12**(1983), 233-245.

[10] G.X.Z. Yuan, *The Study of Minimax Inequalities and Applications to Economies and Variational Inequalities,* Memoirs Amer. Math. Soc., **132**(1988).

[11] G.X.Z. Yuan and E. Taradfar, *Maximal elements and equilibria of generalized games for u-majorized and condensing correspondences,* Internat. J. Math. Sci., **22**(1999), 179-189.

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