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A TOPOLOGICAL PROPERTY OF THE COMMON FIXED POINTS SET OF TWO MULTIVALUED OPERATORS SATISFYING A LATIF-BEG TYPE CONDITION

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Abstract. Let X be a nonempty, closed and convex subset of a Banach space. We prove that the common fixed points set of two lower semicontinuous multivalued operators defined on X with values in the set of all nonempty, closed and convex subsets of X, which satisfy a contraction type condition of Latif-Beg type, is an absolute retract for paracompact spaces. We also present a result regarding the common fixed points set for two multivalued operators satisfying a Latif-Beg type condition, using the notion of selection property.
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1. Introduction

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X, i.e. $P(X) := \{ Y \mid \emptyset \neq Y \subseteq X \}$.
Let $F : X \to P(X)$ be a multivalued operator. We denote by $\mathcal{F}_F$ the fixed points set of F, i.e. $\mathcal{F}_F := \{ x \in X \mid x \in F(x) \}$.
Let $F_1, F_2 : X \to P(X)$ be two multivalued operators. We denote by $(\mathcal{CF})_{F_1,F_2}$ the common fixed points set of F_1 and F_2, i.e. $(\mathcal{CF})_{F_1,F_2} = \{ x \in X \mid x \in F_1(x) \cap F_2(x) \}$.
Let X, Y be two nonempty sets and $F : X \to P(Y)$ a multivalued mapping. A mapping $\varphi : X \to Y$, with the property that $\varphi(x) \in F(x)$, for each $x \in X$, is called a *selection* of F.

We denote by $\mathbb{N}_0$ the set of all nonnegative integer numbers and by $\mathbb{R}_+$ the set of all nonnegative real numbers.

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Further on we shall use the notations $B(x_0, r) := \{ x \in X \mid d(x_0, x) < r \}$ and $P_b(X) = \{ Y \in P(X) \mid Y \text{ is a bounded set} \}$.

Throughout the paper, for a topological space X we denote by $P_{cl}(X)$ the set of all nonempty and closed subsets of X and for a linear space X we denote by $P_{cv}(X)$ the set of all nonempty and convex subsets of X.

Let C be a topological space (metric space). We say that C is an *absolute retract* for paracompact spaces (metric spaces) iff for any paracompact topological space (metric space) T, for any $A \in P_{cl}(T)$ and for any continuous function $\psi : A \to C$, there exists a continuous function $\varphi : T \to C$ such that $\varphi|_A = \psi$.

The fixed points set of a single-valued contraction defined on a complete metric space, with values in this space, has a single element, while in the case of the multivalued contractions this does not happen in general. Therefore, the idea of studying the topological properties of the fixed points set in the multivalued case appears very natural.

Let E be a Banach space, $X \in P_{cl,cv}(E)$ and $F : X \to P_{cl,cv}(X)$ a multivalued contraction. B. Ricceri proved in [9] that $\mathcal{F}_F$ is an absolute retract for paracompact spaces.

A. Bressan, A. Cellina and A. Fryszkowski proved in [4] a result of the above-mentioned type for a multivalued contraction defined on $L^1(T, E)$, for some measure space T, with values in the set of all nonempty, closed, bounded and decomposable subsets of $L^1(T, E)$.

L. Górniewicz, S. A. Marano and M. Ślosarski established in [5] a result which, in the set of multivalued contractions with nonempty, closed and bounded values, unifies and extends to arbitrary absolute retracts the results given in [9] and [4], by defining the notion of selection property with respect to a given family of metric spaces [5, Definition 2.1].

Let (X, d) be a complete metric space and absolute retract for metric spaces. In [13] we proved that the common fixed points set of two multivalued operators defined on X, which have the selection property and satisfy a contraction type condition, is an absolute retract for metric spaces.

Another interesting problem is to see in what conditions the properties of the values of a multivalued operator are inherited by its fixed points set. For some multifunctions, this problem was studied by H. Schirmer in [10] and by M.-C. Alicu and O. Mark in [1]. In [12] we gave a result regarding some properties of the values of two multifunctions satisfying a contraction type condition, which are inherited by their common fixed points set.

Let E be a Banach space and $X \in P_{cl,cv}(E)$. In Section 2 we prove that the common fixed points set of two lower semicontinuous multivalued operators $F_1, F_2 : X \to P_{cl,cv}(X)$, which satisfy a contraction type condition of Latif-Beg type, is an absolute retract for paracompact spaces. We also present a result regarding the common fixed points set of two multivalued operators satisfying a Latif-Beg type condition, using the notion of selection property.

2. A topological property of the common fixed points set of two multivalued operators

The following two results are presented in [9], as an immediate consequence of some results proved by E. Michael in [7] and as a modified version of a result given in [2].

Theorem 2.1 ([9], Theorem 2). *Let T be a paracompact topological space, $(E, \|\cdot\|)$ a Banach space and $G : T \to P_{cl,\,cv}(E)$ a lower semicontinuous multivalued mapping.*

Then for any $A \in P_{cl}(T)$ and for any continuous selection ψ of $G|_A$, there exists a continuous selection φ of G such that $\varphi|_A = \psi$.

Proposition 2.1 ([9], Proposition 3). *Let T be a topological space, (X, d) a metric space, $G : T \to P(X)$ a lower semicontinuous multivalued mapping, $f : T \to X$ a continuous function and $g : T \to \mathbb{R}_+$ a continuous functional.*

We consider the multivalued mapping $Q : T \to P(X)$ defined by

$$Q(t) = \begin{cases} B(f(t), g(t)), & \text{if } t \in T,\ g(t) > 0, \\ \{f(t)\}, & \text{if } t \in T,\ g(t) = 0 \end{cases}$$

and we suppose that $G(t) \cap Q(t) \neq \emptyset$, for each $t \in T$.
Then the multivalued mapping $T \ni t \multimap \overline{G(t) \cap Q(t)}$ is lower semicontinuous.

We give the following theorem concerning the common fixed points set of two multivalued operators, which satisfy a contraction type condition of Latif-Beg type.

Theorem 2.2. *Let* $(E, \|\cdot\|)$ *be a Banach space,* $X \in P_{cl,cv}(E)$ *and* $F_1, F_2 : X \to P_{cl,cv}(X)$ *two lower semicontinuous multivalued operators. We suppose that*:

(i) *there exist* $a_{11}, \ldots, a_{15} \in]0, +\infty[$, *with* $a_{11} + a_{12} + a_{13} + 2\,a_{14} < 1$, *such that for each* $x \in X$, *any* $u_x \in F_1(x)$ *and for all* $y \in X$, *there exists* $u_y \in F_2(y)$ *so that*

$$\|u_x - u_y\| \leq a_{11}\,\|x - y\| + a_{12}\,\|x - u_x\| + a_{13}\,\|y - u_y\| +$$

$$+ a_{14}\,\|x - u_y\| + a_{15}\,\|y - u_x\|;$$

(ii) *there exist* $a_{21}, \ldots, a_{25} \in]0, +\infty[$, *with* $a_{21} + a_{22} + a_{23} + 2\,a_{24} < 1$, *such that for each* $x \in X$, *any* $u_x \in F_2(x)$ *and for all* $y \in X$, *there exists* $u_y \in F_1(y)$ *so that*

$$\|u_x - u_y\| \leq a_{21}\,\|x - y\| + a_{22}\,\|x - u_x\| + a_{23}\,\|y - u_y\| +$$

$$+ a_{24}\,\|x - u_y\| + a_{25}\,\|y - u_x\|.$$

Then $(\mathcal{CF})_{F_1,F_2} = \mathcal{F}_{F_1} = \mathcal{F}_{F_2} \in P_{cl}(X)$ *and* $(\mathcal{CF})_{F_1,F_2}$ *is an absolute retract for paracompact spaces.*

Proof. From a result given by A. Sîntămărian in [11] it follows that $(\mathcal{CF})_{F_1,F_2} = \mathcal{F}_{F_1} = \mathcal{F}_{F_2} \in P_{cl}(X)$.

We shall prove that $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for paracompact spaces.

Let $q \in]1, \min\{(a_{11}+a_{12}+a_{13}+2\,a_{14})^{-1}, (a_{21}+a_{22}+a_{23}+2\,a_{24})^{-1}\}[$ and set $l := \max\{(a_{11}+a_{12}+a_{14})/[1-(a_{13}+a_{14})], (a_{21}+a_{22}+a_{24})/[1-(a_{23}+a_{24})]\} < 1$. It is not difficult to verify that $ql < 1$.

Let T be a paracompact topological space, $A \in P_{cl}(T)$ and $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ a continuous function.

Applying Theorem 2.1, considering the multivalued mapping $G : T \to P_{cl,cv}(X)$ defined by $G(t) = X$, for each $t \in T$, we obtain that there exists a continuous function $\varphi_0 : T \to X$ such that $\varphi_0|_A = \psi$.

We observe that the function $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ is a continuous selection of the multivalued mapping $A \ni t \multimap F_1(\varphi_0(t))$.

Now, applying again Theorem 2.1, considering the multivalued mapping $G : T \to P_{cl,cv}(X)$ defined by $G(t) = F_1(\varphi_0(t))$, for each $t \in T$, we obtain that there exists a continuous function $\varphi_1 : T \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(t) \in F_1(\varphi_0(t))$, for each $t \in T$.

For $t \in T$, taking into account that $\varphi_1(t) \in F_1(\varphi_0(t))$ and the condition (i) from the hypothesis, we have that there exists $y_1(t) \in F_2(\varphi_1(t))$ such that

$$\|\varphi_1(t)-y_1(t)\| \le a_{11}\|\varphi_0(t)-\varphi_1(t)\|+a_{12}\|\varphi_0(t)-\varphi_1(t)\|+a_{13}\|\varphi_1(t)-y_1(t)\|+$$
$$+a_{14}\|\varphi_0(t)-y_1(t)\|+a_{15}\|\varphi_1(t)-\varphi_1(t)\| =$$
$$=(a_{11}+a_{12})\|\varphi_1(t)-\varphi_0(t)\|+a_{13}\|\varphi_1(t)-y_1(t)\|+a_{14}\|\varphi_0(t)-y_1(t)\| \le$$
$$\le (a_{11}+a_{12}+a_{14})\|\varphi_1(t)-\varphi_0(t)\|+(a_{13}+a_{14})\|\varphi_1(t)-y_1(t)\|.$$

So

$$\|\varphi_1(t)-y_1(t)\| \le \frac{a_{11}+a_{12}+a_{14}}{1-(a_{13}+a_{14})}\|\varphi_1(t)-\varphi_0(t)\| \le l\,\|\varphi_1(t)-\varphi_0(t)\|,$$

for each $t \in T$.

We consider the multivalued mapping $Q_1 : T \to P(X)$ defined by

$$Q_1(t) = \begin{cases} B(\varphi_1(t), ql\,\|\varphi_1(t)-\varphi_0(t)\|), & \text{if } t \in T,\ \varphi_1(t) \ne \varphi_0(t), \\ \{\varphi_1(t)\}, & \text{if } t \in T,\ \varphi_1(t) = \varphi_0(t). \end{cases}$$

If $t \in T$ and $\varphi_1(t) \ne \varphi_0(t)$, then we can write that

$$\|\varphi_1(t)-y_1(t)\| \le l\,\|\varphi_1(t)-\varphi_0(t)\| < ql\,\|\varphi_1(t)-\varphi_0(t)\|.$$

So, $y_1(t) \in Q_1(t)$.
If $t \in T$ and $\varphi_1(t) = \varphi_0(t)$, then we have that $\|\varphi_1(t)-y_1(t)\| = 0$, hence $\varphi_1(t) = y_1(t) \in F_2(\varphi_1(t))$.

Therefore $F_2(\varphi_1(t)) \cap Q_1(t) \ne \emptyset$, for each $t \in T$.

We apply Proposition 2.1, taking $G(t) := F_2(\varphi_1(t))$, $f(t) := \varphi_1(t)$ and $g(t) := ql\,\|\varphi_1(t)-\varphi_0(t)\|$, for each $t \in T$. It follows that the multivalued mapping $T \ni t \multimap \overline{F_2(\varphi_1(t)) \cap Q_1(t)}$ is lower semicontinuous.

Considering the multivalued mapping $G : T \to P_{cl,cv}(X)$ defined by $G(t) = \overline{F_2(\varphi_1(t)) \cap Q_1(t)}$, for each $t \in T$, and taking into account the fact that for each $t \in A$ we have that $G(t) = \overline{F_2(\varphi_1(t)) \cap Q_1(t)} = \{\varphi_1(t)\} = \{\psi(t)\}$, we can apply Theorem 2.1 and we obtain that there exists a continuous function $\varphi_2 : T \to X$ such that $\varphi_2|_A = \psi$ and $\varphi_2(t) \in \overline{F_2(\varphi_1(t)) \cap Q_1(t)}$, for each $t \in T$.

So, we can write that

$\varphi_2|_A = \psi$,
$\varphi_2(t) \in F_2(\varphi_1(t))$, for each $t \in T$,
$\|\varphi_2(t) - \varphi_1(t)\| \leq ql \, \|\varphi_1(t) - \varphi_0(t)\|$, for each $t \in T$.

For $t \in T$, taking into account that $\varphi_2(t) \in F_2(\varphi_1(t))$ and the condition (ii) from the hypothesis, we have that there exists $y_2(t) \in F_1(\varphi_2(t))$ such that

$$\|\varphi_2(t)-y_2(t)\| \leq a_{21} \|\varphi_1(t)-\varphi_2(t)\|+a_{22} \|\varphi_1(t)-\varphi_2(t)\|+a_{23} \|\varphi_2(t)-y_2(t)\|+$$
$$+a_{24} \|\varphi_1(t) - y_2(t)\| + a_{25} \|\varphi_2(t) - \varphi_2(t)\| =$$
$$= (a_{21} + a_{22}) \|\varphi_2(t) - \varphi_1(t)\| + a_{23} \|\varphi_2(t) - y_2(t)\| + a_{24} \|\varphi_1(t) - y_2(t)\| \leq$$
$$\leq (a_{21} + a_{22} + a_{24}) \|\varphi_2(t) - \varphi_1(t)\| + (a_{23} + a_{24}) \|\varphi_2(t) - y_2(t)\|.$$

So
$$\|\varphi_2(t) - y_2(t)\| \leq \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})} \|\varphi_2(t) - \varphi_1(t)\| \leq$$
$$\leq l \, \|\varphi_2(t) - \varphi_1(t)\| \leq ql^2 \, \|\varphi_1(t) - \varphi_0(t)\|,$$
for each $t \in T$.

We consider the multivalued mapping $Q_2 : T \to P(X)$ defined by

$$Q_2(t) = \begin{cases} B(\varphi_2(t), (ql)^2 \|\varphi_1(t) - \varphi_0(t)\|), & \text{if } t \in T, \ \varphi_1(t) \neq \varphi_0(t), \\ \{\varphi_2(t)\}, & \text{if } t \in T, \ \varphi_1(t) = \varphi_0(t). \end{cases}$$

If $t \in T$ and $\varphi_1(t) \neq \varphi_0(t)$, then we can write that

$$\|\varphi_2(t) - y_2(t)\| \leq ql^2 \, \|\varphi_1(t) - \varphi_0(t)\| < (ql)^2 \, \|\varphi_1(t) - \varphi_0(t)\|.$$

So, $y_2(t) \in Q_2(t)$.
If $t \in T$ and $\varphi_1(t) = \varphi_0(t)$, then we have that $\|\varphi_2(t) - y_2(t)\| = 0$, hence $\varphi_2(t) = y_2(t) \in F_1(\varphi_2(t))$.

Therefore $F_1(\varphi_2(t)) \cap Q_2(t) \neq \emptyset$, for each $t \in T$.

We apply Proposition 2.1, taking $G(t) := F_1(\varphi_2(t))$, $f(t) := \varphi_2(t)$ and $g(t) := (ql)^2 \|\varphi_1(t) - \varphi_0(t)\|$, for each $t \in T$. It follows that the multivalued mapping $T \ni t \multimap \overline{F_1(\varphi_2(t)) \cap Q_2(t)}$ is lower semicontinuous.

Considering the multivalued mapping $G : T \to P_{cl, cv}(X)$ defined by $G(t) = \overline{F_1(\varphi_2(t)) \cap Q_2(t)}$, for each $t \in T$, and taking into account the fact that for each $t \in A$ we have that $G(t) = \overline{F_1(\varphi_2(t)) \cap Q_2(t)} = \{\varphi_2(t)\} = \{\psi(t)\}$, we can apply Theorem 2.1 and we obtain that there exists a continuous function $\varphi_3 : T \to X$ such that $\varphi_3|_A = \psi$ and $\varphi_3(t) \in \overline{F_1(\varphi_2(t)) \cap Q_2(t)}$, for each $t \in T$.

So, we can write that

$\varphi_3|_A = \psi$,
$\varphi_3(t) \in F_1(\varphi_2(t))$, for each $t \in T$,
$\|\varphi_3(t) - \varphi_2(t)\| \leq (ql)^2 \, \|\varphi_1(t) - \varphi_0(t)\|$, for each $t \in T$.

By induction, we obtain that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$, where $\varphi_n : T \to X$ is a continuous function, for each $n \in \mathbb{N}_0$, with $\varphi_0|_A = \psi$ and with the following properties:

$\varphi_n|_A = \psi$,
$\varphi_{2n-1}(t) \in F_1(\varphi_{2n-2}(t))$ and $\varphi_{2n}(t) \in F_2(\varphi_{2n-1}(t))$, for each $t \in T$,
$\|\varphi_n(t) - \varphi_{n-1}(t)\| \leq (ql)^{n-1}\|\varphi_1(t) - \varphi_0(t)\|$, for each $t \in T$,
for each $n \in \mathbb{N}$.

For each $\lambda > 0$ we put $T_\lambda := \{\, t \in T \mid \|\varphi_1(t) - \varphi_0(t)\| < \lambda \,\}$. The family of sets $\{\, T_\lambda \mid \lambda > 0 \,\}$ is an open covering of T.

For each $\lambda > 0$ the sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges uniformly on T_λ, taking into account that $\|\varphi_n(t) - \varphi_{n-1}(t)\| \leq (ql)^{n-1}\|\varphi_1(t) - \varphi_0(t)\|$, for each $t \in T$ and for each $n \in \mathbb{N}$, that $ql < 1$ and the completeness of X.

Let $\varphi : T \to X$ be the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}_0}$. The function φ is continuous.

Because $\varphi_n|_A = \psi$, for each $n \in \mathbb{N}_0$, we are able to write that $\varphi|_A = \psi$.

We have $\varphi_{2n-1}(t) \in F_1(\varphi_{2n-2}(t))$, for each $t \in T$ and for each $n \in \mathbb{N}$. Letting n tend to the infinity, it follows that $\varphi(t) \in F_1(\varphi(t))$, for each $t \in T$.

We have $\varphi_{2n}(t) \in F_2(\varphi_{2n-1}(t))$, for each $t \in T$ and for each $n \in \mathbb{N}$. Letting n tend to the infinity, it follows that $\varphi(t) \in F_2(\varphi(t))$, for each $t \in T$.

So, we can write that $\varphi : T \to (\mathcal{CF})_{F_1,F_2}$. □

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a Banach space, $X \in P_{cl,\,cv}(E)$ and $F : X \to P_{cl,\,cv}(X)$ a lower semicontinuous multivalued operator with the property that there exist $a_1, \ldots, a_5 \in]0, +\infty[$, with $a_1 + a_2 + a_3 + 2a_4 < 1$, such that for each $x \in X$, any $u_x \in F(x)$ and for all $y \in X$, there exists $u_y \in F(y)$ so that*

$$\|u_x - u_y\| \leq a_1 \,\|x - y\| + a_2 \,\|x - u_x\| + a_3 \,\|y - u_y\| + a_4 \,\|x - u_y\| + a_5 \,\|y - u_x\|.$$

Then $\mathcal{F}_F \in P_{cl}(X)$ and $\mathcal{F}_F$ is an absolute retract for paracompact spaces.

Proof. From a result given by A. Sîntămărian in [11] it follows that $\mathcal{F}_F \in P_{cl}(X)$.

Let $q \in]1, (a_1 + a_2 + a_3 + 2a_4)^{-1}[$ and set $l := (a_1 + a_2 + a_4)/[1 - (a_3 + a_4)]$. We have $ql < 1$.

In order to prove that $\mathcal{F}_F$ is an absolute retract for paracompact spaces, let T be a paracompact topological space, $A \in P_{cl}(T)$ and $\psi : A \to \mathcal{F}_F$ a continuous function.

Using an argument similar to that in the proof of Theorem 2.2, we obtain a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$, where $\varphi_n : T \to X$ is a continuous function, for each $n \in \mathbb{N}_0$, with $\varphi_0|_A = \psi$ and with the following properties:

$\varphi_n|_A = \psi$,

$\varphi_n(t) \in F(\varphi_{n-1}(t))$, for each $t \in T$,

$\|\varphi_n(t) - \varphi_{n-1}(t)\| \leq (ql)^{n-1}\|\varphi_1(t) - \varphi_0(t)\|$, for each $t \in T$,

for each $n \in \mathbb{N}$.

The sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges pointwise on T to a continuous function $\varphi : T \to X$, with $\varphi|_A = \psi$. Also, it can be written that $\varphi : T \to \mathcal{F}_F$. $\square$

Definition 2.1. *Let $X \in \mathcal{M}$, where by $\mathcal{M}$ we denoted the set of all metric spaces, let $\mathcal{D} \in P(\mathcal{M})$ and let $F : X \to P_{b,cl}(X)$ be a lower semi-continuous multivalued operator. We say that F has the* selection property *with respect to $\mathcal{D}$ if for any $Y \in \mathcal{D}$, for any continuous function $f : Y \to X$ and for any continuous functional $g : Y \to]0, +\infty[$ such that $G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$, for each $y \in Y$, and for any $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of G.*
If $\mathcal{D} = \mathcal{M}$, then we say that F has the selection property *(we denote this by $F \in SP(X)$).*

More details on the selection property concept can be found in [5, Remarks 2.2.1 - 2.2.4].

We give the following theorem concerning the common fixed points set of two multivalued operators, using the notion of selection property.

Theorem 2.4. *Let (X, d) be a complete metric space and absolute retract for metric spaces and let $F_1, F_2 \in SP(X)$. We suppose that:*

(*i*) *there exist $a_{11}, \ldots, a_{15} \in]0, +\infty[$, with $a_{11} + a_{12} + a_{13} + 2\,a_{14} < 1$, such that for each $x \in X$, any $u_x \in F_1(x)$ and for all $y \in X$, there exists $u_y \in F_2(y)$ so that*

$$d(u_x, u_y) \leq a_{11}\, d(x, y) + a_{12}\, d(x, u_x) + a_{13}\, d(y, u_y) + a_{14}\, d(x, u_y) + a_{15}\, d(y, u_x);$$

(ii) *there exist* $a_{21}, \ldots, a_{25} \in]0, +\infty[$, *with* $a_{21} + a_{22} + a_{23} + 2\, a_{24} < 1$, *such that for each* $x \in X$, *any* $u_x \in F_2(x)$ *and for all* $y \in X$, *there exists* $u_y \in F_1(y)$ *so that*

$$d(u_x, u_y) \leq a_{21}\, d(x, y) + a_{22}\, d(x, u_x) + a_{23}\, d(y, u_y) + a_{24}\, d(x, u_y) + a_{25}\, d(y, u_x).$$

Then $(\mathcal{CF})_{F_1,F_2} = \mathcal{F}_{F_1} = \mathcal{F}_{F_2} \in P_{cl}(X)$ *and* $(\mathcal{CF})_{F_1,F_2}$ *is an absolute retract for metric spaces.*

Proof. As we specified in the proof of Theorem 2.2, we have that $(\mathcal{CF})_{F_1,F_2} = \mathcal{F}_{F_1} = \mathcal{F}_{F_2} \in P_{cl}(X)$.

We shall prove that $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

Let $q \in]1, \min\{(a_{11}+a_{12}+a_{13}+2\,a_{14})^{-1}, (a_{21}+a_{22}+a_{23}+2\,a_{24})^{-1}\}[$ and set $l := \max\{(a_{11}+a_{12}+a_{14})/[1-(a_{13}+a_{14})], (a_{21}+a_{22}+a_{24})/[1-(a_{23}+a_{24})]\} < 1$. We have $ql < 1$.

Let Y be a metric space, $A \in P_{cl}(Y)$ and $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ a continuous function.

Taking into account the fact that X is an absolute retract for metric spaces it follows that there exists a continuous function $\varphi_0 : Y \to X$ such that $\varphi_0|_A = \psi$.

We consider the functional $g_0 : Y \to]0, +\infty[$ defined by

$$g_0(y) = \sup\{\, d(\varphi_0(y), z) \mid z \in F_1(\varphi_0(y))\,\} + 1,$$

for each $y \in Y$. It is not difficult to see that g_0 is continuous.

We have that $F_1(\varphi_0(y)) \cap B(\varphi_0(y), g_0(y)) = F_1(\varphi_0(y))$, for each $y \in Y$.

Also, we observe that the function $\psi : A \to (\mathcal{CF})_{F_1,F_2}$ is a continuous selection of the multivalued mapping $A \ni y \multimap F_1(\varphi_0(y))$.

Because $F_1 \in SP(X)$, it follows that there exists a continuous function $\varphi_1 : Y \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(y) \in F_1(\varphi_0(y))$, for each $y \in Y$.

For $y \in Y$, taking into account that $\varphi_1(y) \in F_1(\varphi_0(y))$ and the condition (i) from the hypothesis, we have that there exists $u_1(y) \in F_2(\varphi_1(y))$ such that

$$d(\varphi_1(y), u_1(y)) \leq a_{11}\, d(\varphi_0(y), \varphi_1(y)) + a_{12}\, d(\varphi_0(y), \varphi_1(y)) + a_{13}\, d(\varphi_1(y), u_1(y)) +$$

$$+ a_{14}\, d(\varphi_0(y), u_1(y)) + a_{15}\, d(\varphi_1(y), \varphi_1(y)) =$$

$$= (a_{11} + a_{12})\, d(\varphi_0(y), \varphi_1(y)) + a_{13}\, d(\varphi_1(y), u_1(y)) + a_{14}\, d(\varphi_0(y), u_1(y)) \leq$$

$$\leq (a_{11} + a_{12} + a_{14})\, d(\varphi_0(y), \varphi_1(y)) + (a_{13} + a_{14})\, d(\varphi_1(y), u_1(y)).$$

So

$$d(\varphi_1(y), u_1(y)) \leq \frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})} d(\varphi_0(y), \varphi_1(y)) \leq l\, d(\varphi_0(y), \varphi_1(y)) <$$

$$< l\, d(\varphi_0(y), \varphi_1(y)) + l < l\, d(\varphi_0(y), \varphi_1(y)) + q^{-1},$$

for each $y \in Y$.

Hence $G_2(y) := F_2(\varphi_1(y)) \cap B(\varphi_1(y), l\, d(\varphi_0(y), \varphi_1(y)) + q^{-1}) \neq \emptyset$, for each $y \in Y$.

Taking into account that $F_2 \in SP(X)$, we obtain that there exists a continuous function $\varphi_2 : Y \to X$ such that $\varphi_2|_A = \psi$ and $\varphi_2(y) \in \overline{G_2(y)}$, for each $y \in Y$.

So, we can write that

$\varphi_2|_A = \psi$,

$\varphi_2(y) \in F_2(\varphi_1(y))$, for each $y \in Y$,

$d(\varphi_1(y), \varphi_2(y)) \leq l\, d(\varphi_0(y), \varphi_1(y)) + q^{-1}$, for each $y \in Y$.

For $y \in Y$, taking into account that $\varphi_2(y) \in F_2(\varphi_1(y))$ and the condition (ii) from the hypothesis, we have that there exists $u_2(y) \in F_1(\varphi_2(y))$ such that

$$d(\varphi_2(y), u_2(y)) \leq a_{21}\, d(\varphi_1(y), \varphi_2(y)) + a_{22}\, d(\varphi_1(y), \varphi_2(y)) + a_{23}\, d(\varphi_2(y), u_2(y)) +$$

$$+ a_{24}\, d(\varphi_1(y), u_2(y)) + a_{25}\, d(\varphi_2(y), \varphi_2(y)) =$$

$$= (a_{21} + a_{22})\, d(\varphi_1(y), \varphi_2(y)) + a_{23}\, d(\varphi_2(y), u_2(y)) + a_{24}\, d(\varphi_1(y), u_2(y)) \leq$$

$$\leq (a_{21} + a_{22} + a_{24})\, d(\varphi_1(y), \varphi_2(y)) + (a_{23} + a_{24})\, d(\varphi_2(y), u_2(y)).$$

So

$$d(\varphi_2(y), u_2(y)) \leq \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})} d(\varphi_1(y), \varphi_2(y)) \leq$$

$$\leq l\, d(\varphi_1(y), \varphi_2(y)) \leq l^2\, d(\varphi_0(y), \varphi_1(y)) + lq^{-1} < l^2\, d(\varphi_0(y), \varphi_1(y)) + q^{-2},$$

for each $y \in Y$.

Hence $G_3(y) := F_1(\varphi_2(y)) \cap B(\varphi_2(y), l^2\, d(\varphi_0(y), \varphi_1(y)) + q^{-2}) \neq \emptyset$, for each $y \in Y$.

Taking into account that $F_1 \in SP(X)$, we obtain that there exists a continuous function $\varphi_3 : Y \to X$ such that $\varphi_3|_A = \psi$ and $\varphi_3(y) \in \overline{G_3(y)}$, for each $y \in Y$.

So, we can write that

$\varphi_3|_A = \psi$,

$\varphi_3(y) \in F_1(\varphi_2(y))$, for each $y \in Y$,

$d(\varphi_2(y), \varphi_3(y)) \leq l^2\, d(\varphi_0(y), \varphi_1(y)) + q^{-2}$, for each $y \in Y$.

By induction, we obtain that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$, where $\varphi_n : Y \to X$ is a continuous function, for each $n \in \mathbb{N}_0$, with $\varphi_0|_A = \psi$ and with the following properties:

$\varphi_n|_A = \psi$,

$\varphi_{2n-1}(y) \in F_1(\varphi_{2n-2}(y))$ and $\varphi_{2n}(y) \in F_2(\varphi_{2n-1}(y))$, for each $y \in Y$,

$d(\varphi_{n-1}(y), \varphi_n(y)) \leq l^{n-1}\, d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$, for each $y \in Y$,

for each $n \in \mathbb{N}$.

For each $\lambda > 0$ we put $Y_\lambda := \{\, y \in Y \mid d(\varphi_0(y), \varphi_1(y)) < \lambda \,\}$. The family of sets $\{\, Y_\lambda \mid \lambda > 0 \,\}$ is an open covering of Y.

For each $\lambda > 0$ the sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges uniformly on Y_λ, taking into account that $d(\varphi_{n-1}(y), \varphi_n(y)) \leq l^{n-1} d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$, for each $y \in Y$ and for each $n \in \mathbb{N}$, that $l < 1$, $q > 1$ and the completeness of X.

Let $\varphi : Y \to X$ be the pointwise limit of $(\varphi_n)_{n\in\mathbb{N}_0}$. The function φ is continuous.

Because $\varphi_n|_A = \psi$, for each $n \in \mathbb{N}_0$, we are able to write that $\varphi|_A = \psi$.

We have $\varphi_{2n-1}(y) \in F_1(\varphi_{2n-2}(y))$, for each $y \in Y$ and for each $n \in \mathbb{N}$. Letting n tend to the infinity, it follows that $\varphi(y) \in F_1(\varphi(y))$, for each $y \in Y$.

We have $\varphi_{2n}(y) \in F_2(\varphi_{2n-1}(y))$, for each $y \in Y$ and for each $n \in \mathbb{N}$. Letting n tend to the infinity, it follows that $\varphi(y) \in F_2(\varphi(y))$, for each $y \in Y$.

So, we can write that $\varphi : Y \to (\mathcal{CF})_{F_1,F_2}$. □

Remark 2.1. *In the hypotheses of Theorem* 2.2*, we are able to write that* X *is an absolute retract for metric spaces* (*see, for example,* [3, *p.* 85]*,* [5, *p.* 2676]) *and that the multivalued operators* F_1 *and* F_2 *have the selection property, as follows from* [5, *Example* 1.6.1] *and* [5, *Remark* 2.2.3]*. Now, applying Theorem* 2.4*, we obtain that* $(\mathcal{CF})_{F_1,F_2}$ *is an absolute retract for metric spaces.*

We give the following theorem concerning the fixed points set of a multivalued operator, using the notion of selection property.

Theorem 2.5. *Let* (X, d) *be a complete metric space and absolute retract for metric spaces and let* $F \in SP(X)$*, with the property that there exist* $a_1, \ldots, a_5 \in]0, +\infty[$*, with* $a_1 + a_2 + a_3 + 2a_4 < 1$*, such that for each* $x \in X$*,* *any* $u_x \in F(x)$ *and for all* $y \in X$*, there exists* $u_y \in F(y)$ *so that*

$$d(u_x, u_y) \leq a_1\, d(x, y) + a_2\, d(x, u_x) + a_3\, d(y, u_y) + a_4\, d(x, u_y) + a_5\, d(y, u_x).$$

Then $\mathcal{F}_F \in P_{cl}(X)$ *and* $\mathcal{F}_F$ *is an absolute retract for metric spaces.*

Proof. As we specified in the proof of Theorem 2.3, we have that $\mathcal{F}_F \in P_{cl}(X)$.

Let $q \in]1, (a_1 + a_2 + a_3 + 2a_4)^{-1}[$ and set $l := (a_1 + a_2 + a_4)/[1 - (a_3 + a_4)]$. We have $ql < 1$.

In order to prove that $\mathcal{F}_F$ is an absolute retract for metric spaces, let Y be a metric space, $A \in P_{cl}(Y)$ and $\psi : A \to \mathcal{F}_F$ a continuous function.

Using an argument similar to that in the proof of Theorem 2.4, we obtain a sequence $(\varphi_n)_{n\in\mathbb{N}_0}$, where $\varphi_n : Y \to X$ is a continuous function, for each $n \in \mathbb{N}_0$, with $\varphi_0|_A = \psi$ and with the following properties:

$\varphi_n|_A = \psi$,

$\varphi_n(y) \in F(\varphi_{n-1}(y))$, for each $y \in Y$,

$d(\varphi_{n-1}(y), \varphi_n(y)) \leq l^{n-1} d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$, for each $y \in Y$,

for each $n \in \mathbb{N}$.

The sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ converges pointwise on Y to a continuous function $\varphi : Y \to X$, with $\varphi|_A = \psi$. Also, it can be written that $\varphi : Y \to \mathcal{F}_F$. □

Remark 2.2. *In the hypotheses of Theorem* 2.3*, as we motivated in Remark* 2.1*, we can write that* X *is an absolute retract for metric spaces and that the multivalued operator* F *has the selection property. Now, applying Theorem* 2.5*, we obtain that* $\mathcal{F}_F$ *is an absolute retract for metric spaces.*

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