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APPROXIMATING COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. Two algorithmic frameworks for finding a common fixed point of a finite collection of nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces, respectively, are proposed. Corresponding weak and strong convergence theorems are established.
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1. Introduction

Let X be a real Banach space, C a closed and convex subset of X, and let $\{T_i\}_{i=1}^m$ be a finite collection of nonexpansive self-mappings of C such that their common fixed point set, F, is nonempty. In this paper we address the problem, which is sometimes referred to as the convex feasibility problem, of finding a point in F. There is a considerable body of work on this problem in the framework of Hilbert spaces which captures applications in various areas: image restoration [5], computer tomography [12] and approximation theory [23, 24], to name a few. Projection methods dominate the iterative approaches to this problem because in a Hilbert space H the nearest point projection P_K from H onto a closed and convex subset K of H is nonexpansive. We aim to solve the problem in the framework of Banach spaces where the situation is quite different because the nearest point projections there are not nonexpansive.

The most straightforward attempt to solve the fixed point problem for a single nonexpansive mapping T is to iterate it cyclically, namely, to consider $\{T^n x\}_{n=1}^{\infty}$. However, this sequence may not converge even in the weak topology. One way to overcome this difficulty is to use Mann's iteration method [17] that produces a sequence $\{x_n\}$ via the following recursion:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1}$$

for all $n \in \mathbb{N}$, where the initial point x_1 is arbitrary. Reich [21] proved that if X is uniformly convex with a Fréchet differentiable norm and the sequence of parameters $\{\alpha_n\} \subset [0, 1]$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1) converges weakly to a fixed point of T. Some modifications of Mann's iteration scheme (1) for a single nonexpansive mapping were proposed in the Hilbert space setting by Ishikawa [13] and later his result was extended to Banach spaces by Tan and Xu [26]. Furthermore, several attempts to extend Ishikawa's result to finitely many mappings have also been made. Das and Debata [8] studied an Ishikawa-like scheme defined by

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n, \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, where $0 < a < b < 1$, and T and S are two quasi-nonexpansive mappings. They proved that under certain conditions the sequence $\{x_n\}$ defined by (2) converges strongly to a common fixed point of S and T in real strictly convex Banach spaces. Khan and Fukhar-ud-din [14] used the iteration method (2) with bounded errors to establish weak and strong convergence results for two nonexpansive mappings in the setting of uniformly convex Banach spaces which satisfy Opial's condition. More recently, Chidume and Ali [6] proposed an algorithmic scheme for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^{m}$ of C into X with respective sequences $\{k_{i,n}\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} k_{i,n} = 1$ and $\sum_{n=1}^{\infty} (k_{i,n} - 1) < \infty$, $1 \leq i \leq m$, which is generated for $m \geq 2$ by

$$\begin{cases} x_{n+1} = (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{n+m-2} \\ y_{n+m-2} = (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{n+m-3}, \\ \ldots \\ y_n = (1-\alpha_{mn})x_n + \alpha_{mn}T_m^n x_n, \end{cases} \tag{3}$$

where $\{\alpha_{i,n}\}_{n=1}^{\infty} \subset [\epsilon, 1-\epsilon]$, $\epsilon > 0$, for each $1 \leq i \leq m$, respectively. They proved that if X is a uniformly convex Banach space the dual space X^* of which has the Kadec-Klee property, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=1}^m$.

Another direction of research with the aim of achieving strong convergence to a fixed point of a nonexpansive mapping was initiated by Halpern [11] who introduced the following iterative scheme in a Hilbert space setting, now known as the anchor point method. This method is defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{4}$$

where the point $u \in C$ is the anchor and $T : C \to C$ a nonexpansive mapping. He proved that under appropriate conditions on the sequence $\{\alpha_n\}$, the process $\{x_n\}$ defined by (4) converges strongly to the projection of the anchor u onto the set of fixed points of T, namely, $P_{\mathrm{Fix}(T)}u$. Subsequently, Lions [16] and Wittmann [27] extended the class of admissible sequences $\{\alpha_n\}$. Bauschke [1] considered the case of a finite collection of nonexpansive mappings in Hilbert space, and quite recently, O'Hara, Pillay and Xu [18] extended Bauschke's result to Banach spaces. In another recent paper, Kim and Xu [15] proposed the following combination of Mann's and Halpern's methods:

$$x_{n+1} = \beta_n u + (1-\beta_n)(\alpha_n x_n + (1-\alpha_n)Tx_n), \tag{5}$$

where $T : C \to C$ is a nonexpansive mapping, $u \in C$ is arbitrary and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $(0,1)$. They proved that under some appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, the process defined by (5) converges strongly to a fixed point of T. Their scheme consists of a convex combination of a point in C and Mann's iteration method (1), and works in uniformly smooth Banach spaces.

In the present paper we introduce two iterative procedures for approximating common fixed points of finitely many nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces, respectively, where, at each iteration, the mappings are chosen in cyclic order. We note that our convergence theorems originate in the results established by Tan and Xu [26], and Kim and Xu [15], respectively, for a single nonexpansive mapping, and that our algorithmic schemes can be viewed as natural extensions of their methods to a finite collection of mappings. Comparing our first scheme to that proposed by Chidume and Ali [6], we see that no additional substep is involved in our iteration process and that it contains the case of a single nonexpansive mapping as well. The proof of our first result is different from that presented by Tan and Xu [26]. As a matter of fact, we impose different restrictions on the parameters even for the case of a single mapping. To prove the convergence of our second scheme we partially combine the ideas of Kim and Xu [15], and of O'Hara, Pillay and Xu [18].

2. Preliminaries

Throughout this paper we assume that X is a real Banach space with norm $\|\cdot\|$. Given a nonempty, closed and convex subset C of X, a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$. The fixed point set of T is denoted by $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. The normalized duality map J from X into X^*, the dual space of X, is given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in X$. A Banach space X is said to be *uniformly convex* if

$$\inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \le 1, \|y\| \le 1, \text{ and } \|x - y\| \ge \epsilon \right\} > 0$$

for all positive ϵ. It is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{6}$$

exists for all $x, y \in U$, where $U = \{x \in X : \|x\| = 1\}$ is the unit sphere of X. It is said to be *uniformly smooth* if the limit (6) exists and is attained uniformly for all $x, y \in U$. It is known [7] that a Banach space X is uniformly smooth if and only if the duality map J is single-valued and norm-to-norm uniformly continuous on bounded subsets of X. The norm of X is said to be

Fréchet differentiable if for each $x \in U$, the limit (6) is attained uniformly for $y \in U$. In this case we have

$$\frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle \leq \frac{1}{2}\|x+h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, J(x)\rangle + \tilde{g}(\|h\|), \qquad (7)$$

for all bounded x, h in X, where in this case $J(x)$ coincides with the Fréchet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $x \in X$, and $\tilde{g}(\cdot)$ is a function defined on $[0,\infty)$ such that $\lim\limits_{t\to 0^+}\dfrac{\tilde{g}(t)}{t} = 0$. It is said to satisfy Opial's condition if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $\bar{x}$, it follows that $\limsup\limits_{n\to\infty}\|x_n - \bar{x}\| < \limsup\limits_{n\to\infty}\|x_n - y\|$ for all $y \in X$, $y \neq \bar{x}$. It is known that l^p spaces for $1 < p < \infty$ enjoy this property and that any separable Banach space can be equivalently renormed so that it satisfies Opial's condition [9, 19].

Proposition 2.1. [4] Let X be a uniformly convex Banach space, $C \subset X$ a bounded, closed and convex subset of X, and let $T : C \to X$ be a nonexpansive mapping. Then there exists a strictly increasing continuous function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$g(\|T(tx + (1-t)y) - (tTx + (1-t)Ty)\| \leq \|x - y\| - \|Tx - Ty\|$$

for all $x, y \in C$ and $0 \leq t \leq 1$.

Proposition 2.2. [28] Let $\{p_n\}$ be a sequence of nonnegative real numbers satisfying

$$p_{n+1} \leq (1-\gamma_n)p_n + \gamma_n\sigma_n, \text{ for all } n \geq 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ are two real sequences such that

(i) $\lim\limits_{n\to\infty}\gamma_n = 0$ and $\sum\limits_{n=0}^{\infty}\gamma_n = \infty$,

either

(ii) either $\limsup\limits_{n\to\infty}\sigma_n \leq 0$ or $\sum\limits_{n=0}^{\infty}|\gamma_n\sigma_n| < \infty$.

Then the sequence $\{p_n\}$ converges to zero.

Proposition 2.3. [25] Let X be a uniformly convex Banach space and let b, c be two constants such that $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$, and $\{x_n\}$, $\{y_n\}$ are two sequences in X. Then the following three conditions

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d, \quad \limsup_{n\to\infty} \|x_n\| \le d, \quad \limsup_{n\to\infty} \|y_n\| \le d,$$

(where $d \ge 0$ is some constant), together imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

The following fact is sometimes called the subdifferential inequality.

Proposition 2.4. In any Banach space, we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $x, y \in X$, where $j(x+y) \in J(x+y)$.

Given a nonempty subset D of C, a map $Q : C \to D$ is called a *retraction* from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is called *sunny* if $Q(x+t(x-Qx)) = Qx$ for all $x \in C$ and $t \ge 0$ whenever $x+t(x-Qx) \in C$; it is said to be *sunny nonexpansive* if it is both sunny and nonexpansive. Reich [22] showed that if X is uniformly smooth, $C \subset X$ is bounded, closed and convex and $T : C \to C$ is nonexpansive mapping then, for each $u \in C$, the implicit process defined by $x_t = tu + (1-t)Tx_t, 0 < t \le 1$, converges strongly to a fixed point of T when t tends to zero. Defining $Q : C \to \mathrm{Fix}(T)$ by $Qu =$ the strong $\lim_{t\to\infty} x_t$, we obtain that Q is the unique sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. Sunny nonexpansive retractions enjoy in terms of duality mappings some of the properties the nearest point projections have in Hilbert space. We will need in the sequel the following one.

Proposition 2.5. [3, 10, 20] Let C be a closed and convex subset of a smooth Banach space X, D a subset of C and $Q : C \to D$ a retraction. Then Q is sunny nonexpansive if and only if

$$\langle x - Qx, J(y - Qx)\rangle \le 0, \text{ for all } x \in C \text{ and } y \in D.$$

3. Convergence Theorems

In this section we establish a weak convergence theorem for the Ishikawa iteration process in a uniformly convex Banach space and a strong convergence theorem for a modified Mann-Halpern iteration process in uniformly smooth Banach spaces, respectively.

Let C be a closed and convex subset of uniformly convex Banach space X and let $\{T_i\}_{i=1}^m$ be a finite collection of nonexpansive self-mappings of C having a nonempty common fixed point set F. Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be two sequences in $[0,1]$ such that $\{s_n\} \subset [0, 1-\epsilon]$, $\epsilon > 0$, and $\{t_n\} \subset [b, c]$, $0 < b < c < 1$, respectively. We define our first iteration process as follows:

$$x_{n+1} = t_n T_n(s_n T_n x_n + (1-s_n)x_n) + (1-t_n)x_n \tag{8}$$

for all $n \in \mathbb{N}$. Here we set $T_n = T_{n\ (mod\ m)}$, where we let the $mod\, m$ function take values in $\{1, 2, ..., m\}$.

Theorem 3.1. Assume that X either satisfies Opial's condition or that its norm is Fréchet differentiable. Then for any initial point x_1 in C, the iteration process defined by (8) converges weakly to a point in F.

Proof. We begin our proof by showing that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and that $\lim\limits_{n\to\infty} \|x_n - f\|$ exists for all $f \in F$. For each $n \in \mathbb{N}$ and $x \in C$, we let $S_n x = t_n T_n(s_n T_n x + (1-s_n)x) + (1-t_n)x$. Then $S_n : C \to C$ is nonexpansive because, for all $x, y \in C$, we have:
$\|S_n x - S_n y\| =$
$= \|t_n T_n(s_n T_n x + (1-s_n)x) + (1-t_n)x - t_n T_n(s_n T_n y + (1-s_n)y) - (1-t_n)y\|$
$\le t_n\|T_n(s_n T_n x + (1-s_n)x) - T_n(s_n T_n y + (1-s_n)y)\| + (1-t_n)\|x-y\|$
$\le t_n\|(s_n T_n x + (1-s_n)x) - (s_n T_n y + (1-s_n)y)\| + (1-t_n)\|x-y\|$
$\le t_n s_n\|x-y\| + t_n(1-s_n)\|x-y\| + (1-t_n)\|x-y\| = \|x-y\|$.

Note that $\mathrm{Fix}(T_n) \subset \mathrm{Fix}(S_n)$. Hence it follows that for all $f \in F$,

$$\|x_{n+1} - f\| = \|S_n x_n - S_n f\| \le \|x_n - f\|,$$

that is, the sequence $\{x_n\}_{n=1}^{\infty}$ is Fejér monotone [2, Definition 2.15] with respect to F, the sequence $\{\|x_n - f\|\}_{n=1}^{\infty}$ is decreasing, and therefore $\lim\limits_{n\to\infty} \|x_n - f\|$ exists. Set $d = \lim\limits_{n\to\infty} \|x_n - f\|$. Since we have shown that the sequence $\{x_n\}$ is bounded, we may assume from now on that so is C.
Next, we show that $\lim\limits_{n\to\infty} \|T_n x_n - x_n\| = 0$. To this end, set $y_n = s_n T_n x_n + (1-s_n)x_n$. Then $x_{n+1} = t_n T_n y_n + (1-t_n)x_n$.
We have, for all $f \in F$
$\|x_{n+1} - f\| = \|t_n T_n y_n + (1-t_n)x_n - f\| = \|t_n(T_n y_n - f) + (1-t_n)(x_n - f)\|$
and
$\|T_n y_n - f\| \le \|y_n - f\| = \|s_n T_n x_n + (1-s_n)x_n - f\| = \|s_n(T_n x_n - f) + (1-s_n)(x_n - f)\| \le s_n\|x_n - f\| + (1-s_n)\|x_n - f\| = \|x_n - f\|$.

Consequently, since

$$\lim_{n\to\infty} \|t_n(T_n y_n - f) + (1-t_n)(x_n - f)\| = d,$$

$$\limsup_{n\to\infty} \|T_n y_n - f\| \le d \text{ and } \lim_{n\to\infty} \|x_n - f\| = d,$$

it follows from Proposition 2.3 that

$$\lim_{n\to\infty} \|T_n y_n - x_n\| = 0. \tag{9}$$

Now
$\|T_n x_n - x_n\| \le \|T_n x_n - T_n y_n\| + \|T_n y_n - x_n\| \le \|x_n - y_n\| + \|T_n y_n - x_n\|$
$= s_n\|T_n x_n - x_n\| + \|T_n y_n - x_n\|$,
that is,

$$\|T_n x_n - x_n\| \le \frac{1}{1-s_n}\|T_n y_n - x_n\|,$$

and it follows that

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0, \tag{10}$$

as claimed. Since

$$\|x_{n+1} - x_n\| = \|t_n T_n y_n + (1-t_n)x_n - x_n\| = t_n\|T_n y_n - x_n\|,$$

we also obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{11}$$

Since
$\|x_n - T_{n+i}x_n\| \le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\|$
$\le 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\|$, we conclude that

$$\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0, \text{ for all } 1 \le i \le m. \tag{12}$$

Let p be a weak cluster point of $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $w\text{-}\lim_{k\to\infty} x_{n_k} = p$. We may assume (after passing to another subsequence if necessary) that $n_k\ (mod\, m) = i$ for some $1 \le i \le m$. For any fixed $l \in \{1, 2, ..., m\}$ we can find $j \in \{1, 2, ..., m\}$, independent of k, such that $(n_k + j)\ (mod\, m) = l$ for all $k \in \mathbb{N}$. Then it follows from (12) that

$$\lim_{k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0.$$

Since l is arbitrary, we get $p \in F$ by the demiclosedness principle [2]. It remains to be shown that the entire sequence $\{x_n\}$ converges weakly to p. To

this end, assume q is another weak cluster point of $\{x_n\}$ and $w\text{-}\lim_{k\to\infty} x_{m_k} = q$. We may repeat the above argument and obtain that $q \in F$.
Hence, by Opial's condition, we have:
$\lim_{n\to\infty} \|x_n - p\| = \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| = \lim_{k\to\infty} \|x_{m_k} - q\|$
$< \lim_{k\to\infty} \|x_{m_k} - p\| = \lim_{n\to\infty} \|x_n - p\|$, which contradicts our assumption regarding the existence of different weak cluster points. Therefore $\{x_n\}$ converges weakly to a point in F.
Now assume that X has a Fréchet differentiable norm. Then by (7) we obtain for all $f_1, f_2 \in F$ and $0 < t < 1$,

$$\frac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle \le \frac{1}{2}\|tx_n + (1-t)f_1 - f_2\|^2$$

$$\le \frac{1}{2}\|f_1 - f_2\|^2 + t\langle x_n - f_1, J(f_1 - f_2)\rangle + \tilde{g}(t\|x_n - f_1\|). \tag{13}$$

We now claim that $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ exists (*cf.* [21]). To see this, set $W_{n,m} = S_{n+m-1}S_{n+m-2}...S_{n+1}S_n$. Then $W_{n,m}$ is nonexpansive and $x_{n+m} = W_{n,m}x_n$. It follows from Proposition 2.1 that:

$$g(\|W_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\|)$$

$$\le \|x_n - f_1\| - \|W_{n,m}x_n - W_{n,m}f_1\| = \|x_n - f_1\| - \|x_{n+m} - f_1\|.$$

Since $\lim_{n\to\infty} \|x_n - f_1\|$ exists, we conclude that

$$\lim_{n,m\to\infty} \|W_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\| = 0.$$

Consequently, since

$$\|tx_{n+m} + (1-t)f_1 - f_2 \ \le \|W_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\|$$

$$+\|W_{n,m}(tx_n + (1-t)f_1) - f_2\| \le \|W_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\|$$

$$+\|tx_n + (1-t)f_1 - f_2\|,$$

it follows that

$$\limsup_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$$

$$\le \lim_{n,m\to\infty} \|W_{n,m}(tx_n+(1-t)f_1)-(tx_{n+m}+(1-t)f_1)\|+\liminf_{n\to\infty} \|tx_n+(1-t)f_1-f_2\|$$

$$= \liminf_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|,$$

that is, $\lim_{n\to\infty} \|tx_n + (1-t)f_1 - f_2\|$ exists. Returning to (13), we get

$$\frac{1}{2}\|f_1 - f_2\|^2 + t\limsup_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle \le \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)f_1 - f_2\|^2$$

$$\le \frac{1}{2}\|f_1 - f_2\|^2 + t\liminf_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle + o(t).$$

Hence

$$\limsup_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle \le \liminf_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle + \frac{o(t)}{t}.$$

Letting t tend to zero, we see that $\lim_{n\to\infty}\langle x_n - f_1, J(f_1 - f_2)\rangle$ exists. Since all weak cluster points of $\{x_n\}$ are in F, we obtain

$$\|p - q\|^2 = \langle p - q, J(p - q)\rangle = 0,$$

that is, $\{x_n\}$ converges weakly to a point in F. □

Our second convergence theorem is a modification of Mann's and Halpern's iteration methods. We continue to consider the case of finitely many nonexpansive mappings. Let C be a closed and convex subset of a uniformly smooth Banach space X, and let $\{T_i\}_{i=1}^m$ be a finite collection of nonexpansive mappings having a nonempty common fixed point set F such that

$$F = \mathrm{Fix}(T_m T_{m-1} \ldots . T_2 T_1) = \mathrm{Fix}(T_1 T_m \ldots . T_3 T_2) = \ldots = \mathrm{Fix}(T_{m-1} \ldots . T_1 T_m).$$

Given $u, x_1 \in C$, we define the sequence $\{x_n\}_{n=1}^\infty \subset C$ by

$$x_{n+1} = \beta_n u + (1-\beta_n)(\alpha_n x_n + (1-\alpha_n)T_n x_n) \tag{14}$$

for all $n \in \mathbb{N}$, where once again $T_n = T_{n\ (mod\ m)}$, the $mod\, m$ function takes values in $\{1, 2, ..., m\}$, and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ are sequences in $(0, 1)$.

Theorem 3.2. Assume that the sequences $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$;

(ii) $\sum_{n=1}^\infty \alpha_n = \infty$, $\sum_{n=1}^\infty \beta_n = \infty$;

(iii) $\sum_{n=1}^\infty |\alpha_{n+m} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+m} - \beta_n| < \infty$.

Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (14) converges strongly to Qu, where Q is the unique sunny nonexpansive retraction of C onto F.

Proof. First we observe that $\{x_n\}_{n=1}^{\infty}$ is bounded. Setting $y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n$, we may write (14) as $x_{n+1} = \beta_n u + (1-\beta_n)y_n$. Since for all $f \in F$,

$$\|y_n - f\| \le \alpha_n \|x_n - f\| + (1-\alpha_n)\|T_n x_n - f\| \le \|x_n - f\|,$$

we have, by induction,

$$\|x_{n+1} - f\| \le \beta_n \|u - f\| + (1-\beta_n)\|y_n - f\| \le \beta_n \|u - f\| + (1-\beta_n)\|x_n - f\|$$
$$\le \max\{\|u - f\|, \|x_n - f\|\} \le \max\{\|u - f\|, \|x_1 - f\|\}, \text{ for all } n \in \mathbb{N}.$$

Hence $\{x_n\}$ is indeed bounded and so is $\{y_n\}$. Therefore

$$\lim_{n\to\infty} \|x_{n+1} - T_n x_n\| \le \lim_{n\to\infty} (\beta_n \|u - y_n\| + \alpha_n \|x_n - T_n x_n\|) = 0. \tag{15}$$

Next, we show that $\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$. Indeed, after some calculations we get

$$x_{n+m} - x_n = (\beta_{n+m-1} - \beta_{n-1})(u - y_{n-1}) + (1-\beta_{n+m-1})(y_{n+m-1} - y_{n-1})$$
$$= (\beta_{n+m-1} - \beta_{n-1})(u - \alpha_{n-1}x_{n-1} - (1-\alpha_{n-1})T_{n-1}x_{n-1})$$
$$+(1-\beta_{n+m-1})(\alpha_{n+m-1}x_{n+m-1} + (1-\alpha_{n+m-1})T_{n+m-1}x_{n+m-1} - \alpha_{n-1}x_{n-1}$$
$$-(1-\alpha_{n-1})T_{n-1}x_{n-1}) = (\beta_{n+m-1}-\beta_{n-1})(u-T_{n-1}x_{n-1}) - \alpha_{n-1}(\beta_{n+m-1}-\beta_{n-1})$$
$$(x_{n-1}-T_{n-1}x_{n-1}) + (1-\beta_{n+m-1})\alpha_{n+m-1}(x_{n+m-1}-x_{n-1}) + (1-\beta_{n+m-1})(\alpha_{n+m-1}$$
$$-\alpha_{n-1})(x_{n-1}-T_{n-1}x_{n-1}) + (1-\beta_{n+m-1})(1-\alpha_{n+m-1})(T_{n+m-1}x_{n+m-1}-T_{n-1}x_{n-1}),$$

so that,

$$\|x_{n+m} - x_n\| \le (1-\alpha_{n+m-1})(1-\beta_{n+m-1})\|T_{n+m-1}x_{n+m-1} - T_{n-1}x_{n-1}\|$$
$$+(1-\beta_{n+m-1})\alpha_{n+m-1}\|x_{n+m-1} - x_{n-1}\| + |(\alpha_{n+m-1} - \alpha_{n-1})(1-\beta_{n+m-1})$$
$$-(\beta_{n+m-1}-\beta_{n-1})\alpha_{n-1}|\|x_{n-1}-T_{n-1}x_{n-1}\| + (\beta_{n+m-1}-\beta_{n-1})\|u-T_{n-1}x_{n-1}\|.$$

Consequently,

$$\begin{aligned} \|x_{n+m} - x_n\| &\le (1-\beta_{n+m-1})\|x_{n+m-1} - x_{n-1}\| + \gamma(|\alpha_{n+m-1} - \alpha_{n-1}| \\ &+ 2|\beta_{n+m-1} - \beta_{n-1}|), \end{aligned}$$

where γ is a constant such that $\gamma \ge \max\{\|u-T_{n-1}x_{n-1}\|, \|x_{n-1}-T_{n-1}x_{n-1}\|\}$ for all $n \in \mathbb{N}$. Proposition 2.2 now implies that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0. \tag{16}$$

Next, we claim that $\lim_{n\to\infty} \|x_{n+m} - T_{n+m-1}...T_n x_n\| = 0$. Indeed,
$\|x_{n+m} - T_{n+m-1}...T_n x_n\| = \|x_{n+m} - T_{n+m-1}x_{n+m-1} + T_{n+m-1}x_{n+m-1}$
$-T_{n+m-1}T_{n+m-2}x_{n+m-2} + \cdots + T_{n+m-1}...T_{n+1}x_{n+1} - T_{n+m-1}...T_n x_n\|$.

By (15), we have

$$x_{n+m} - T_{n+m-1}x_{n+m-1} \to 0.$$

We also have

$$x_{n+m-1} - T_{n+m-2}x_{n+m-2} \to 0,$$

and since T_{n+m-1} is nonexpansive, we get

$$T_{n+m-1}x_{n+m-1} - T_{n+m-1}T_{n+m-2}x_{n+m-2} \to 0.$$

Similarly, $T_{n+m-1}T_{n+m-2}x_{n+m-2} - T_{n+m-1}T_{n+m-2}T_{n+m-3}x_{n+m-3} \to 0$, and so on. It follows that

$$\lim_{n\to\infty} \|x_{n+m} - T_{n+m-1}...T_n x_n\| = 0, \tag{17}$$

as claimed. So, from (16) and (17) we obtain

$$\begin{aligned}\lim_{n\to\infty} \|x_n - T_{n+m-1}...T_n x_n\| \le \lim_{n\to\infty} \|x_n - x_{n+m}\| \\ + \lim_{n\to\infty} \|x_{n+m} - T_{n+m-1}...T_n x_n\| = 0.\end{aligned} \tag{18}$$

Next, we show that $\limsup_{n\to\infty}\langle u - Qu, J(x_n - Qu)\rangle \le 0$. We may choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - Qu, J(x_n - Qu)\rangle = \lim_{j\to\infty}\langle u - Qu, J(x_{n_j} - Qu)\rangle,$$

where $\{x_{n_j}\}$ converges weakly to $\bar{x}$, and $T_{n_j} = T_i$ for some $i \in \{1, 2, ..., m\}$. Then $T_{n_j+m-1}...T_{n_j} = T_{i+m-1}...T_i$ for all $j \in \mathbb{N}$. Set $S = T_{i+m-1}...T_i$. Then S is nonexpansive and $\mathrm{Fix}(S) = F$ by assumption. Now define the implicit process $z_t = tu+(1-t)Sz_t$ $0 < t \le 1$. Since the sunny nonexpansive retraction of C onto F is unique, we have $Qu = \lim_{t\to 0^+} z_t$ [22]. By the subdifferential inequality,

$$\begin{aligned}
\|z_t - x_{n_j}\|^2 &= \|(1-t)(Sz_t - x_{n_j}) + t(u - x_{n_j})\|^2 \\
&\leq (1-t)^2\|Sz_t - x_{n_j}\|^2 + 2t\langle u - x_{n_j}, J(z_t - x_{n_j})\rangle \\
&\leq (1-t)^2(\|Sz_t - Sx_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|)^2 \\
&+ 2t\langle u - z_t + z_t - x_{n_j}, J(z_t - x_{n_j})\rangle \\
&= (1-t)^2(\|Sz_t - Sx_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|)^2 \\
&+ 2t(\|z_t - x_{n_j}\| + \langle u - z_t, J(z_t - x_{n_j})\rangle) \\
&\leq (1+t^2)\|z_t - x_{n_j}\|^2 \\
&+ (1-t)^2(2\|Sx_{n_j} - x_{n_j}\|\,\|z_t - x_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|^2) \\
&+ 2t\langle u - z_t, J(z_t - x_{n_j})\rangle.
\end{aligned}$$

Thus

$$\|z_t - x_{n_j}\|^2 \leq (1+t^2)\|z_t - x_{n_j}\|^2 + (1-t)^2(2\|Sx_{n_j} - x_{n_j}\|\,\|z_t - x_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|^2)$$
$$+2t\langle u - z_t, J(z_t - x_{n_j})\rangle.$$

Consequently,

$$\langle u - z_t, J(x_{n_j} - z_t)\rangle$$
$$\leq \frac{t}{2}\|z_t - x_{n_j}\|^2 + \frac{1}{2t}(1-t)^2(2\|Sx_{n_j} - x_{n_j}\|\,\|z_t - x_{n_j}\| + \|Sx_{n_j} - x_{n_j}\|^2).$$

It now follows from (18) that

$$\lim_{j\to\infty}\langle u - z_t, J(x_{n_j} - z_t)\rangle \leq \frac{t}{2}M$$

for some $M > 0$. Since J is norm-to-norm uniformly continuous on bounded subsets of X, letting t tend to zero we obtain

$$\lim_{j\to\infty}\langle u - Qu, J(x_{n_j} - Qu)\rangle \leq 0.$$

Using once again the subdifferential inequality, we now see that

$$\begin{aligned}
\|x_{n+1} - Qu\|^2 &= \|\beta_n(u - Qu) + (1-\beta_n)(y_n - Qu)\|^2 \\
&\leq (1-\beta_n)^2\|y_n - Qu\|^2 + 2\beta_n\langle u - Qu, J(x_{n+1} - Qu)\rangle \\
&\leq (1-\beta_n)\|x_n - Qu\|^2 + 2\beta_n\langle u - Qu, J(x_{n+1} - Qu)\rangle.
\end{aligned}$$

It now follows from Proposition 2.2 that $\{x_n\}$ converges strongly to Qu, where Q is the unique sunny nonexpansive retraction of C onto F. □

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