

AN \mathcal{E} -LATTICE STRUCTURE ASSOCIATED TO SOME CLASSES OF FINITE GROUPS

MARIUS TĂRNĂUCEANU

Faculty of Mathematics

"Al.I. Cuza" University

Iași, Romania

E-mail: tarnauc@uaic.ro

Abstract. In the present paper we introduce and study a canonical \mathcal{E} -lattice structure on the set of element orders of some finite groups. We show that a finite abelian group is uniquely determined by this canonical \mathcal{E} -lattice.

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1. PRELIMINARIES

The notion of \mathcal{E} -lattice has been introduced in [6]. So, given a nonvoid set L and a map $\varepsilon : L \rightarrow L$, we denote by $\text{Ker } \varepsilon$ the kernel of ε (i.e. $\text{Ker } \varepsilon = \{(a, b) \in L \times L \mid \varepsilon(a) = \varepsilon(b)\}$), by $\text{Im } \varepsilon$ the image of ε (i.e. $\text{Im } \varepsilon = \{\varepsilon(a) \mid a \in L\}$) and by $\text{Fix } \varepsilon$ the set consisting of all fixed points of ε (i.e. $\text{Fix } \varepsilon = \{a \in L \mid \varepsilon(a) = a\}$). We say that L is an \mathcal{E} -lattice (relative to ε) if there exist two binary operations $\wedge_\varepsilon, \vee_\varepsilon$ on L which satisfy the following properties:

- a) $a \wedge_\varepsilon (b \wedge_\varepsilon c) = (a \wedge_\varepsilon b) \wedge_\varepsilon c$, $a \vee_\varepsilon (b \vee_\varepsilon c) = (a \vee_\varepsilon b) \vee_\varepsilon c$, for all $a, b, c \in L$;
- b) $a \wedge_\varepsilon b = b \wedge_\varepsilon a$, $a \vee_\varepsilon b = b \vee_\varepsilon a$, for all $a, b \in L$;
- c) $a \wedge_\varepsilon a = a \vee_\varepsilon a = \varepsilon(a)$, for any $a \in L$;
- d) $a \wedge_\varepsilon (a \vee_\varepsilon b) = a \vee_\varepsilon (a \wedge_\varepsilon b) = \varepsilon(a)$, for all $a, b \in L$.

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Clearly, in an \mathcal{E} -lattice L (relative to ε) the map ε is idempotent and $\text{Im } \varepsilon = \text{Fix } \varepsilon$. Moreover, for any $a, b \in L$, we have:

$$\begin{aligned} a \wedge_{\varepsilon} \varepsilon(a) &= a \vee_{\varepsilon} \varepsilon(a) = \varepsilon(a), \\ a \wedge_{\varepsilon} \varepsilon(b) &= \varepsilon(a) \wedge_{\varepsilon} b = \varepsilon(a) \wedge_{\varepsilon} \varepsilon(b) = \varepsilon(a \wedge_{\varepsilon} b), \\ a \vee_{\varepsilon} \varepsilon(b) &= \varepsilon(a) \vee_{\varepsilon} b = \varepsilon(a) \vee_{\varepsilon} \varepsilon(b) = \varepsilon(a \vee_{\varepsilon} b). \end{aligned}$$

Also, note that the set $\text{Fix } \varepsilon$ is closed under the binary operations $\wedge_{\varepsilon}, \vee_{\varepsilon}$ and, denoting by \wedge, \vee the restrictions of $\wedge_{\varepsilon}, \vee_{\varepsilon}$ to $\text{Fix } \varepsilon$, we have that $(\text{Fix } \varepsilon, \wedge, \vee)$ is a lattice. The connection between the \mathcal{E} -lattice concept and the lattice concept is very powerful. Thus, if $(L, \wedge_{\varepsilon}, \vee_{\varepsilon})$ is an \mathcal{E} -lattice and \sim is an equivalence relation on L such that $\sim \subseteq \text{Ker } \varepsilon$, then the factor set L/\sim is a lattice isomorphic to the lattice $\text{Fix } \varepsilon$. Conversely, if L is a nonvoid set and \sim is an equivalence relation on L having the property that the factor set L/\sim is a lattice, then the set L can be endowed with an \mathcal{E} -lattice structure (relative to a map $\varepsilon : L \rightarrow L$) such that $\sim \subseteq \text{Ker } \varepsilon$ and $L/\sim \cong \text{Fix } \varepsilon$.

If $(L, \wedge_{\varepsilon}, \vee_{\varepsilon})$ is an \mathcal{E} -lattice and for every $x \in L$ we denote by $[x]$ the equivalence class of x modulo $\text{Ker } \varepsilon$ (i.e. $[x] = \{y \in L \mid \varepsilon(x) = \varepsilon(y)\}$), then we have $a \wedge_{\varepsilon} b \in [\varepsilon(a) \wedge \varepsilon(b)]$ and $a \vee_{\varepsilon} b \in [\varepsilon(a) \vee \varepsilon(b)]$, for all $a, b \in L$. We say that L is *canonical \mathcal{E} -lattice* if $a \wedge_{\varepsilon} b, a \vee_{\varepsilon} b \in \text{Fix } \varepsilon$, for all $a, b \in L$. Three fundamental types of canonical \mathcal{E} -lattices have been identified and studied in [6].

Let $(L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1}), (L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})$ be two \mathcal{E} -lattices and, for every element $a \in L_i$, denote by $[a]_i$ the equivalence class of a modulo $\text{Ker } \varepsilon_i$, $i = 1, 2$. A map $f : L_1 \rightarrow L_2$ is called an *\mathcal{E} -lattice homomorphism* if:

- a) $f \circ \varepsilon_1 = \varepsilon_2 \circ f$;
- b) for all $a, b \in L_1$, we have:
 - i) $f(a \wedge_{\varepsilon_1} b) = f(a) \wedge_{\varepsilon_2} f(b)$;
 - ii) $f(a \vee_{\varepsilon_1} b) = f(a) \vee_{\varepsilon_2} f(b)$.

Moreover, if the map f is one-to-one and onto, then it is called an *\mathcal{E} -lattice isomorphism*. \mathcal{E} -lattice isomorphisms between canonical \mathcal{E} -lattices have been investigated in [7]. Mention here only the following characterization of these, established in Proposition 1, § 2.1: a map $f : L_1 \rightarrow L_2$ is an \mathcal{E} -lattice isomorphism if and only if its restriction to the set $\text{Fix } \varepsilon_1$ is a lattice isomorphism

from $\text{Fix } \varepsilon_1$ to $\text{Fix } \varepsilon_2$ and $f|[a]_1 : [a]_1 \rightarrow [f(a)]_2$ is one-to-one and onto, for each $a \in \text{Fix } \varepsilon_1$.

An interesting example of a canonical \mathcal{E} -lattice structure on the set of subgroups of a group G has been presented in [7]. Another canonical \mathcal{E} -lattice can be associated to some classes of finite groups G , too. Its study is the main goal of this paper.

Most of our notations are standard and will usually not be repeated here. Basic definitions and results on lattices can be found in [1] or [2]. For group theory concepts we refer the reader to [3] and [4].

2. MAIN RESULTS

Let G be a finite group of order n and L_n be the lattice consisting of all divisors of n . We define on G the following equivalence relation

$$a \sim a' \text{ iff } \text{ord}(a) = \text{ord}(a')$$

and take a set of representatives $\{a_i \mid i \in I\}$ for the equivalence classes modulo \sim . There exist many classes of finite groups such that the set $\{\text{ord}(a_i) \mid i \in I\}$ forms a sublattice of L_n , as finite p -groups or finite abelian groups. For such a group G , this induces a lattice structure on $\{a_i \mid i \in I\}$, which is isomorphic to that of $\{\text{ord}(a_i) \mid i \in I\}$. As we have seen in Section 1, G becomes a canonical \mathcal{E} -lattice with respect to the idempotent map

$$\varepsilon : G \rightarrow G, \varepsilon(a) = a_i \text{ iff } a \in [a_i],$$

where the binary operations $\wedge_\varepsilon, \vee_\varepsilon$ are defined by

$$a \wedge_\varepsilon b = \varepsilon(a) \wedge \varepsilon(b) \text{ and } a \vee_\varepsilon b = \varepsilon(a) \vee \varepsilon(b).$$

We shall call it the *order canonical \mathcal{E} -lattice* associated to G .

Our first aim is to give a detailed description of the above \mathcal{E} -lattice in the case when G is a finite abelian group. Then, from the fundamental theorem of finitely generated abelian groups, there are (uniquely determined by G) the numbers $m \in \mathbb{N}^*, d_1, d_2, \dots, d_m \in \mathbb{N} \setminus \{0, 1\}$ satisfying $d_1/d_2/\dots/d_m, \prod_{i=1}^m d_i = n$ and $G \cong \prod_{i=1}^m \mathbb{Z}_{d_i}$. Clearly, the element orders of G are all divisors of the

exponent d_m of G , therefore the following lattice isomorphism holds:

$$\text{Fix } \varepsilon \cong L_{d_m}. \tag{*}$$

In order to complete our description we need to count the elements of an arbitrary equivalence class $[a_i]$ modulo \sim , i.e. to determine the number of elements of a given order in G . First of all, we focus on the particular situation in which G is a finite abelian p -group. Then it has a direct decomposition of type $G \cong \prod_{i=1}^m \mathbb{Z}_{p^{\alpha_i}}$, where p is a prime and $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m$. A recurrence method to compute the number of elements of order p^α ($\alpha \in \mathbb{N}$) in G is indicated in the next lemma.

Lemma 1. *Let p be a prime, $(\alpha_m)_{m \in \mathbb{N}^*}$ be a chain of positive integers such that $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m \leq \dots$ and $(f_m^p)_{m \in \mathbb{N}^*}$ be the chain of functions defined by:*

$$f_m^p : \mathbb{N} \rightarrow \mathbb{N},$$

$$f_m^p(\alpha) = \left| \left\{ x \in \prod_{i=1}^m \mathbb{Z}_{p^{\alpha_i}} \mid \text{ord}(x) = p^\alpha \right\} \right|, \quad (\forall) \alpha \in \mathbb{N}.$$

Then $f_m^p(0) = 1$, $f_m^p(\alpha) = 0$ for all $\alpha > \alpha_m$ and the chain $(f_m^p)_{m \in \mathbb{N}^*}$ satisfies the equality

$$f_{m+1}^p(\alpha) = p^\alpha g_m^p(\alpha) - p^{\alpha-1} g_m^p(\alpha - 1), \quad (\forall) \alpha \in \mathbb{N}^*,$$

where

$$g_m^p(\alpha) = \begin{cases} p^{m\alpha}, & 0 \leq \alpha \leq \alpha_1 \\ p^{(m-1)\alpha + \alpha_1}, & \alpha_1 \leq \alpha \leq \alpha_2 \\ \vdots \\ p^{\alpha_1 + \alpha_2 + \dots + \alpha_m}, & \alpha_m \leq \alpha. \end{cases}$$

Proof. Let $x = (x_1, x_2, \dots, x_{m+1}) \in \prod_{i=1}^{m+1} \mathbb{Z}_{p^{\alpha_i}}$. Then $\text{ord}(x)$ is the maximum between the order of (x_1, x_2, \dots, x_m) in $\prod_{i=1}^m \mathbb{Z}_{p^{\alpha_i}}$ and the order of x_{m+1} in

$\mathbb{Z}_p^{\alpha_{m+1}}$, and so, $\text{ord}(x) = p^\alpha$ iff:

$$\left\{ \begin{array}{l} \text{ord}(x_1, x_2, \dots, x_m) = p^\alpha \text{ and } \text{ord}(x_{m+1}) = p^\beta \text{ for some } \beta \leq \alpha \\ \text{or} \\ \text{ord}(x_1, x_2, \dots, x_m) = p^\beta \text{ for some } \beta < \alpha \text{ and } \text{ord}(x_{m+1}) = p^\alpha. \end{array} \right.$$

This shows that the following recurrence relation holds

$$\begin{aligned} f_{m+1}^p(\alpha) &= p^\alpha f_m^p(\alpha) + (p^\alpha - p^{\alpha-1})(f_m^p(0) + f_m^p(1) + \dots + f_m^p(\alpha-1)) = \\ &= p^\alpha (f_m^p(0) + f_m^p(1) + \dots + f_m^p(\alpha)) - p^{\alpha-1}(f_m^p(0) + f_m^p(1) + \dots + f_m^p(\alpha-1)), \end{aligned} \tag{1}$$

for all $\alpha \in \mathbb{N}^*$. Putting $g_m^p(\alpha) = \sum_{\beta=0}^{\alpha} f_m^p(\beta)$, the equality (1) becomes:

$$g_{m+1}^p(\alpha) = p^\alpha g_m^p(\alpha) - p^{\alpha-1} g_m^p(\alpha - 1), \quad (\forall) \alpha \in \mathbb{N}^*. \tag{2}$$

Summing up these equalities for $\alpha = 1, 2, \dots$, we obtain:

$$g_{m+1}^p(\alpha) = p^\alpha g_m^p(\alpha), \quad (\forall) \alpha \in \mathbb{N}. \tag{3}$$

Since

$$\begin{aligned} g_1^p(\alpha) &= \sum_{\beta=0}^{\alpha} f_1^p(\beta) = 1 + \sum_{\beta=1}^{\min\{\alpha, \alpha_1\}} f_1^p(\beta) = 1 + \sum_{\beta=1}^{\min\{\alpha, \alpha_1\}} (p^\beta - p^{\beta-1}) = \\ &= \begin{cases} p^\alpha, & 0 \leq \alpha \leq \alpha_1 \\ p^{\alpha_1}, & \alpha_1 \leq \alpha, \end{cases} \end{aligned}$$

by (3) we infer that $g_m^p(\alpha)$ is of the indicated form. □

Example. We want to determine the number of elements of a given order in the group $\mathbb{Z}_4 \times \mathbb{Z}_{16}$. By applying Lemma 1 for $m = 1$, we obtain:

$$f_2^p(\alpha) = \begin{cases} 1, & \alpha = 0 \\ p^{2\alpha} - p^{2(\alpha-1)}, & 1 \leq \alpha \leq \alpha_1 \\ p^{\alpha+\alpha_1} - p^{\alpha-1+\alpha_1}, & \alpha_1 < \alpha \leq \alpha_2 \\ 0, & \alpha_2 < \alpha. \end{cases}$$

Taking $p = 2$, $\alpha_1 = 2$, $\alpha_2 = 4$, it results that

$$f_2^2(\alpha) = \begin{cases} 1, & \alpha = 0 \\ 2^{2\alpha} - 2^{2(\alpha-1)}, & 1 \leq \alpha \leq 2 \\ 2^{\alpha+2} - 2^{\alpha+1}, & 2 < \alpha \leq 4 \\ 0, & 4 < \alpha, \end{cases}$$

and thus $\mathbb{Z}_4 \times \mathbb{Z}_{16}$ has 1 element of order 2^0 , $2^2 - 2^0 = 3$ elements of order 2^1 , $2^4 - 2^2 = 12$ elements of order 2^2 , $2^5 - 2^4 = 16$ elements of order 2^3 and $2^6 - 2^5 = 32$ elements of order 2^4 .

Next, we return to the general case in which G is a finite abelian group. Because such a group can be uniquely written as a direct product of finite abelian p -groups, by using Lemma 1 we are able to establish an explicit formula for the number of elements of an arbitrary order in G .

Theorem 2. *Let G be a finite abelian group of order n , $\{p_1, p_2, \dots, p_k\}$ be the set of all distinct prime divisors of n and $G \cong \prod_{i=1}^k G_i$ be the direct decomposition of G as product of finite abelian p -groups ($G_i = p_i$ -group, $i = \overline{1, k}$). Then, for any $(\beta_1, \beta_2, \dots, \beta_k) \in \mathbb{N}^k$, the number of elements of order $p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$ in G is*

$$\prod_{i=1}^k f_{m_i}^{p_i}(\beta_i),$$

where m_i denotes the number of direct factors of G_i , $i = \overline{1, k}$.

Proof. Since every element x of G can be uniquely written as a product $x_1 x_2 \dots x_k$ with $x_i \in G_i$, $i = \overline{1, k}$, and $(\text{ord}(x_i), \text{ord}(x_j)) = 1$ for all $i \neq j = \overline{1, k}$, we get $\text{ord}(x) = \prod_{i=1}^k \text{ord}(x_i)$. So, $\text{ord}(x) = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$ if and only if $\text{ord}(x_i) = p_i^{\beta_i}$, $i = \overline{1, k}$. Now, our statement follows immediately from Lemma 1. □

Example. We want to determine the number of elements of order 120 in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{720}$. First we write G as a product of finite abelian p -groups

$$G = G_1 \times G_2 \times G_3,$$

where $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_{16}$, $G_2 = \mathbb{Z}_3 \times \mathbb{Z}_9$ and $G_3 = \mathbb{Z}_5$. Since $120 = 2^3 3^1 5^1$, we must compute the numbers $f_2^2(3), f_2^3(1), f_1^5(1)$ of elements of orders $2^3, 3^1, 5^1$ in G_1, G_2, G_3 , respectively. We already know that $f_2^2(3) = 16$. Similarly, by Lemma 1, we find $f_2^3(1) = 8$ and $f_1^5(1) = 4$. Hence $\mathbb{Z}_{12} \times \mathbb{Z}_{720}$ has $16 \cdot 8 \cdot 4 = 512$ elements of order 120.

From Theorem 2 we easily obtain the number of elements whose orders are prime powers in a finite abelian group.

Corollary 3. *Under the hypotheses of Theorem 2, for each $i \in \{1, 2, \dots, k\}$, the number of elements of order p_i^α in G is $f_{m_i}^{p_i}(\alpha)$. In particular, G possesses $f_{m_i}^{p_i}(1) = p_i^{m_i} - 1$ elements of order p_i .*

Since the structure of the order canonical \mathcal{E} -lattice of a finite abelian group is completely determined, it appears the following question: what can be said about two finite abelian groups whose order canonical \mathcal{E} -lattices are isomorphic? In order to answer this question, we need first to prove an auxiliary result.

Lemma 4. *Let G and G' be two finite abelian p -groups. If the numbers of elements of order p^α in G and G' are equal for all $\alpha \in \mathbb{N}$, then G and G' are isomorphic.*

Proof. Suppose that $G = \prod_{i=1}^{r+1} \mathbb{Z}_{p^{\alpha_i}}$ and $G' \cong \prod_{i=1}^{s+1} \mathbb{Z}_{p^{\beta_i}}$, where $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{r+1}$ and $1 \leq \beta_1 \leq \beta_2 \leq \dots \leq \beta_{s+1}$. By Corollary 3, G and G' have $p^{r+1} - 1$ and $p^{s+1} - 1$ elements of order p , respectively, so $r + 1 = s + 1$. Suppose that $\alpha_1 \neq \beta_1$ and assume $\alpha_1 < \beta_1$. Then, by Lemma 1, we obtain that the numbers of elements of order p^{α_1+1} in G and G' are $p^{(r+1)\alpha_1+r} - p^{(r+1)\alpha_1}$ and $p^{(r+1)(\alpha_1+1)} - p^{(r+1)\alpha_1}$, fact which contradicts our hypothesis. Thus $\alpha_1 = \beta_1$. Now, it is clear that a standard induction argument will show that $\alpha_i = \beta_i$, for all $i = \overline{1, r + 1}$. Hence $G \cong G'$. \square

Finally, we are able to show that the order canonical \mathcal{E} -lattice of a finite abelian group determines uniquely the structure of this group, and so to answer the above question.

Theorem 5. *Two finite abelian groups G and G' are isomorphic if and only if their order canonical \mathcal{E} -lattices are isomorphic.*

Proof. Clearly, the order canonical \mathcal{E} -lattices of two isomorphic finite abelian groups are also isomorphic. Conversely, assume that the order canonical \mathcal{E} -lattices of $G \cong \prod_{i=1}^m \mathbb{Z}_{d_i}$ and $G' \cong \prod_{i=1}^{m'} \mathbb{Z}_{d'_i}$ (where $d_i, i = \overline{1, m}$, and $d'_i, i = \overline{1, m'}$, are given by the fundamental theorem of finitely generated abelian groups) are isomorphic, and take an isomorphism f between these. Then, by using the lattice isomorphism (*), we get:

$$L_{d_m} \cong L_{d'_{m'}}. \tag{4}$$

This shows that $n = |G|$ and $n' = |G'|$ have the same number of prime divisors. Put $n = p_1^{\mu_1} p_2^{\mu_2} \dots p_k^{\mu_k}$ and $n' = q_1^{\eta_1} q_2^{\eta_2} \dots q_k^{\eta_k}$, where all p_i and q_i are prime, $i = \overline{1, k}$. The relation (4) implies also that $d_m = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_k^{\gamma_k}$ and $d'_{m'} = q_1^{\gamma_1} q_2^{\gamma_2} \dots q_k^{\gamma_k}$ for some $\gamma_1, \gamma_2, \dots, \gamma_k \in \mathbb{N}^*$. Then $f(p_i^\alpha) = q_i^\alpha$, for all $i = \overline{1, k}$ and $\alpha \in \mathbb{N}^*$. Because an \mathcal{E} -lattice isomorphism induces a bijection between the corresponding equivalence classes modulo \sim , it results that the numbers of elements of orders p_i^α and q_i^α in G and G' , respectively, are equal.

On the other hand, G and G' can be written as direct products of finite abelian p -groups

$$G \cong \prod_{i=1}^k G_i, \quad G' \cong \prod_{i=1}^k G'_i,$$

where $|G_i| = p_i^{m_i}$ and $|G'_i| = q_i^{m'_i}$, $i = \overline{1, k}$. From Corollary 3 we know that the numbers of elements of orders p_i in G_i and q_i in G'_i are $p_i^{m_i} - 1$ and $q_i^{m'_i} - 1$ (m_i and m'_i being the numbers of direct factors in G_i and G'_i). It follows that $p_i^{m_i} - 1 = q_i^{m'_i} - 1$, so $p_i = q_i$, $i = \overline{1, k}$. Now, Lemma 4 implies that $G_i \cong G'_i$, for all $i = \overline{1, k}$, and hence $G \cong G'$. \square

We finish our paper by the remark that the detailed study of order canonical \mathcal{E} -lattices can be extended from finite abelian groups to another classes of finite groups.

Open problem. *Find other finite groups for which the previous canonical \mathcal{E} -lattice structure can be defined (i.e. the orders of elements form, with respect*

to divisibility, a lattice) and described (i.e. we can determine the number of elements of an arbitrary given order).

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